SOME THEOREMS ON OPEN RIEMANN SURFACES

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Let F be an open Riemann surface spread over the z-plane. We say that F is of positive or null boundary, according as there exists a Green's function on F or not. Let $u(z)$ be a harmonic function on F and

$$D(u)=\iint_F\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy \qquad (z=x+iy)$$

be its Dirichlet integral. As R. Nevanlinna[1] proved, if F is of null boundary, there exists no one-valued non-constant harmonic function on F, whose Dirichlet integral is finite. This Nevanlinna's theorem was proved very simply by Kuroda.[2] By this mehod, we will prove

THEOREM 1. *Let F be an open Riemann surface with null boundary and Δ be a non-compact domain on F, whose houndary Λ consists of (compact or non-compact) analytic curves. Let $u(z)$ be a one-valued harmonic function in Δ, such that $u(z)=0$ on Λ and its Dirichlet integral in Δ*

$$D(u)=\iint_\Delta\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy$$

is finite. Then $u(z)\equiv 0$.

This theorem was proved by R. Nevanlinna[3] under the condition that $u(z)$ is harmonic outside a compact domain F_0 and its Dirichlet integral in $F-F_0$ is finite.

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1) (a) R. Nevanlinna: Quadratisch integrierbare Differentiale auf einer Riemannschen Mannigfaltigkeit. Annales Acad. Sci. Fenn. Series A, Mathematica-Physica **1** (1941).
(b) Über das Anwachsen des Dirichletintegrals einer analytischen Funktion auf einer offenen Riemannschen Fläche. Annales Acad. Sci. Fenn. Series A, Mathematica-Physica **45** (1948).

2) T. Kuroda: Some remarks on an open Riemann surface. To appear in the Tohoku Math. Journ.

3) R. Nevanlinna. l.c. 1) (a).

Proof. We choose a schlicht disc F_0 in $\varDelta$, whose boundary is Γ_0. We appoximate F by a sequence of compact Riemann surfaces F_n, $\bar{F}_n \subset F_{n+1}$ $(n=0, 1, 2, \ldots)$, $F_n \to F$, whose boundary $\Gamma_0+\Gamma_n$ consists of a finite number of analytic Jordan curves.

Let

(1) $$\omega_n(z)=\omega(z, \Gamma_n, F_n-F_0)$$

be the harmonic measure of Γ_n with respect to F_n-F_0, such that $\omega_n(z)$ is harmonic in F_n-F_0 and $\omega_n(z)=0$ on Γ_0, $\omega_n(z)=1$ on Γ_n.

Let $\bar{\omega}_n(z)$ be its conjugate harmonic function, then

$$D(\omega_n)=\iint_{F_n-F_0}\left(\left(\frac{\partial \omega_n}{\partial x}\right)^2+\left(\frac{\partial \omega_n}{\partial y}\right)^2\right)dxdy=\int_{\Gamma_n}\omega_n d\bar{\omega}_n=\int_{\Gamma_n}d\bar{\omega}_n=\int_{\Gamma_0}d\bar{\omega}_n.$$

As Nevanlinna[4] proved, $D(\omega_n)\to 0$ as $n\to\infty$, so that

(2) $$\mu_n=\frac{2\pi}{D(\omega_n)}\to\infty \quad \text{as} \quad n\to\infty.$$

We put

(3) $$u_n(z)=\mu_n\omega_n(z), \qquad v_n(z)=\mu_n\bar{\omega}_n(z),$$

then $u_n(z)=0$ on Γ_0, $u_n(z)=\mu_n$ on Γ_n and $v_n(z)$ is its conjugate harmonic function, such that

(4) $$\int_{\Gamma_0}dv_n(z)=2\pi.$$

Let D_λ $(0\leqq\lambda\leqq\mu_n)$ be the domain, such that $0\leqq u_n(z)\leqq\lambda$ and $\varDelta_\lambda$ be the common part of $\varDelta$ and $D_\lambda+F_0$. Let Γ_λ be the niveau curve $u_n(z)=\lambda$ and $\Gamma_\lambda(\varDelta)$ be *its part contained in* $\varDelta$.

To prove $u(z)\equiv 0$, we assume that $u(z)\not\equiv 0$ and let $v(z)$ be its conjugate harmonic function.

