OPEN RIEMANN SURFACE WITH NULL BOUNDARY

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1. Recently the writer has obtained some results concerning meromorphic or algebroidal functions with the set of essential singularities of capacity zero,[1] with an aid of a theorem of Evans.[2] In the present paper, suggested from recent interesting papers of Sario[3] and Pfluger,[4] the writer will extend his results to single-valued analytic functions defined on open abstract Riemann surfaces with null boundary in the sense of Nevanlinna,[5] using a lemma instead of Evans' theorem.

2. Let F be an arbitrary open Riemann surface of finite or infinite genus and $\{F_n\}$ $(n=0, 1, \ldots)$ be a sequence of compact domains of F which satisfies the following conditions:

i) F_0 is simply connected,

ii) the boundary Γ_n of F_n consists of a finite number of simple closed analytic curves,

iii) $\overline{F}_n \subset F_{n+1}$ $(n=0, 1, \ldots)$ where $\overline{F}_n$ denotes the closure of F_n,

iv) every component of the open set $F-\overline{F}_n$ consists of a finite number of non-compact domains,

v) $\bigcup_{n=0}^{\infty} F_n = F$.

Received April 23, 1951.

1) K. Noshiro: [1] Contributions to the theory of the singularities of analytic functions, Jap. Journ. of Math. **19** (1948), pp. 299-327; [2] Note on the cluster sets of analytic functions, Journ. Math. Soc. Japan, **1** (1950), pp. 275-281; [3] A theorem on the cluster sets of pseudo-analytic functions, Nagoya Math. Journ. **1** (1950), pp. 83-89.

2) G. C. Evans: Potentials and positively infinite singularities of harmonic functions, Monatshefte für Math. und Phys. **43** (1936), pp. 419-424.

3) Leo Sario: [1] Über Riemannsche Flächen mit hebbarem Rand, Ann. Acad. Sci. Fenn. A. I. 50 (1948), 79 pp.; [2] Sur les problèmes du type des surfaces de Riemann, Comptes Rendus, Paris, **229** (1949), pp. 1109-1111; [3] Questions d'existence au voisinage de la frontière d'une surface de Riemann, Comptes Rendus, Paris, **230** (1950), pp. 269-271.

4) A. Pfluger: Über das Anwachsen eindeutiger analytischer Funktion auf offenen Riemannschen Fläche, Ann. Acad. Sci. Fenn. A. I. 64 (1949), 18 pp.

5) R. Nevanlinna: Quadratisch integrierbare Differentiale auf einer Riemannschen Mannigfaltigkeit, Ann. Acad. Sci. Fenn. A. I. 1 (1941), 34 pp.

Then the sequence $\{F_n\}$ is said to be an exhaustion of F.

Consider the open set $F_n-\overline{F}_{n-1}$ which consists of a finite number of connected components and the harmonic measure

(1) $$\omega_n=\omega(p, \Gamma_n, F_n-\overline{F}_{n-1}) \quad (n=1, 2, \ldots)$$

of $F_n-\overline{F}_{n-1}$ with boundary values 1 on Γ_n and 0 on Γ_{n-1}. We denote by d_n the total variation of the conjugate harmonic function $\bar{\omega}_n$ along Γ_n:

(2) $$\int_{\Gamma_n} d\bar{\omega}_n=d_n,$$

where the sense of Γ_n is positive with respect to $F_n-\overline{F}_{n-1}$.

We define the *modulus* μ_n of $F_n-\overline{F}_{n-1}$ by the quantity[6]

(3) $$\mu_n=2\pi/d_n.$$

Select suitably an additive constant of $\bar{\omega}_n$ for each connected component of $F_n-\overline{F}_{n-1}$, then the function

