ON THE CHARACTERS OF SOLUBLE GROUPS

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The theory of representations of finite groups, which was originated by G. Frobenius, has been developed by I. Schur to become popular and studied more and more profoundly by R. Brauer even in the current stage of modern algebra. However, it seems that its applications to the structure theory of finite groups are still far from being satisfactory, partially because these two theories for structure and for representation are not firmly tied up.

In this paper I want to clarify the relationship of the structure and the representation theory in the case of soluble groups, though the case may be rather trivial.

The results are described as follows;

Let p be a fixed prime number and we use in the following modular terminologies for this p. After R. Brauer we say that a class of conjugate elements in a group is of defect d if the order of the centralizer of the element of the class is divisible exactly by p^d and that a block of characters in a group is of defect d if the degrees of all the characters of the block is divisible by p^{a-d} and at least one of them is not divisible by p^{a-d+1}, where p^a is the highest power of p which divides the group order.

In § 1 we shall prove that there exists a block of characters of defect 0 in a soluble group G, if G has no normal p-subgroup which is distinct from $\{e\}$ and has no group of the first kind as an associated group (as for definitions, see below). Since the order of the group of the first kind is even, the last condition is always satisfied by a soluble group of odd order. Further since a p-Sylow subgroup of the group of the first kind is irregular in the sense of P. Hall, the last condition is also always satisfied by a soluble group whose p-Sylow subgroup is regular. In § 2 we shall treat the case of positive defects and prove the similar theorems.

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§ 1.

We refer to an absolutely irreducible ordinary character simply as a character.

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LEMMA 1. Let N be a normal subgroup of a soluble group G, and let the order of N be prime to p, where p is a fixed prime number. Let $K_1, K_2, \ldots, K_r$ be the classes of conjugate elements of G of defects 0 such that they are contained in N. Then G has at least r characters of defects 0 which are linearly independent mod. $\mathfrak{p}$ on $K_1, K_2, \ldots, K_r$, where $\mathfrak{p}$ is one of the prime divisors of p in the algebraic number field generated by characters of G and its subgroups over the rational number field.

Proof. Applying the induction argument with respect to the order of G, we assume that the assertion is true for all groups of smaller orders.

If $G=N$, then it is well known that the assertion is valid, e.g. as in I. Schur,[1] by a result of R. Brauer and C. Nesbitt.[2]

Let us assume $G \neq N$ and let H be a normal maximal subgroup of G over N. Let s be the index of H in G. Then s is a prime. If $s=p$, then K_i $(1 \leqq i \leqq r)$ is divided into the p classes L_{ij} $(j=1, \ldots, p)$ of conjugate elements of H of defects 0. It follows by the induction hypothesis that H has at least rp characters φ of defects 0 which are linearly independent mod. $\mathfrak{p}$ on $L_{11}, \ldots, L_{rp}$. We consider the matrix

$$\begin{pmatrix} \varphi_1(L_{11}) & \ldots & \varphi_1(L_{rp}) \\ \vdots & & \vdots \\ \varphi_{rp}(L_{11}) & \ldots & \varphi_{rp}(L_{rp}) \end{pmatrix},$$

which is non-singular mod. $\mathfrak{p}$. Further we consider the $rp \times r$ matrix

$$\begin{pmatrix} \varphi_1(L_{11})+\ldots+\varphi_1(L_{1p}) & \ldots & \varphi_1(L_{r1})+\ldots+\varphi_1(L_{rp}) \\ \vdots & & \vdots \\ \varphi_{rp}(L_{11})+\ldots+\varphi_{rp}(L_{1p}) & \ldots & \varphi_{rp}(L_{r1})+\ldots+\varphi_{rp}(L_{rp}) \end{pmatrix},$$

which of course has rank r mod. $\mathfrak{p}$. Let φ^* be a character of G induced by φ. Then it can be easily seen that $\varphi(L_{i1})+\ldots+\varphi(L_{ip})=\varphi^*(K_i)$ for $i=1, \ldots, r$. Rewriting the components of the above matrix, we have the following $rp \times r$ matrix of rank r mod. $\mathfrak{p}$.

$$\begin{pmatrix} \varphi_1{}^*(K_1) & \ldots & \varphi_1{}^*(K_r) \\ \vdots & & \vdots \\ \varphi_{rp}{}^*(K_1) & \ldots & \varphi_{rp}{}^*(K_r) \end{pmatrix}.$$

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[1] I. Schur, Neue Begründung der Theorie der Gruppencharaktere, Sitz. Berlin (1905), pp. 406–432.

[2] R. Brauer and C. Nesbitt, On the modular representations of groups of finite order I, Univ. Toronto studies Math. Series 4 (1937), pp. 20.

This shows that among φ_i^* there exist r linearly mod. $\mathfrak{p}$ independent characters, say, $\varphi_1^*, \ldots, \varphi_r^*$. If φ^* is not irreducible, we can readily see that $\varphi^*(K_i) \equiv 0$ (mod. $\mathfrak{p}$) for $i=1, \ldots, r$. Therefore $\varphi_1^*, \ldots, \varphi_r^*$ are all irreducible. Considering the degrees, we can conclude by a theorem of R. Brauer and C. Nesbitt[3] that $\varphi_1^*, \ldots, \varphi_r^*$ are all of defects 0. If $s \nmid p$, then K_i $(1 \leqq i \leqq r)$ is divided into s classes of conjugate elements of H of defects 0 or coincides with one of them. Let us suppose $K_1 = L_{11} + \ldots + L_{1s}, \ldots, K_t = L_{t1} + \ldots + L_{ts}, K_{t+1} = L_{t+1}, \ldots, K_r = L_r$. H has then at least $ts + (r-t)$ characters φ of defects 0 which are linearly independent mod. $\mathfrak{p}$ on $L_{11}, \ldots, L_{ts}, L_{t+1}, \ldots, L_r$ by the induction hypothesis. We consider the matrix

$$\begin{pmatrix} \varphi_1(L_{11}) & \ldots & \varphi_1(L_r) \\ \vdots & & \vdots \\ \varphi_{ts+(r-t)}(L_{11}) & \ldots & \varphi_{ts+(r-t)}(L_r) \end{pmatrix},$$

which is of degree $ts+(r-t)$ and non-singular mod. $\mathfrak{p}$. Let φ^* be a character of G induced by φ. Then it can be easily seen that $\varphi(L_{i1}) + \ldots + \varphi(L_{is}) = \varphi^*(K_i)$ for $i=1, \ldots, t$ and $s\ \varphi(L_i) = \varphi^*(K_i)$ for $i=t+1, \ldots, r$. Rewriting the components of the matrix, we have a matrix

$$\begin{pmatrix} \varphi_1^*(K_1) & \ldots & \varphi_1^*(K_r) \\ \vdots & & \vdots \\ \varphi^*_{ts+(r-t)}(K_1) & \ldots & \varphi^*_{ts+(r-t)}(K_r) \end{pmatrix}$$

of type $(ts+(r-t), r)$ and of rank r mod. $\mathfrak{p}$. Therefore r induced characters, say, $\varphi_1^*, \ldots, \varphi_r^*$ are linearly independent mod. $\mathfrak{p}$ on $K_1, \ldots, K_r$. We put $\varphi_i^{**} = \varphi_i^*$ if φ_i^* is irreducible and put $\varphi_i^{**} = (1/s)\varphi_i^* = \varphi_i$, if φ_i^* is reducible. Evidently each φ_i^{**} is of defect 0 by a theorem of R. Brauer and C. Nesbitt.[4] Thus Lemma 1 is completely proved.