Since $u(z)=0$ on Λ,

(5) $$D(\lambda)=\iint_{\Delta_\lambda}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy=\int_{\Gamma_\lambda(\Delta)}u\frac{\partial u}{\partial \nu}ds=\int_{\Gamma_\lambda(\Delta)}u\frac{\partial u}{\partial u_n}dv_n,$$

where ν is the inner normal and ds is the arc element on Γ_λ.
From

(6) $$D(\lambda)=\int_0^\lambda d\lambda\int_{\Gamma_\lambda(\Delta)}\left(\left(\frac{\partial u}{\partial u_n}\right)^2+\left(\frac{\partial u}{\partial v_n}\right)^2\right)dv_n+\text{const.},$$

we have

$$D'(\lambda)=\int_{\Gamma_\lambda(\Delta)}\left(\left(\frac{\partial u}{\partial u_n}\right)^2+\left(\frac{\partial u}{\partial v_n}\right)^2\right)dv_n,$$

so that from (5),

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[4] R. Nevanlinna. l.c. [1] (a).

$$D^2(\lambda) \leqq \int_{\Gamma_\lambda(\Delta)} u^2 dv_n \int_{\Gamma_\lambda(\Delta)} \left(\frac{\partial u}{\partial u_n}\right)^2 dv_n \leqq D'(\lambda) \int_{\Gamma_\lambda(\Delta)} u^2 dv_n.$$

We put

(7) $$m(\lambda) = \int_{\Gamma_\lambda(\Delta)} u^2 dv_n,$$

then

(8) $$D^2(\lambda) \leqq m(\lambda) D'(\lambda).$$

Since $u=0$ on Δ,

(9) $$m'(\lambda) = 2\int_{\Gamma_\lambda(\Delta)} u \frac{\partial u}{\partial u_n} dv_n = 2\,D(\lambda) > 0,$$

(10) $$m''(\lambda) = 2\,D'(\lambda).$$

Since by (9), $m'(\lambda)$ (>0) is an increasing function of λ, $m(\lambda)$ is an increasing convex function of λ.

Since $m'(\lambda)>0$, we have from (8), (9), (10),

$$\frac{m'(\lambda)}{m(\lambda)} \leqq 2\frac{m''(\lambda)}{m'(\lambda)}.$$

Hence integrating, we have

$$m(\lambda) \leqq k[m'(\lambda)]^2, \qquad k = \frac{m(0)}{(m'(0))^2} = \frac{m(0)}{4(D(0))^2},$$

so that from (8), (9),

$$\frac{D^2(\lambda)}{D'(\lambda)} \leqq 4kD^2(\lambda), \quad \text{or} \quad d\lambda \leqq 4kdD(\lambda).$$

Hence integrating on $[0, \mu_n]$, we have

$$\mu_n \leqq 4k(D(\mu_n) - D(0)) \leqq 4kD(\mu_n).$$

Since by the hypothesis, $D(\mu_n)$ is bounded and $\mu_n \to \infty$ as $n \to \infty$, this is absurd. Hence $u(z) \equiv 0$. q.e.d.

THEOREM 2. *Let F be an open Riemann surface with null boundary and Δ be a non-compact domain on F, whose boundary Γ consists of (compact or non-compact) analytic curves. Let $u(z)$ be a one-valued harmonic function in Δ, whose Dirichlet integral in Δ is finite. If $u(z)$ is bounded on Γ, then $u(z)$ is bounded in Δ, such that*

$$m \leqq u(z) \leqq M \quad \textit{in} \quad \Delta,$$

where

$$m = \inf_{\Gamma} u(z), \qquad M = \sup_{\Gamma} u(z).$$[5]

[5] c. f. R. Nevanlinna. l.c. [1] (a).

Proof. Suppose that there exists a point z_0 in Δ, such that $u(z_0)>M$. We choose K, such that $u(z_0)>K>M$ and let $\Delta(K)$ be the sub-domain of Δ, such that $u(z)\geqq K$ and Λ be its boundary. Then $u(z)=K$ on Λ, so that Λ has no common points with Γ, hence $\Delta(K)$ is non-compact. Since $u(z)=K$ on Λ and its Dirichlet integral in $\Delta(K)$ is finite, we have by Theorem 1, $u(z)\equiv K$, which is absurd, since $u(z_0)>K$. Hence $u(z)\leqq M$ in Δ. Similarly $u(z)\geqq m$ in Δ. q.e.d.