(4) $$z_n=\frac{2\pi}{d_n}(\omega_n+i\bar{\omega}_n)+r_{n-1}=x_n+iy_n,$$

where

(5) $$r_n=\sum_{\nu=1}^{n}\mu_\nu \quad (n\geqq 1), \qquad r_0=0,$$

maps the open set $F_n-\overline{F}_{n-1}$ with a finite number of suitable slits onto a slit-rectangle $K_n: r_{n-1}<x_n<r_n$, $0<y_n<2\pi$ in a one-one conformal manner.[7] Accordingly, the function $z=x+iy$ defined by z_n for each $F_n-\overline{F}_{n-1}$ $(n=1, 2, \ldots)$ maps the subsurface $F-\overline{F}_0$ with a finite or enumerable number of suitable slits onto a union of slit-rectangles: $K=\bigcup_{n=1}^{\infty}K_n$, lying in the domain $0<x<R=\lim_{n\to\infty}r_n=\sum_{\nu=1}^{\infty}\mu_\nu$, $0<y<2\pi$, in a one-one conformal manner. For convenience, we shall call the figure K a *graph* of $F-\overline{F}_0$ by the exhaustion $\{F_n\}$. Similarly we can also define the graph of $F_n-\overline{F}_0$.

We first prove the following

LEMMA. *Let $\{F_n\}$ be an exhaustion of a Riemann surface F with null boundary and k_ν $(\nu=1, 2, \ldots)$ be an arbitrary sequence of positive numbers. Then there exists a subsequence $\{F_{n_\nu}\}$ $(\nu=0, 1, \ldots)$ which is an exhaustion such that*

$$\mu_{n_\nu}\geqq k_\nu \quad (\nu=1, 2, \ldots),$$

where $F_{n_0}=F_0$ and μ_{n_ν} denotes the modulus of the open set $F_{n_\nu}-\overline{F}_{n_{\nu-1}}$.

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[6] For the definition of modulus, cf. Sario: loc. cit. [3]; Pfluger: loc. cit.

[7] Cf. Sario: loc. cit. [1], p. 11.

Proof. It is obvious that

$$F_n-\overline{F}_j \subset F_n-\overline{F}_0 \quad \text{for} \quad 0<j<n.$$

Consider two harmonic measures

$$\text{(6)} \qquad \omega_n^{(j)}=\omega(p,\ \Gamma_n,\ F_n-\overline{F}_j) \quad \text{and} \quad \omega_n^{(0)}=\omega(p,\ \Gamma_n',\ F_n-\overline{F}_0).$$

Then, by the maximum principle, we have

$$\text{(7)} \qquad \omega_n^{(j)}(p)<\omega_n^{(0)}(p).$$

Since F has a null boundary, $\omega_n^{(0)}$ $(n=1, 2, \ldots)$ converges to a constant zero on F. Consequently, for fixed j, $\omega_n^{(j)}\to 0$ as $n\to\infty$. Denote by $\overline{\omega}_n^{(j)}$ the conjugate harmonic function of $\omega_n^{(j)}$ and put

$$\text{(8)} \qquad \int_{\Gamma_j} d\overline{\omega}_n^{(j)}=d_n^{(j)}>0.$$

Then, it is easily seen that the modulus $\mu_n^{(j)}=2\pi/d_n^{(j)}$ of the open set $F_n-\overline{F}_j$ tends to infinity as $n\to\infty$. Accordingly, for any positive number k, we can find a number n such that $\mu_n^{(j)}\geqq k$. Repeating the same argument, our assertion is proved.

As an application of the graph of $F_n-\overline{F}_0$ by an exhaustion $\{F_n\}$, we can state

THEOREM 1. *Let μ_n^* and μ_n be the moduli of $F_n-\overline{F}_0$ and $F_n-\overline{F}_{n-1}$ respectively. Then, there exists*

$$\text{(9)} \qquad \mu_n^*\geqq\mu_1+\mu_2+\ \cdots\ +\mu_n.$$

Proof. Consider $w=\omega_n^{(0)}+i\overline{\omega}_n^{(0)}$(cf. (6)) as a function of $z=x+iy$ in the graph of $F_n-\overline{F}_0$. Then, it is clear that

$$d_n^{(0)}=\int_{x=\lambda} d\overline{\omega}_n^{(0)}\leqq\int_{x=\lambda}|dw| \quad (0<\lambda<r_n=\sum_{\nu=1}^{n}\mu_\nu).$$

Schwarz's inequality gives

$$[d_n^{(0)}]^2\leqq\left(\int_{x=\lambda}\left|\frac{dw}{dz}\right|^2 dy\right)\cdot\left(\int_{x=\lambda}dy\right)=2\pi\int_{x=\lambda}\left|\frac{dw}{dz}\right|^2 dy,$$

whence, by integration

$$r_n[d_n^{(0)}]^2\leqq2\pi\int_0^{r_n}\int_{x=\lambda}\left|\frac{dw}{dz}\right|^2 dyd\lambda=2\pi\int_{\Gamma_0}d\overline{\omega}_n^{(0)}=2\pi d_n^{(0)}.$$