Remark. The above proof of Lemma 1 holds good, without modification, under a weaker condition that G/N is soluble.

LEMMA 2. The equation

$$q^s - 1 = p^t$$

is satisfied by positive integers s, t and rational primes p, q, only when

(1) $q=2;\ t=1,$ or

(2) $s=1;\ p=2,$ or $q=3;\ s=2;\ p=2,\ t=3.$

[3] R. Brauer and C. Nesbitt, On the modular characters of groups, Ann. Math. **42** (1941), pp. 556-590.

[4] See [3].

Proof. First we suppose $p>2$. Then $q=2$ and $s\geqq 2$. If $t\equiv 0$ (mod. 2), then $-1\equiv 2^s-1=p^t\equiv 1$ (mod. 2^2), which shows a contradiction. Therefore $t\equiv 1$ (mod. 2) and $2^s=p^t+1=(p+1)(p^{t-1}-\dots+1)$. Since p and t are odd, $p^{t-1}-\dots+1$ is odd. This shows that $p^{t-1}-\dots+1=1$, i.e., $t=1$.

Secondly we suppose $p=2$. If $s\equiv 0$ (mod. 2), say $s=2u$, then $q^s-1=(q^u-1)\times(q^u+1)=2^t$ and $[q^u-1, q^u+1]=2$. Therefore $q^u-1=2$ and $q=3$, $u=1$, $s=2$. If $s\equiv 1$ (mod. 2), then $q^s-1=(q-1)(q^{s-1}+\dots+1)=2^t$, whence $q^{s-1}+\dots+1=1$, i.e., $s=1$.

Groups of the first kind

Let p and q be a pair of primes in Lemma 2. Let K be a holomorph of an abelian group of order q^s and of type $(q,\dots,q)$ by a cyclic group of automorphisms of order p^t. Let $H=K_1\times\dots\times K_p$ (p-ple product of K) and let each K_i be isomorphic to K. We fix an isomorphism σ_i between K and K_i for each i. We denote generally $\sigma_i(a)$ by a_i for an element a of K. Let G be a holomorph of H by a cyclic group $\{\prod_{e\neq a\in K}(a_1,\dots,a_p)\}$ of automorphisms of order p. We define such a G as the group of the first kind.

We describe some properties of the structures of such groups. G is a soluble group of rank 3. A q-Sylow subgroup S_q is a normal and abelian subgroup of type $(q,\dots,q)$. A p-Sylow subgroup S_p has no Ω-property in the sense of P. Hall, that is, the totality of elements of order p of S_p with e forms no subgroup. G has no normal p-subgroup distinct from $\{e\}$. G is of order divisible by 2 by Lemma 2. G has no class of conjugate elements of defect 0. To show this, it may be sufficient to remark that all the elements $\neq e$ of $S_q(K)$ are conjugate in K with one another. Since the group ring of G is primary decomposable for p in the sense of M. Osima,[5] we can analyse in detail the modular properties of characters of G for p. E.g. G has no block of defect 0 and only one block of defect 1.

Example. Let G be a group of the first kind. Then the n-ple product G_n of G gives us an example such that G_n has no normal p-subgroup distinct from $\{e\}$, but the defects of blocks of G_n are not smaller than n.

Let a group H be homomorphic to some subgroup of a group G. We call H an associated group of G.

THEOREM 1. *A soluble group G with no normal p-subgroup $\neq\{e\}$ has a character of defect 0 if G has no group of the first kind as an associated group.*

[5] M. Osima, On primary decomposable group rings, Proc. Phys.-Math. Soc. Japan **24** (1942), pp. 1-9.

In other words, in such a group $\{e\}$ *is a defect group.*

Proof. Let $g=p^a g'$, $(p, g')=1$, be the order of G. If $g'=1$, the theorem is evident. So we assume the theorem is valid for groups with smaller value of g'.

Let N be the largest normal subgroup of G with order prime to p. We may assume that G has a p-Sylow complement H_p of G and $N=H_p$. In fact, if $N \neq H_p$ then $N \cdot S_p \neq G$. Therefore $N \cdot S_p$ has a character of defect 0 by the induction hypothesis, if $N \cdot S_p$ has no normal p-subgroup $\neq \{e\}$. Let us now suppose that $N \cdot S_p$ has a normal p-subgroup $\neq \{e\}$. Then the centralizer $\mathfrak{Z}(N)$ of N in G is of order divisible by p, as is readily seen. Let N_1 be the largest normal subgroup of G which is contained in $\mathfrak{Z}(N)$ and of order prime to p. Obviously $N_1 \leqq N$. Let $P \cdot N_1$ be a normal subgroup of G which is contained in $\mathfrak{Z}(N)$ and is also minimal over N_1. Since $P \leqq \mathfrak{Z}(N)$ and $N_1 \leqq N$, we have $P \cdot N_1 = P \times N_1$. Therefore P is a normal p-subgroup $\neq \{e\}$ of G. This is a contradiction. Therefore $N \cdot S_p$ has actually a character of defect 0. Then, as is readily seen, G has at least one character of defect 0 by Lemma 1, because there exists in G at least one class of conjugate elements of defect 0 which is contained in N.[6] So we consider the case where $N=H_p$. Then the group ring of G is primary decomposable in the sense of M. Osima. We remark that in such a group the existence of the class of conjugate elements of defect 0 is equivalent to that of the character of defect 0, by a theorem of M. Osima.[7]