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We will prove

THEOREM 3. *Let F be an open Riemann surface and $z=0$ be contained in F. We approximate F by a sequence of compact Riemann surfaces F_n, $\bar{F}_n\subset F_{n+1}$ $(n=0,1,2....)$, $F_n\rightarrow F$, whose boundary Γ_n consists of a finite number of analytic Jordan curves and F_0 contains $z=0$. Let $g_n(z, 0)$ be the Green's function of F_n, with $z=0$ as its pole and let at $z=0$,*

$$g_n(z, 0)=\log\frac{1}{|z|}+\gamma_n+\varepsilon_n(z)\qquad(\varepsilon_n(0)=0),$$

where γ_n is the Robin's constant. Then

$$g_n(z, 0)-\gamma_n\qquad(n=0,1,2,...)$$

is uniformly bounded in any compact domain of F, which does not contain $z=0$.

Proof. Let $F_0: |z|\leqq\rho<1$ be contained in F and $\Gamma_0: |z|=\rho$.
We put

(1) $$M_n=\underset{\Gamma_0}{\mathrm{Max.}}\ g_n(z, 0).$$

Then Heins[6] proved that

(2) $$u_n(z)=M_n-g_n(z, 0)$$

is uniformly bounded in any compact domain of F, which does not contain $z=0$. For the sake of completeness, we will reproduce his proof. Now

(3) $$u_n(z)>0\quad\text{in}\quad F_n-F_0$$

and since $u_1(z)-u_n(z)$ is harmonic in F_1 and at some point z_0 on Γ_0, $u_n(z_0)=0$, $u_1(z_0)\geqq 0$, by the maximum principle, we have

$$\underset{\Gamma_1}{\mathrm{Max.}}\ (u_1(z)-u_n(z))\geqq 0.$$

Since $u_1(z)=M_1$ on Γ_1, we have

[6] M. Heins: The conformal mapping of simply connected Riemann surfaces. Annals of Math. **50** (1949).

(4) $$\underset{\Gamma_1}{\mathrm{Min.}}\, u_n(z) \leqq M_1,$$

Let $|z| \leqq \rho_1$ $(\rho<\rho_1<1)$ be contained in F_1, then from (3), (4) and Harnack's theorem on positive harmonic functions, we conclude that for any compact domain $\varDelta$ of F, which contains $|z| \leqq \rho_1$, there exists a constant $K=K(\varDelta)$, such that for $n \geqq n_0$,

(5) $$|g_n(z, 0)-M_n| \leqq K \quad \text{in} \quad \varDelta \quad \text{outside} \quad |z|=\rho_1.$$

Hence

$$|v_n(z)|=|g_n(z, 0)-M_n-\log\frac{1}{|z|}| \leqq K+\log\frac{1}{\rho_1} \quad \text{on} \quad |z|=\rho_1.$$

Since $v_n(z)$ is harmonic in $|z| \leqq \rho_1$,

$$|v_n(0)|=|\gamma_n-M_n| \leqq K+\log\frac{1}{\rho_1},$$

so that from (5),

(6) $$|g_n(z, 0)-\gamma_n| \leqq K+|\gamma_n-M_n| \leqq 2K+\log\frac{1}{\rho_1} \quad \text{in} \quad \varDelta \quad \text{outside} \quad |z|=\rho_1.$$

Hence

(7) $$|g_n(z, 0)-\log\frac{1}{|z|}-\gamma_n| \leqq 2K+2\log\frac{1}{\rho_1} \quad \text{on} \quad |z|=\rho_1.$$

Since the left hand side of (7) is harmonic in $|z| \leqq \rho_1$, (7) holds in $|z| \leqq \rho_1$, so that

(8) $$|g_n(z, 0)-\gamma_n| \leqq 2K+2\log\frac{1}{\rho_1}+\log\frac{1}{|z|} \quad \text{in} \quad |z| \leqq \rho_1.$$

From (6), (8), we have the theorem. q.e.d.

By Theorem 3, we can find a partial sequence n_κ, such that

(9) $$\lim_{\kappa}\,(g_{n_\kappa}(z, 0)-\gamma_{n_\kappa})=u(z, 0)$$

uniformly in any compact domain of F, which does not contain $z=0$. $u(z, 0)$ is harmonic on F, except at $z=0$, where it has a logarithmic singularity. Hence we have

THEOREM 4. *Let F be an open Riemann surface and z_0 be any point of F. then there exists a potential function $u(z, z_0)$, which is harmonic on F, except at z_0, where it has a logarithmic singularity, such that*

$$u(z, z_0)-\log\frac{1}{|z-z_0|}$$

is harmonic at z_0.

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