Therefore

$$r_n\leqq2\pi/d_n^{(0)}=\mu_n^*.$$

Combining Lemma with Sario's theorem which is easily deduced from (9) by Nevanlinna's theorem, we can complete Sario's result in the following

THEOREM 2. *In order that an open Riemann surface F has a null boundary it is necessary and sufficient that there exists an exhaustion $\{F_n\}$ such that $\sum_{n=1}^{\infty}\mu_n=\infty$ where μ_n denotes the modulus of the open set $F_n-\overline{F}_{n-1}$.*[8)]

3. Let F be an open Riemann surface with null boundary. Then, by Theorem 2, we can select an exhaustion $\{F_n\}$ of F such that $\sum_{n=1}^{\infty}\mu_n=\infty$, μ_n denoting the modulus of $F_n-\overline{F}_{n-1}$. Suppose that $w=f(p)$ is non-constant, single-valued and meromorphic on the surface F. Then, the space formed by the elements $q=[p, f(p)]$, where p varies on F, defines a conformally equivalent covering surface Φ of the w-plane. Clearly the mapping $p\longleftrightarrow q$, where $q=[p, f(p)]$, is topological and conformal.

We first give a proof for Yûjôbô's theorem which is an extension of a theorem of Tsuji.[9)]

THEOREM 3 (Yûjôbô). *The covering surface Φ has Gross' property.*[10)]

Proof. Let $q_0=[p_0, f(p_0)]$ be an arbitrary point on Φ with projection $w_0=f(p_0)$. Consider the star-region H formed by the segments from q_0 to singular points (algebraic branch-points or accessible boundary points of Φ) along all rays: $\arg(w-w_0)=\varphi$ $(0\leqq\varphi<2\pi)$ on Φ. We shall show that the linear measure of the set E of arguments φ of singular rays (by which we understand rays meeting singular points in finite distances) is equal to zero. Denote by H_ρ the part of H above a circular disc $|w-w_0|<\rho$ and by Δ_ρ the image of H_ρ by the mapping $p\longleftrightarrow q$. Then Δ_ρ is a simply connected domain on the surface F. We select as F_0 the image of a small circular disc with centre q_0.

Now, we shall use the graph K, lying in the half-strip: $0<x<\infty$, $0<y<2\pi$, of the subsurface $F-\overline{F}_0$ by the exhaustion $\{F_n\}$ with $\sum_{n=1}^{\infty}\mu_n=\infty$. In the graph K we consider the image $\tilde{\Delta}_\rho$ of $\Delta_\rho-\overline{F}_0$ by the function $z(p)=x(p)+iy(p)$, defined in **2**, and the composed function $w=w(z)=f(p(z))$ defined on $\tilde{\Delta}_\rho$. Let $\tilde{\Theta}_\lambda$ be the image of the intersection Θ_λ of the niveau curve C_λ: $x(p)=\lambda$ $(0<\lambda<\infty)$[11)] and

8) Sario stated only the sufficient condition. Cf. loc. cit. [3]; R. Nevanlinna: loc. cit. Moreover, Sario remarked that a graph K of finite length can be constructed by a suitable choice of an exhaustion of F, in the case when F is simply connected and of parabolic type. Cf. loc. cit. [2].

9) Z. Yûjôbô reported this result at the annual meeting of the Math. Soc. of Japan in 1948. However, his proof has been published nowhere. M. Tsuji: On the behaviour of a meromorphic function in the neighbourhood of a closed set of capacity zero, Proc. Imp. Acad. **18** (1942), pp. 213-219.

10) W. Gross: Über die Singularitäten analytischer Funktionen, Monatshefte für Math. und Phys. **29** (1918), pp. 1-47.

11) Evidently the niveau curve C_λ coincides with Γ_n when $\lambda=r_n$ $(n=0, 1, \ldots)$.