Now suppose that the theorem is not valid for G; G has no character of defect 0. Then G has no class of conjugate elements of defect 0 from the above remark. Under these circumstances we can assume that H_p is the join of some minimal normal subgroups of G; H_p is completely reducible. In fact, let M_1 be any minimal normal subgroup of G. Then obviously M_1 is contained in H_p and G/M_1 has no class of conjugate elements of defect 0. On the other hand, G/M_1 has a character of defect 0 by the induction hypothesis, if G/M_1 has no normal p-subgroup $\neq \{e\}$. Then G/M_1 has a normal p-subgroup $\neq \{e\}$. Therefore let $P_1 M_1/M_1$ be a minimal normal p-subgroup $\neq \{e\}$ of G/M_1. Since obviously $H_p/M_1 \cdot P_1 M_1/M_1 = H_p/M_1 \times P_1 M_1/M_1$, $P_1 M_1/M_1$ is contained in the centre of G/M_1. Therefore P_1 is of order p and we can put $P_1=\{A_1\}$, $A_1{}^p=e$. Moreover it is easily seen that $\mathfrak{Z}_{P_1 A_1}(A_1)=P_1$, therefore $G=M_1\mathfrak{Z}(A_1)$ and $M_1 \frown \mathfrak{Z}(A_1) = e$ as in P. Hall.[8] Now we suppose that there exist r independent minimal

[6] R. Brauer, On the arithmetic in a group ring, Proc. Nat. Acad. Sci. U.S.A. (1944), pp. 109-114.

[7] See [5].

[8] P. Hall, A note on soluble groups, Jour. London Math. Soc., 3 (1928), pp. 98-105.

normal subgroups $M_1, M_2, \ldots, M_r$ of G and that $G/M_1, G/M_2, \ldots, G/M_r$ have respectively minimal normal p-subgroups $P_1M_1/M_1, P_2M_2/M_2, \ldots, P_rM_r/M_r$, where $P_1=\{A_1\}, P_2=\{A_2\}, \ldots, P_r=\{A_r\}$ and $A_1^p=A_2^p= \ldots =A_r^p=e$, such that $G=M_1M_2 \ldots M_r \cdot \mathfrak{Z}(A_1A_2 \ldots A_r)$ and $M_1M_2 \ldots M_r \frown \mathfrak{Z}(A_1A_2 \ldots A_r)=e$. If $M_1M_2 \ldots M_r \neq H_p$, then $S_p \cdot M_1M_2 \ldots M_r \neq G$ and obviously $S_p \cdot M_1M_2 \ldots M_r$ has no class of conjugate elements of defect 0. On the other hand, $S_p \cdot M_1M_2 \ldots M_r$ has a character of defect 0 by the induction hypothesis, if $S_p \cdot M_1M_2 \ldots M_r$ has no normal p-subgroup $\neq \{e\}$. So $S_p \cdot M_1M_2 \ldots M_r$ has a normal p-subgroup $\neq \{e\}$ which is obviously contained in $\mathfrak{Z}(A_1A_2 \ldots A_r)$. Since any conjugate subgroup of $\mathfrak{Z}(A_1A_2 \ldots A_r)$ can be obtained by transforming with some element of $M_1M_2 \ldots M_r$, $\mathfrak{Z}(A_1A_2 \ldots A_r)$ contains at least one normal subgroup $\neq \{e\}$ of G. Therefore let M_{r+1} be any minimal normal subgroup of G which is contained in $\mathfrak{Z}(A_1A_2 \ldots A_r)$. Then it is obvious that $M_1, M_2, \ldots, M_r$ and M_{r+1} are all independent one another. Moreover it is easily seen, as is shown above, that G/M_{r+1} has a minimal normal p-subgroup $P_{r+1}M_{r+1}/M_{r+1}$ of order p, and we may put $P_{r+1}=\{A_{r+1}\}$, $A_{r+1}^p=e$. Since obviously $P_1P_2 \ldots P_r \cdot M_1M_2 \ldots M_r \cap P_{r+1} \cdot M_{r+1}=e$, we have $P_1P_2 \ldots P_r \cdot M_1M_2 \ldots M_r \cdot P_{r+1}M_{r+1}=P_1P_2 \ldots P_r \cdot M_1M_2 \ldots M_r \times P_{r+1}M_{r+1}$. Therefore $A_1A_2 \ldots A_{r+1}$ has $(M_1M_2 \ldots M_{r+1} : e)$-distinct conjugate elements. From this we can readily see that $G=M_1M_2 \ldots M_{r+1} \cdot \mathfrak{Z}(A_1A_2 \ldots A_{r+1})$ and $M_1M_2 \ldots M_{r+1} \frown \mathfrak{Z}(A_1A_2 \ldots A_{r+1})=e$. Thus the induction argument gives us that H_p is the join of some minimal normal subgroups of G, i.e. H_p is completely reducible. In particular, H_p is abelian.

Next we can assume that $H_p=S_q$ where S_q is the q-Sylow subgroup of G and of type $(q, q, \ldots, q)$. In fact, suppose that $H_p \neq S_q$. Then G/S_q has no class of conjugate elements of defect 0 as in the case of G. On the other hand, G/S_q has a character of defect 0 by the induction hypothesis, if G/S_q has no normal p-subgroup $\neq \{e\}$. So G/S_q has a normal p-subgroup $\neq \{e\}$. Therefore let PS_q/S_q be the largest normal p-subgroup of G/S_q. Then G/PS_q has no normal p-subgroup $\neq \{e\}$. Therefore G/PS_q has a character of defect 0 by the induction hypothesis and again has a class of conjugate elements K of defect 0. We can readily see that K contains an element A of G which is p- and q-regular and we may assume that $S_p(\mathfrak{Z}(A)) \leqq P$. On the other hand, PS_q has no normal p-subgroup $\neq \{e\}$. Therefore PS_q has a character of defect 0 by the induction hypothesis and again has a class L of conjugate elements of defect 0. Let B be an element of L and consider AB. Since the order of A is prime to that of B, $\mathfrak{Z}(AB)$ is contained in $\mathfrak{Z}(A)$ and $\mathfrak{Z}(B)$. Then it is easily seen that AB is an element of defect 0. This proves a contradiction. So we consider the case where $H_p=S_q$ i.e. H_p is a vector space over the prime field

$GF[q]$ of characteristic q.

Moreover we can assume that S_q is a minimal normal subgroup of G. To show this we regard $S_q = V$ as a representation space of S_p over $GF[q]$. Then it is obvious that V is completely reducible since $p \neq q$. Thererfore let $V = V_1 + V_2 + \ldots + V_r$ be a decomposition of V into its irreducible subspaces $V_1, V_2, \ldots, V_r$: Adapting to this decomposition we designate

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix} \text{ and } x = x_1 \dot{+} x_2 \dot{+} \ldots \dot{+} x_r, \text{ where } A \text{ is an element of } S_p$$

and x is a vector of V. Now suppose that $r > 1$. Consider $S_p \cdot V_i$ and denote by P_i the kernel, i.e. the totality of elements of S_p represented by E_i, where E_i is the unit matrix. Then $(S_p/P_i) \cdot V_i$ has no normal p-subgroup $\neq \{e\}$. Therefore $(S_p/P_i) \cdot V_i$ has a character of defect 0 by the induction hypothesis and again has a class of conjugate elements of defect 0. Therefore let x_i be a vector of V_i such that $A_i x_i = x_i$ implies $A_i = E_i$. And put $x = x_1 \dot{+} x_2 \dot{+} \ldots \dot{+} x_r$. Then $Ax = x$ implies $A_i x_i = x_i$ for every $i = 1, 2, \ldots, r$, whence $A_i = E_i$ for every $i = 1, 2, \ldots, r$. Therefore $A = E$ and x is an element of defect 0 which is a contradiction. Thus $V = V_1$, i.e. S_q is a minimal normal subgroup of G.

