

SOME STUDIES ON SEMI-LOCAL RINGS

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Introduction. The concept of semi-local rings was introduced by C. Chevalley [1]⁰⁾, which the writer has generalized in a recent paper [7] by removing the chain condition. The present paper aims mainly at the study of completions of semi-local rings. First in § 1 we investigate semi-local rings which are subdirect sums of semi-local rings, and we see in § 2 that a Noetherian semi-local ring R is complete if (and only if) R/\mathfrak{p} is complete for every minimal prime divisor \mathfrak{p} of zero ideal, together with some other properties. Further we consider in § 3 subrings of the completion of a semi-local ring. § 4 gives some supplementary remarks to [7], Chapter II, Proposition 8.

TERMS. A ring means a commutative ring with identity and under the term "subring" we mean a subring having the same identity. Semi-local rings or local rings are those in the sense of Nagata [7] (or [6]). So, (semi-)local rings in the sense of Chevalley [1] (or Cohen [2]) are called Noetherian (semi-)local rings.

1. Subdirect sums of semi-local rings.

LEMMA 1.1. Let R and R^* be a subdirect sum and the direct sum of rings R_1, R_2, \dots, R_n respectively, and suppose that R is quasi-semi-local.¹⁾ We denote by φ_i the natural homomorphism of R onto R_i , by \mathfrak{n}_i the kernel of φ_i and by $\mathfrak{m}, \mathfrak{m}^*, \mathfrak{m}_i$ the J-radicals²⁾ of R, R^*, R_i respectively ($i=1, 2, \dots, n$). Then we have (1) $\mathfrak{m}^* = \mathfrak{m}R^*$, (2) $\mathfrak{m}^* \cap R = \mathfrak{m}$, (3) $\mathfrak{m}^* = \mathfrak{m}_1 + \mathfrak{m}_2 + \dots + \mathfrak{m}_n$, (4) $\varphi_i(\mathfrak{m}^k) = \mathfrak{m}_i^k$ ($k=1, 2, \dots$), (5) $(\mathfrak{n}_1 + \mathfrak{n}_2) ((\mathfrak{m}^*)^k \cap R) \subseteq (\mathfrak{n}_1 + \mathfrak{n}_2)\mathfrak{m}^k$ ($k=1, 2, \dots$) provided $n=2$.

Proof. (1), (2) and (3) are almost evident.³⁾ To prove (4), let $\mathfrak{p}_1, \dots, \mathfrak{p}_h$ be the totality of maximal ideals of R . Then it follows $\mathfrak{m}^k = \mathfrak{p}_1^k \cap \dots \cap \mathfrak{p}_h^k = \mathfrak{p}_1^k \dots \mathfrak{p}_h^k$ and $\varphi_i^{-1}(\mathfrak{m}_i^k) = (\mathfrak{p}_1^k + \mathfrak{n}_i) \cap \dots \cap (\mathfrak{p}_h^k + \mathfrak{n}_i) = (\mathfrak{p}_1^k + \mathfrak{n}_i) \dots (\mathfrak{p}_h^k + \mathfrak{n}_i)$. This proves (4). Finally, assume that $n=2$, and consider an element b_1 of

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⁰⁾ Numbers in brackets refer to the bibliography at the end.

¹⁾ A quasi-semi-local ring is a ring which has only a finite number of maximal ideals; cf. [9].

²⁾ J-radical (Jacobson radical) of a ring is the intersection of all maximal ideals in the ring.

³⁾ Cf. [9, Lemma 2].

$(m^*)^k \cap R$. Then we can choose an element b_2 from m^k so that $\varphi_1(b_1) = \varphi_1(b_2)$, i.e., $b = b_1 - b_2 \in (m^k + n_2) \cap n_1$ (by virtue of (3) just above). Then we have $(n_1 + n_2)b = n_1 b \subseteq n_1((m^k + n_2) \cap n_1) \subseteq n_1 m^k$, which proves (5).