Δ_ρ by the function $z(p)=x(p)+iy(p)$. We denote by $\theta(\lambda)$ the total length of $\tilde{\Theta}_\lambda$ and $L(\lambda)$ that of the image of $\tilde{\Theta}_\lambda$ by $w=w(z)$. Then we can apply the method in proving a well-known theorem of Gross. It is clear that

$$L(\lambda)=\int_{\tilde{\Theta}_\lambda} |w'(z)|\,dy.$$

By Schwarz's inequality

$$[L(\lambda)]^2 \leqq \int_{\tilde{\Theta}_\lambda} |w'(z)|^2 dy \cdot \int_{\tilde{\Theta}_\lambda} dy = \theta(\lambda)\int_{\tilde{\Theta}_\lambda} |w'(z)|^2 dy = \theta(\lambda)\frac{dA(\lambda)}{d\lambda},$$

where

$$A(\lambda)=\int_0^\lambda\int_{\tilde{\Theta}_\lambda} |w'(z)|^2 dx\,dy.$$

Hence

$$\int_{\lambda_0}^\lambda \frac{[L(\lambda)]^2}{\theta(\lambda)}\,d\lambda \leqq A(\lambda)-A(\lambda_0)\leqq \pi\rho^2.$$

Since $\theta(\lambda)\leqq 2\pi$,

$$\int_{\lambda_0}^\lambda [L(\lambda)]^2 d\lambda \leqq 2\pi^2\rho^2,$$

whence follows $\varliminf_{\lambda\to\infty} L(\lambda)=0$. Accordingly we easily see that our assertion is true.

Remark. It is well-known that Iversen's property is a direct result from Gross' property. Thus Theorem 3 includes a theorem due to Stoïlow.[12] Next consider a connected piece Φ_ρ of Φ above any circular disc (c) : $|w-w_0|<\rho$. Denote by $n(w)$ the number of sheets above w inside (c) and put $N=\sup_{w\in(c)} n(w)$, $(0<N\leqq\infty)$. Then, the set E of points w such that $n(w)<N$, $w\in(c)$ is of capacity zero.[13] Consequently, the spherical area of Φ is infinite, provided that Φ has an infinite number of sheets. It is also known that a Riemann surface on which no Green's function exists coincides with a Riemann surface with null bounbary.[14]

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12) S. Stoïlow: Sur les singularités des fonctions analytiques multiformes dont la surface de Riemann a sa frontière de mesure harmonique null, Mathematica, **19** (1943), pp. 126-138.

13) Y. Nagai: On the behaviour of the boundary of Riemann surfaces, II, Proc. Jap. Acad. **26** (1950), pp. 10-16; M. Tsuji: Some metrical theorems on Fuchsian groups, Kôdai Math. Sem. Rep. Nos. 4-5 (1950), pp. 89-93; A. Mori: On Riemann surfaces, on which no bounded harmonic function exists, which will appear in Journ. Math. Soc. Japan.

14) K. I. Virtanen: Über die Existenz von beschränkten harmonischen Funktionen auf offenen Riemannschen Flächen, Ann. Acad. Sci. Fenn. A. I. 75 (1950), 8 pp.

If we use the Lemma instead of Evans' theorem, in the same way in proving Theorem 3, and apply Ahlfors' theory of covering surfaces,[15] the arguments in a previous paper of the writer (loc. cit. [1]) will give the following theorems.

THEOREM 4. *Φ is regularly exhaustible in the sense of Ahlfors.*[16]

THEOREM 5. *Denote by Δ_λ the compact domain of F bounded by the niveau curve: $x(p)=\lambda$ $(0<\lambda<\infty)$ and by Φ_λ the image of Δ_λ on Φ by the mapping $p\longleftrightarrow q$. Let $D_1, D_2, \ldots, D_m$ $(m\geqq 2)$ be m closed disjoint circular discs on the Riemann w-sphere. We define the defect $\delta(D_j)$, the ramification index $\vartheta(D_j)$ and a quantity ξ by*

$$\delta(D_j)=\varliminf_{\lambda\to\infty}\left[1-\frac{n(\lambda, D_j)}{S(\lambda)}\right],\quad \vartheta(D_j)=\varliminf_{\lambda\to\infty}\frac{n_1(\lambda, D_j)}{S(\lambda)},\quad \xi=\varlimsup_{\lambda\to\infty}\frac{\rho^+(\Delta_\lambda)}{S(\lambda)},$$

where $n(\lambda, D_j)$ denotes the number of sheets of all islands above D_j, $n_1(\lambda, D_j)$ the number of orders of the branch-points of the islands above D_j, $S(\lambda)$ the average number of sheets of Φ_λ with respect to the w-sphere, $\rho(\Delta_\lambda)$ the Euler characteristic of Δ_λ and $\rho^+=\max(0, \rho)$. Then, there exists

$$\sum_{j=1}^{m}\delta(D_j)+\sum_{j=1}^{m}\vartheta(D_j)\leqq 2+\xi.$$[17]