The centre $C_1(S_p)$ of S_p is cyclic. In fact, if $C_1(S_p)$ is not cyclic, then there exists an element $C \neq e$ of $C_1(S_p)$ aud an element $Q \neq e$ of S_q such that $CQ = QC$ by a theorem of W. Burnside-H. Zassenhaus.[9] Since $\{Q^P;\ P \in S_p\} = S_q$ by the minimality of S_q, $CQ = QC$ implies that C is contained in the centre of G which is a contradiction. Thus $C_1(S_p)$ is cyclic. (Or, since V is faithful for S_p, this assertion is a special case of a theorem of Y. Akizuki-K. Shoda.[10])

S_p is not abelian. In fact, if S_p is abelian, then $S_p = C_1(S_p)$ is cyclic. And it can be readily seen that any element of S_p with order p is contained in the centre of G by supposition, which is a contradiction. Thus S_p is not abelian.

Next we show that $V = S_q$ is reducible for a suitable maximal subgroup M of S_p. We designate the representation of S_p by V with A for the clarity of description. To do this, let $GF[q^f]$ be the minimal splitting field of A and extend V to $V_{GF[q^f]}$. Then A is decomposable, by a theorem of I. Schur,[11] into its irreducible parts: $A = m \sum_{\kappa} A_\kappa$, where A_κ's are all algebraically conjugate to each other with respect to $GF[q]$. Let p^e be the degree of A_κ. Then $e > 0$.

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9) H. Zassenhaus, Über endliche Fastkörper, Hbg. Abh. **11** (1935), pp. 187-220.

10) K. Shoda, Über direkt zerlegbare Gruppen, Jour. Fac. Sci., Tokyo **2** (1930).

11) I. Schur, Arithmetische Untersuchungen über endliche Gruppen lineare Substitutionen, Sitz. Berlin (1906), pp. 164-184.

For; $e=0$ implies that S_p is abelian and this is not the case. Therefore every A_κ is reducible for a suitable maximal subgroup M_κ of S_p by a lemma of R. Brauer[12]: $A_\kappa(M_\kappa)=\sum_{\lambda=1}^{p}A_{\kappa\lambda}$. Now since A_κ's are all algebraically conjugate to each other, we may assume that M_κ's are all equal to each other: $M_\kappa=M$. Furthermore $A_\kappa(M)=\sum_{\lambda=1}^{p}A_{\kappa\lambda}$ and $A_{\kappa\lambda}$ and $A_{\kappa'\lambda'}$ are algebraically conjugate one another if $\lambda=\lambda'$. Then $A=\sum_{\lambda=1}^{p}(m\sum_{\kappa}A_{\kappa\lambda})$, where $m\sum_{\kappa}A_{\kappa\lambda}$ is realizable in $GF[q]$ by a theorem of I. Schur.[13] Thus A is reducible for M. Since the degree of $A_{\kappa\lambda}$ is p^{e-1}, we have that $V(M)=V_1+V_2+\dots+V_p$. And it can be easily seen that if X is any element of S_p which is not contained in M, then $V_2=V_1^X,\dots,V_p=V_{p-1}^X$, $V_1=V_p^X$. Moreover it can also be easily seen that V is considered as a representation space of S_p induced by a representation space V_1 of M by a theorem of G. Frobenius.[14]

Last, we designate the representation of M by V_1 with B. Since the representation A of S_p is the induced representation of the representation B of M and $V_2=V_1^X,\dots,V_p=V_{p-1}^X$, $V_1=V_p^X$, we can describe:

$$A(Y)=\begin{pmatrix} B(Y) & & & \\ & B(Y^X) & & \\ & & \ddots & \\ & & & B(Y^{X^{p-1}}) \end{pmatrix} \text{ for any element } Y \text{ of } M, \text{ and } A(X)=\begin{pmatrix} O & E & & \\ & \ddots & \ddots & \\ & & \ddots & E \\ E & & & O \end{pmatrix},$$

where E is the unit matrix of the same degree as B. And we designate a vector x of V in the form:

$$x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$$ adapted to this realization. Then it is obvious that $A(Y)(x)$

$$=\begin{pmatrix} B(Y)x_1 \\ B(Y^X)x_2 \\ \vdots \\ B(Y^{X^{p-1}})x_p \end{pmatrix} \text{ and } A(X)x=\begin{pmatrix} x_2 \\ \vdots \\ x_p \\ x_1 \end{pmatrix}.$$ Now consider $B\cdot V_1$. Then we see that $B\cdot V_1$ has no normal p-subgroup $\neq\{e\}$. Therefore $B\cdot V_1$ has a character of defect 0 by the induction hypothesis and again has a class of conjugate elements of defect 0. Therefore let x_1 be an element of V_1 of defect 0. Next consider the group of B-automorphisms of V_1 and denote it by C. Then C is the centralizer of B in the general linear homogeneous group of the same dimension

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12) R. Brauer, On Artin's L-series with general group characters, Ann. Math. 48 (1947), pp. 502-514.

13) See 11).

14) G. Frobenius, Über Relationen zwischen den Charakteren einer Gruppen und denen ihrer Untergruppen, Sitz. Berlin (1898), pp. 501-515.

with that of V_l over $GF[q]$. And since V_l is irreducible for B, the ring of B-endomorphisms of V_l is a field $GF[q^s]$ with a suitable $s>0$. Then, since C is the multiplicative group of $GF[q^s]$, C is cyclic and of order q^s-1. And we may designate: $C=\{Z\}, Z^{q^s-1}=e$. Since $C\geqq C_l(B)$ and the group generated by B and C is clearly irreducible, it can be readily seen that C is faithful for any vector $y\neq 0$ of V_l, i.e. $y^{Z^n}=y$ implies $n\equiv 0$ (mod. q^s-1). Clearly x_1^Z is of defect 0 with x_1. Now suppose that there is no element of B which translates x_1 to x_1^Z. Then it can be easily seen that

$$x=\begin{pmatrix} x_1^Z \\ x_1 \\ \vdots \\ x_1 \end{pmatrix}$$ is of defect 0 in G. This is a contradiction. Therefore $x_1^Y=x_1^Z$ for some element Y of B. Then Y and Z are of the same order. Moreover, applying the induction hypothesis, we have $Y=Z$, i.e. $C=C_1(B)$. On the other hand, we consider a vector x of V with the form:

$$x=\begin{pmatrix} x_2 \\ x_1 \\ \vdots \\ x_1 \end{pmatrix},$$ where x_2 is a vector of V_l with a positive defect. Let $W\neq E$ be a matrix of A which fixes x. Then it can be easily seen that W is of the form:

$$W=\begin{pmatrix} W_1 & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{pmatrix},$$ and the totality of matrices of such a form forms a normal subgroup of M. Therefore A contains a matrix D of the form:

$$D=\begin{pmatrix} C_1 & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{pmatrix},$$ where C_1 is a generating element of $C_1(B)$. Next consider $C_1(B)\cdot V_1$. And let $V_1=V_{11}^*+V_{12}^*+\dots+V_{1u}^*$ be a decomposition of V_1 into its $C_l(B)$-irreducible subspaces. V_{11}^* is an s-dimensional subspace of V_l. We designate a vector y_1 of V_1 in the form:

$$y_1=\begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1u} \end{pmatrix},$$ adapted to this decomposition. Then the totality of vectors of V_1 of the form:

$$y_1=\begin{pmatrix} y_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ forms a $C_1(B)$-subspace $V_1^*=V_{11}^*$ of V_1. Then it can be readily

seen that $V^* = V_1^* + V_2^* + \dots + V_p^*$ is allowable

$$\text{by} \left\{ \begin{pmatrix} C_1 & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{pmatrix}, \begin{pmatrix} E & & & \\ & C_1 & & \\ & & \ddots & \\ & & & E \end{pmatrix}, \dots, \begin{pmatrix} E & & & \\ & E & & \\ & & \ddots & \\ & & & C_1 \end{pmatrix} \right\} \text{ and also by } A(X).$$

And it can again be readily seen that the group generated by $\begin{pmatrix} C_1 & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{pmatrix}$, $A(X)$ and V^* is of the first kind. This is a contradiction. Thus Theorem 1 is finally proved.

Example. Put $G = \{(12), (13), (14), (56), (57), (58), (9, 10), (9, 11), (9, 12), (159)(2, 6, 10)(3, 7, 11)(4, 8, 12)\}$. Then it is verified by a comparatively simple calculation that G has a character of defect 0. But $\{(123), (124), (567), (568), (9, 10, 11), (9, 10, 12), (159)(2, 6, 10)(3, 7, 11)(4, 8, 12)\}$ is a group of the first kind. This shows us that the condition in Theorem 1 is not necessary. Furthermore it is easily seen that $\{(12)(34), (13)(24), (56)(78), (57)(68), (9, 10)(11, 12), (9, 11)(10, 12)\}$ is the largest normal subgroup of G with order prime to 3 and there is no class of conjugate elements of defect 0 in this group. This shows us that the converse of Lemma 1 is not true in general.

Example. Let K be a holomorph of quaternion group by a group of automorphisms of the order $p=3$.

Put $H = K_1 \times K_2 \times K_3$ where each K_i is isomorphic with K. Let φ_i be a fixed isomorphism from K to K_i and we denote $\varphi_i(a)$ by a_i for each element a of K and for each i.

Let G be a holomorph of H by $\{\prod_{a \neq e \in K} (a_1 a_2 a_3)\}$. It is clear that G has a factor group aud no subgroup isomorphic to a group of the first kind.

Remark. Let G be a group such that G has no normal p-subgroup distinct from $\{e\}$ and has a class of conjugate elements of defect 0. G has not always a block of defect 0.

§ 2.

Let $l > 2$ be a prime. Let G be a l-group of exponent l and of class 2, whose centre is of order l. Then it can be easily shown that G can be constructed in the following manner. We designate by L a non-abelian l-group of order l^3 and of exponent l. Consider $L \times \dots \times L$ (m-ple product of L) and identify centres of all the component groups of this direct product. We denote the last group by $L \dot{\times} \dots \dot{\times} L$ (m-ple product of L). Then G is isomorphic to $L \dot{\times} \dots \dot{\times} L$ for some m. Now let G be a 2-group of exponent 2^2 and of class 2, whose centre is of order 2. We designate by Q and D respectively quaternion and dihedral groups of order 2^3. In the same way as for the case $l>2$, we can readily show that G is isomorphic to $Q\dot{\times}\dots\dot{\times}D$ for some m. Since it can be easily seen that $Q\dot{\times}Q\cong D\dot{\times}D$, we may say exactly that G is isomorphic to either $Q\dot{\times}\dots\dot{\times}Q$ or $Q\dot{\times}\dots\dot{\times}Q\dot{\times}D$.

Now we count the number of elements of order 2^2 in $Q\dot{\times}\dots\dot{\times}Q$ and $Q\dot{\times}\dots\dot{\times}Q\dot{\times}D$ respectively. This number, as is easily seen, equals, in $Q\dot{\times}\dots\dot{\times}Q$, to

$$2\Big\{\sum_{r\equiv 1\ (\mathrm{mod.}\ 2)} 3^r\binom{m}{r}\Big\}=2^m(2^m-(-1)^m)$$

and in $Q\dot{\times}\dots\dot{\times}Q\dot{\times}D$, to

$$2\Big\{3\sum_{r\equiv 1\ (\mathrm{mod.}\ 2)} 3^r\binom{m-1}{r}+\sum_{r\equiv 0\ (\mathrm{mod.}\ 2)} 3^r\binom{m-1}{r}\Big\}=2\Big\{\sum_{r\equiv 0\ (\mathrm{mod.}\ 2)} 3^r\binom{m}{r}\Big\}=2^m(2^m+(-1)^m).$$

Next let G have an automorphism of prime order p for which all the elements of $C_1(G)$ are fixed and $G/C_1(G)$ is irreducible, where $C_1(G)$ is the centre of G and we regard $G/C_1(G)$ as a vector space over the prime field of characteristic l or 2 respectively. Then it is clear that the exponent of 2 with respect to p equals to $2m$:

$$2^{2m}\equiv 1 \ (\mathrm{mod.}\ p).$$

Therefore m cannot be even for the case $Q\dot{\times}\dots\dot{\times}Q$ and odd for the case $Q\dot{\times}\dots\dot{\times}Q\dot{\times}D$, as is readily seen. Moreover m cannot be 3, since there is clearly no prime for which 2 belongs to exponent 6. Similarly we must exclude the case where $l=2^r-1$, when $m=1$.