THEOREM 6. *Suppose that the covering surface Φ has an accessible boundary point Ω with projection w_0. Denote by Φ_ρ the ρ-neighbourhood of Ω which is a covering surface of the circular disc $(c): |w-w_0|<\rho$. We suppose further that Φ_ρ is simply connected. Then Φ_ρ covers every point infinitely often inside (c) with one possible exception.*[18]

To prove Theorem 6, it is necessary to notice that Φ_ρ has an infinite number of sheets. Denote by $n(w)$ the number of sheets of above w inside (c) and put $\sup_{w\in(c)} n(w)=N$. Then we have necessarily $N=\infty$. It is known that the set E of points w such that $n(w)<N$, $w\in(c)$ is of capacity zero (cf. Remark). Accordingly, we can draw a circle $c_1: |w-w_0|=\rho_1$ $(0<\rho_1<\rho)$ such that $c_1\cap E=0$ and Φ_ρ has no algebraic branch-point above the circle c_1. Suppose that $N<\infty$, contrary to the assertion, and consider all loop-cuts of Φ_ρ above c_1. Then there would exist at least one loop-cut by which Φ_ρ is decomposed into two multiply con-

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15) L. Ahlfors: Zur Theorie der Überlagerungsflächen, Acta Math. **65** (1935), pp. 157–194.

16) Cf. Noshiro: loc. cit. [1], p. 307. A. Mori kindly remarked that in the case when Φ has a finite number of sheets, the assertion is directly proved by the fact that a bounded closed set of capacity zero is of linear measure zero.

17) Cf. Noshiro: loc. cit. [1], p. 310.

18) Compare with Noshiro: loc. cit. [1], Theorem 3, p. 315 and Theorem 4, p. 327.

nected pieces. This contradicts to the assumption that Φ_p is simply connected.

Now, we shall use Pfluger's theorm[19]: Suppose that there is a quasi-conformal mapping between two open Riemann surfaces F and F'. Then F' has a null boundary if and only if F has a null boundary.

As an immediate consequence, we obtain

THEOREM 7. *Theorem 3, 4, 5, 6 remain true in the case when $w=f(p)$ is a quasi-analytic function on an open Riemann surface F with null boundary.*

4. Finally we shall give a remark and propose a problem. Let F be an open abstract Riemann surface. Suppose that there exists a function $u(p)$ harmonic at every point on F except a single point p_0 such that

(1)′ $$u = \log|t| + \text{a harmonic function}$$

in a neighbourhood of p_0, t being a local parameter at p_0, and

(2)′ $u(p)$ tends to $+\infty$, as p converges to the ideal boundary Γ of F.

Then it is easily concluded that F has a null boundary. To prove this, denote F_λ the compact domain bounded by the niveau curve C_λ: $u(p)=\lambda$ $(-\infty<\lambda<+\infty)$. Then, the harmonic measure $\omega_\lambda(p)=\omega(p, C_\lambda, F_\lambda-\overline{F}_{\lambda_0})$, $(-\infty<\lambda_0<\lambda<+\infty)$, will be written in the form:

$$\omega_\lambda(p)=\frac{u(p)-\lambda_0}{\lambda-\lambda_0}.$$

Therefore, keeping λ_0 fixed and letting λ tend to $+\infty$, we see that $\omega_\lambda(p)\to 0$ on $F-\overline{F}_{\lambda_0}$.

Problem. Is the converse true? More precisely: Does there exist a function which is harmonic at every point on F except a single point p_0 and satisfies (1)′ and (2)′ when F is an open Riemann surface with null boundary? (An extension of Evans' theorem).

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[19] A. Pfluger: Sur une propriété de l'application quasi-conforme d'une surface de Riemann ouverte, Comptes Rendus, Paris, **227** (1948), pp. 25-26.