In other cases, on the contrary, we can show that there exist actually such automorphisms. First we remark that G is homogeneous in the following sense: Let R and S be subgroups of G which are both isomorphic to either L or Q respectively. Then there exists an automorphism of G by which R is translated to S. Secondly we count the number of subgroups of G which are isomorphic to either L or Q respectively. As it is easily seen, this number equals, in $L\dot{\times}\dots\dot{\times}L$, to

$$\frac{(l^{2m+1}-l)l(l^{2m}-l^{2m-1})}{(l^3-l)(l^3-l^2)}=\frac{l^{2m-2}(l^{2m}-1)}{l^2-1}$$

and, in $Q\dot{\times}\dots\dot{\times}Q$, to

$$\frac{2^m(2^m+1)2^2\Big\{\sum_{r\equiv 0\ (\mathrm{mod.}\ 2)} 3^r\binom{m-1}{r}\Big\}}{6\cdot 4}=\frac{2^{2m-3}(2^m+1)(2^{m-1}+1)}{3}$$

and, in $Q\dot{\times}\dots\dot{\times}Q\dot{\times}D$, to

$$\frac{2^m(2^m+1)2^2\left\{3\sum_{r\equiv 0\ (\mathrm{mod.}\ 2)}3^r\binom{m-2}{r}+\sum_{r\equiv 1\ (\mathrm{mod.}\ 2)}3^r\binom{m-2}{r}\right\}}{6\cdot 4}=\frac{2^{2m-3}(2^m+1)(2^{m+1}+1)}{3}.$$

On the other hand, the group of automorphisms of L which fix all the elements of the centre of L is of order divisible by $l+1$, as it is easily seen. It is well known that the same holds good for Q. Moreover there exists always a prime p for which l belongs to exponent $2m$ and the same holds good for 2 since $m\neq 3$ by a lemma of C. Chevalley and G. Azumaya.[15] Therefore G has an automorphism of order p which fixes all the elements of the centre of G. Thus we have

LEMMA 3. Let G be a group in question. Then G has an automorphism of prime order for which all the elements of $C_l(G)$ are fixed and $G/C_l(G)$ is irreducible.

Remark. It is easily seen that G has l^{2m} characters of degree 1 and $l-1$ faithful characters of degree l^m. The same holds good for 2.

Groups of the second kind. Groups of this kind is divided into two subfamilies.

(1) Let p be a prime and let q be a prime for which p belongs to exponent f. Let Q be an abelian group of order q^f and of type $(q, \ldots, q)$. Then Q has an automorphism π of order p. Let H be a holomorph of Q by $\{\pi\}$. Since clearly H is Frobeniusean type, it can be easily seen that H has a faithful irreducible representation of degree p which is realizable in $GF[p^f]$. Let P be an abelian group of the least order p^e $(e\leqq pf)$ and of type $(p, \ldots, p)$ which has an automorphisms group isomorphic to H. Let G be the holomorph of P by H. The totality of such G's forms the one subfamily of groups of the second kind.

(2) Let Q be a groups of order q^{2m+1} in Lemma 3. Then Q has an automorphism π of order p for which q belongs to exponent $2m$ by Lemma 3. Let H be a holomorph of Q by $\{\pi\}$. Since the centre of Q is of order q, H has a faithful p-modular irreducible representation, by a theorem of T. Nakayama,[16] which is of degree q^m and is realizable in $GF[p^{2m}]$ or $GF[p]$ according as $q>2$

[15] G. Azumaya, Elementary proof of a theorem in number theory, Z-S-S-D. 1187 (1944), pp. 189-196 (in Japanese).
C. Chevalley, Sur la théorie du corps de classes dans les corps finis et les corps locaux, Jour. Coll. Sci., Tokyo **2** (1933), pp. 365-476.

[16] T. Nakayama, Finite groups with faithful irreducible and directly indecomposable modular representations, Proc. Acad. Japan **23** (1947), pp. 22-25.

or $q=2$, as it is easily seen. More finely all the irreducible representations of H are p-modular irreducible by a theorem of M. Osima,[17] as is also easily seen. Let P be an abelian group of the least order p and of type $(p, \ldots, p)$ which has a group of automorphisms isomorphic to H. Let G be a holomorph of P by H. The totality of such G's forms the other subfamily of groups of the second kind.

We describe some properties of a group G belonging to the family of the second kind. G has no normal p-Sylow subgroup. But $\mathfrak{Z}(P)=P$. Therefore G has only one block i.e. 1-block by a Lemma of R. Brauer.[18] G is not p-normal in the sense of O. Grün.[19] In fact, if G is p-normal then we see readily that S_p is abelian. This is absurd.

Example. Let G be a group of the second kind. Let G_n be the n-ple direct product of G. Let P_n be the largest normal p-subgroup of G. Then G has only one block i.e. 1-block, but $S_p : P_n = p^n$.

THEOREM 2. *Let P be the largest normal p-subgroup of a soluble group G and distinct from a p-Sylow subgroup S_p of G. Then the centralizer $\mathfrak{Z}(P)$ of P in G is not a p-subgroup, if G has no group of the second kind as an associated group.*

Proof. We apply the induction argument over the order of G and assume that the assertion is valid for all groups of smaller orders.

Let H be a normal maximal subgroup of G over P. Then the largest normal p-subgroup of H is again P. If P is not a p-Sylow subgroup of H, then the centralizer of P in H is not a p-subgroup by the induction hypothesis, and, of course, $\mathfrak{Z}(P)$ is not a p-subgroup. So we may assume that there is no such a normal maximal subgroup of G over P. Therefore we can readily see that the factor commutator group of G is a p-group and S_p/P is a cyclic group of order p.

Next we may assume that there is one and only one normal maximal subgroup $H=P\cdot S_q$ of G over P, where S_q is a q-Sylow subgroup of G and $q \neq p$. Suppose $G \neq S_p \cdot S_q$. If S_p is not normal in $S_p \cdot S_q$ for some prime divisor q of the order of G, then the centralizer of P in $S_p \cdot S_q$ is not a p-subgroup by the induction hypothesis, and, of course, $\mathfrak{Z}(P)$ is not a p-subgroup. If S_p is normal in $S_p \cdot S_q$ for every prime q, then S_p is normal in G, which is a contradiction.

17) See 5).

18) R. Brauer, On a conjecture by Nakayama, Trans. R. S. of Canada (1947).

19) H. Zassenhaus, Lehrbuch der Gruppentheorie I, (1937).

Therefore $G=S_p\cdot S_q$ and $H=P\cdot S_q$ is the normal maximal subgroup of G containing P.

P may be assumed to be a minimal normal subgroup of G. In fact, let P_1 be a minimal normal subgroup of G which is contained in the centre of P. And let A be an element of S_p which is not contained in P. Consider the group $K=P_1\cdot\{A\}\cdot S_q$, joining P_1 and $\{A\}\cdot S_q$, for clearly we may assume that A is contained in the normalizer of S_q in G. First we suppose $K=G$. If $A^p\neq e$, then A^p is clearly contained in the centre of G. Since $G/\{A^p\}$ has no normal p-Sylow subgroup as in G, $\mathfrak{Z}(P/\{A^p\})$ is not a p-subgroup by the induciton hypothesis. Let $\overline{B}$ be an element $\neq e$ of $\mathfrak{Z}(P/\{A^p\})$ of order prime to p. Then $\overline{B}$ contains an element B of G of order prime to p, such that $[P, B]\subseteq\{A^p\}$. Since $\{A^p\}$ is contained in the centre of G, we have $[P, B^{q^\beta}]=[P, B]^{q^\beta}=e$. Since $[P, B]\subseteq\{A^p\}$, we have $[P, B]=e$. Thus B is contained in $\mathfrak{Z}(P)$ and therefore $\mathfrak{Z}(P)$ is not a p-subgroup. If $A^p=e$, then $P=P_1$ and P is a minimal normal subgroup of G. Secondly we suppose $K\neq G$. Put $K\cap P=P_2$. Since clearly K has no normal p-Sylow subgroup as in G, the centralizer $\mathfrak{Z}_K(P_2)$ of P_2 in K is not a p-subgroup by the induction hypothesis. Put $Q=S_q(\mathfrak{Z}_K(P_2))$. Then it can be easily shown that Q is the largest normal q-subgroup of K. First we suppose $Q=S_q$. Since clearly G/P_1 has no normal p-Sylow subgroup as in G, $\mathfrak{Z}(P/P_1)$ is not a p-subgroup of G/P_1 by the induction hypothesis. Let $\overline{B}\neq e$ be an element of $\mathfrak{Z}(P/P_1)$ of order prime to p. Then $\overline{B}$ clearly contains an element B of G of order prime to p, such that $[P, B]\subseteq P_1$. Since $P_1\subseteq P_2$ and $P_2\cdot\{B\}=P_2\times\{B\}$, it can be easily shown $[P, B^{q^\beta}]=[P, B]^{q^\beta}=e$. Since $[P, B]\subseteq P_1$, we have $[P, B]=e$. Thus B is contained in $\mathfrak{Z}(P)$ and therefore $\mathfrak{Z}(P)$ is not a p-subgroup. Secondly we suppose $Q\neq S_q$. Consider K/Q. If K/Q has no normal p-Sylow subgroup, it can be easily shown as above that K/Q has a normal q-subgroup $\neq\{e\}$, which is not the case by the maximality of Q. Therefore K/Q has the normal p-Sylow subgroup. Then $P_1\cdot\{A\}\cdot Q$ is normal in K and therefore $P\cdot\{A\}\cdot Q\neq G$ is normal in G. Since the index of $P\cdot\{A\}\cdot Q$ in G is prime to p, this is a contradiction. Thus we can assume that P is a minimal normal subgroup of G.

Let $Q\cdot P/P$ be a normal q-subgroup of G/P and be distinct from a q-Sylow subgroup. If $Q\cdot S_p$ has no normal p-Sylow subgroup, then the centralizer of P in $Q\cdot S_p$ is not a p-subgroup by the induction hypothesis, and, of course, $\mathfrak{Z}(P)$ is not a p-subgroup. Now we suppose that $Q\cdot S_p$ has the normal p-Sylow subgroup. Then the index of $\mathfrak{Z}(Q\cdot P/P)$ in G/P is prime to p. Therefore $\mathfrak{Z}(Q\cdot P/P)$ coincides with G/P, i.e. $Q\cdot P/P$ is contained in the centre of G/P. Therefore we have $[G, Q]\subseteq P$. In particular $[S_q, Q]\subseteq P$. Since clearly $[S_q, Q]\subseteq S_q$, we have $[S_q, Q]=e$, i.e. Q is contained in the centre of S_q. Since we may take the last but one term of the upper central series of S_q as such a Q, S_q may be assumed to be at most class 2. Therefore if $q>2$ the totality of elements of orders at most q of S_q forms a characteristic subgroup $\Omega_1(S_q)$ of S_q, which is verified by a comparatively simple direct calculation or by a theorem in the theory of regular p-groups of P. Hall.

Now it does not happen the case that S_q is cyclic and not of order q. In fact, let $S_q=\{A\}$ be of order q^n where $n\geqq 2$. And let AP be transformed into A^xP by an element BP of order p of G/P. Then A^qP is transformed into $A^{qx}P$. Since $\{A^q\}P/P$ is contained in the centre of G/P, we have $qx\equiv q \pmod{q^n}$, whence $x\equiv 1 \pmod{q^{n-1}}$ and therefore $x^q\equiv 1 \pmod{q^n}$. On the other hand, AP is transformed into $A^{x^p}P$ by $B^pP=P$, so we have $x^p\equiv 1 \pmod{q^n}$. Then we have $x\equiv 1 \pmod{q^n}$. This is a contradiction.

We suppose that the assertion is not true for G. Then we have $\mathfrak{Z}(P)=P$. And we regard P as a vector space over the prime field $GF[p]$ of characteristic p. Then $\mathfrak{Z}(P)=P$ shows that G/P has a faithful representation U in $GF[p]$, where P is the representation module. Since the order of G/P is divisible by p, we stand on the modular case. Since P is minimal, the representation U is irreducible in $GF[p]$. Thus U is an irreducible, faithful representation of G/P in $GF[p]$. Now it can be easily shown that the join of minimal normal subgroups of G/P is $\Omega_1(C_1(S_q))P/P$. First we suppose $S_q\neq\Omega_1(C_1(S_q))$. Then $\Omega_1(C_1(S_q))P/P$ is contained in the centre of G/P, and therefore is of order q by a theorem of T. Nakayama.[20] Thus, in such a case, $C_1(S_q)$ is cyclic. In particular, if S_q is abelian, then S_q is cyclic which is not the case. Therefore S_q is not abelian. If $q>2$ and $S_q\neq\Omega_1(S_q)$, $\Omega_1(S_q)$ is contained in the centre of S_q as above, and therefore is of order q. Thus S_q is cyclic by a well known theroem, which is not the case. Therefore if $q>2$ we have $S_q=\Omega_1(S_q)$. And since clearly $S_q\ P/P$ has no normal proper subgroup of G/P distinct from $C_1(S_q)\cdot P/P$, G is a group of the second kind by Lemma 3. This is a contradiction. If $q=2$, we put $C_1(S_2)=\{A\}$. Suppose $A^2\neq e$. Since the commutator subgroup of S_2 is of order 2 as it can be readily shown, we have either $\{\Omega_2(S_2)\}\neq S_2$ where $\Omega_2(S_2)$ is the totality of elements of order at most 4 of S_2 or that S_2P/P is the join of proper normal subgroups of G/P. The latter case clearly does not occur. Then $\{\Omega_2(S_2)\}$ is contained in the centre of S_2 and therefore is a cyclic group of order 4. Then S_2 is cyclic by a well knwon theorem, which is not the case. Therefore $A^2=e$. Then as in the case $q>2$, Lemma 3 can be applied and we see that G is a group of the second kind.

[20] See [16].

This is a contradiction. Secondly we suppose $S_q=\Omega_1(C_1(S_q))$. Then clearly S_q is abelian and is of type $(q, \ldots, q)$. If $S_q \cdot P/P$ is not a minimal normal subgroup of G/P, we can readily see that S_qP/P is the join of smaller normal subgroups of G/P. It is absurd. Therefore S_qP/P is minimal. Then G is a group of the second kind. This is a contradiction. Thus Theorem 2 is completely proved.

LEMMA 4. Let P be the largest normal p-subgroup, whose order is p^d, of a soluble group G. Let PN/P be a normal subgroup, whose order is prime to p, of G/P. Let $K_1, \ldots, K_r$ be the classes of conjugate elements of G of defect d which are contained in PN and $\mathfrak{Z}(P)$. Then G has at least r characters of defect d which are linearly independent mod. $\mathfrak{p}$ on $K_1, \ldots, K_r$, where $\mathfrak{p}$ is one of the prime ideal divisors of p in the algebraic number field generated by characters of G and its subgroups over the rational number field.

Proof. The case $P=e$ is proved by Lemma 1, and the case $P=S_p$ is known.[21] Suppose $P \neq S_p$. Then $PN \neq G$. Let H be a normal maximal subgroup of G over PN. As for the remainder, we have a proof by the same way as in Lemma 1, applying a theorem of R. Brauer[22] in place of the applied theorem of R. Brauer and C. Nesbitt.

THEOREM 3. *Let P be the largest normal p-subgroup of order p^d of a soluble group G. Then G has at least one character of defect d with P as its defect group, if G has no group of the first and the second kinds as asscciated groups.*

Proof. By a theorem of R. Brauer and C. Nesbitt[23] the assertion is trivial for the case $P=S_p$. Omitting this case, we suppose $P \neq S_p$. Applying the induction argument over the order of G, we assume that the assertion is valid for all groups of smaller orders.

Let N/P be the largest normal subgroup of G/P with order prime to p. We can assume that G/P has a p-Sylow complement $H_p(G/P)$, and $N/P = H_p(G/P)$, in the same way as in the proof of Theorem 1. In fact, if $N/P \neq H_p(G/P)$, then $NS_p \neq G$. Therefore NS_p has a character of defect d by the induction hypothesis, if NS_p has P as its largest normal p-subgroup. Now suppose that the largest normal p-subgroup of NS_p contains properly P. Then the centralizer of N/P in G/P is of order divisible by p, as it can be readily

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21) See 3).

22) See 6).

23) See 3).

seen. Let N_1/P be the largest normal subgroup of G/P with order prime to p which is contained in $\mathfrak{Z}(N/P)$. Obviously $N_1/P \subseteqq N/P$. Let P_1N_1/P be a normal subgroup of G/P which is contained in $\mathfrak{Z}(N/P)$ and is minimal over N_1/P. Since $P_1/P \subseteqq \mathfrak{Z}(N/P)$ and $N_1/P \subseteqq N/P$, $P_1N_1/P = P_1/P \times N_1/P$. Then P_1 is a normal p-subgroup of G containing P properly. This is a contradiction. Therefore NS_p actually has a character of defect d with P as its defect groups. Then G has at least one character of defect d with P as its defect group by Lemma 4, because there is in NS_p, and therefore in G, at least one class of conjugate elements of defect d which is contained in $N \cap \mathfrak{Z}(P)$ by a theorem of R. Brauer.[24] So we consider the case where $H_p(G/P) = N/P$.

Since $\mathfrak{Z}(P)$ is not a p-subgroup by Theorem 2, we may denote $\mathfrak{Z}(P) = L \cdot P^{*}$, where $L \neq \{e\}$ is of order prime to p. Clearly $\mathfrak{Z}(P)$ is normal in G. On the other hand, $L \cdot P$ is normal in G, since G/P is primary decomposable. Therefore $\mathfrak{Z}(P) \cap L \cdot P = L \cdot P^{**} = L \times P^{**}$ is normal in G, whence L is normal in G. Thus L is a normal subgroup $\neq \{e\}$ of G whose order is prime to p. It is obvious that L is the largest of normal subgroups of G whose orders are prime to p.

We can assume that G has a p-Sylow complement H_p and $L = H_p$. In fact, suppose $L \neq H_p$ and consider G/L. If S_pL/L is normal in G/L, the index of S_pL in G is prime to p. Obviously P is the largest normal p-subgroup in S_pL, too, as in G. Therefore S_pL has a character of defect d with P as its defect group by the induction hypothesis. Then we have shown in a previous paper[25] that G also has a character of defect d with P as its defect group in such a case. Therefore we may suppose that S_pL/L is not normal in G/L. Let P_1L/L be the largest normal p-subgroup of G/L. Obviously $P_1 \supseteqq P$. By the induction hypothesis G/L has a character which has P_1L/L as its defect group. Therefore G/L has a p-regular element $\bar{x} \neq L$ which is contained in $\mathfrak{Z}(P_1L/L)$ by a theorem of R. Brauer.[26] Let x be an element of $\bar{x}$. Then $[x, P_1L] \subseteqq L$. Therefore $[x, P_1] \subseteqq L$ whence $[x, P] \subseteqq L$. Since obviously $[x, P] \subseteqq P$, we have finally $[x, P] = e$; i.e. x is contained in $\mathfrak{Z}(P)$. Then $\bar{x} = L$, which is a contradiction. Therefore it can be assumed to be $H_p = L$. In particular $G = \mathfrak{Z}(P)S_p$.

Last we consider G/P. Since G/P has no normal p-subgroup $\neq \{e\}$, by the maximality of P, G/P has a character χ of defect 0 by Theorem 1. Applying

24) See 6).

25) N. Itô, Some studies on group characters, Nagoya Math. J. 2 (1951), pp. 17-28.

26) See 6).

a theorem of R. Brauer,[27] we can readily see that χ is a character of G of defect d with P as its defect group. Thus Theorem 3 has been completely proved.

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[27] See [6].