Next we cite lemmas due to Chevalley :

LEMMA 1.2. Let R be a complete Noetherian semi-local ring. If every a_n is an open ideal ($n=1, 2, \dots$) and if $\bigcap_{n=1}^{\infty} a_n = (0)$, then $\{a_n; n=1, 2, \dots\}$ is a system of neighbourhoods of zero. [1, § II, Lemma 7]

LEMMA 1.3. Let R be a Noetherian semi-local ring with J-radical m and let c be an element of R which is not a zero divisor. Then $\{m^n : cR; n=1, 2, \dots\}$ forms a system of neighbourhoods of zero. [1, § II, Lemma 9]

Now we prove

THEOREM 1. Let a Noetherian semi-local ring R be a subdirect sum of two rings R_1 and R_2 . Let n_i be the kernel of natural homomorphism φ_i of R onto R_i ($i=1, 2$). If $n_1 + n_2$ contains a non-zero-divisor c , then R is a subspace of the direct sum R^* of R_1 and R_2 . (R^* is clearly a Noetherian semi-local ring.)

Proof. Let m and m^* be the J-radicals of R and R^* respectively. Then we have $m^k \subseteq (m^*)^k \cap R$, since $m = m^* \cap R$ by Lemma 1.1. On the other hand, it follows from Lemma 1.1 also that $c((m^*)^k \cap R) \subseteq m^k$, i.e., $(m^*)^k \cap R \subseteq m^k : cR$. These prove our assertion by virtue of Lemma 1.3.

COROLLARY. Let R be a Noetherian semi-local ring. If the intersection of ideals q_1, \dots, q_n are zero and if $q_i : q_j = q_i$ for every pair $i \neq j$, then R is a subspace of the direct sum of rings $R/q_1, \dots, R/q_n$; in fact, these assumptions for q_1, \dots, q_n are satisfied if $q_1 \cap \dots \cap q_n$ is a shortest representation of zero ideal as an intersection of primary ideals and if zero ideal has no imbedded prime divisor.

On the other hand, we have

THEOREM 2. Let a Noetherian semi-local ring R be a subdirect sum of (Noetherian semi-local) rings R_1, \dots, R_n . We denote by n_i the kernel of natural homomorphism φ_i of R onto R_i for each i . Let \bar{R} be the completion of R . Then R is a subspace of the direct sum R^* of R_1, \dots, R_n if and only if $\bigcap_{i=1}^n n_i \bar{R} = (0)$.

Proof. We denote by \bar{R}^* the completion of R^* and by $m, \bar{m}, m^*, \bar{m}^*$ the J-radicals of $R, \bar{R}, R^*, \bar{R}^*$ respectively.

If R is a subspace of R^* , it is evident that $\bigcap_{i=1}^n n_i \bar{R} = (0)$. Conversely, assume that $\bigcap_{i=1}^n n_i \bar{R} = (0)$. Then \bar{R} is a subdirect sum of completions \bar{R}_i of R_i ($i=1, 2, \dots, n$)

by the natural way.⁴⁾ Whence $\{(\bar{m}^*)^k \cap \bar{R}; k=1, 2, \dots\}$ forms a system of neighbourhoods of zero in \bar{R} by virtue of Lemma 1.2, that is, for any positive integer k there exists a positive integer $n(k)$ such that $(\bar{m}^*)^{n(k)} \cap \bar{R} \subseteq \bar{m}^k$. Whence $(m^*)^{n(k)} \cap R \subseteq m^k$, which shows that R is a subspace of R^* .

COROLLARY 1. If a Noetherian semi-local ring R is complete and if n_1, \dots, n_n are ideals in R such that $\bigcap_{i=1}^n n_i = (0)$, then R is a subspace of the direct sum of $R/n_1, \dots, R/n_n$.

COROLLARY 2. Let R be a Noetherian semi-local ring, and let there be ideals q_i ($i=1, 2, \dots, n$) in R such that $q_i \supseteq q_j$ for every pair $i \neq j$. Then we have $(\bigcap_{i=1}^n q_i) \bar{R} = \bigcap_{i=1}^n q_i \bar{R}$, where \bar{R} denotes the completion of R .

Proof. This is an immediate consequence of our Theorem 2 and Corollary to Theorem 1.

THEOREM 3. Let a semi-local ring R be a subdirect sum of semi-local rings R_1, \dots, R_n . If R is a subspace of the direct sum R^* of R_1, \dots, R_n , then R is a closed subspace of R^* . In particular, if moreover R^* is complete, i.e., if every R_i is complete, then so is R too.⁵⁾

Proof. Let $(a_i = a_{i1} + \dots + a_{in})$ ($a_i \in R, a_{ik} \in R_k$) ($i=1, 2, \dots$) be a convergent sequence in R with limit $c = c_1 + \dots + c_n$ ($c_k \in R_k$) in R^* . Suppose that $c \notin R$. Let c'_i be, for each i , an element of R which is mapped on c_i by the natural homomorphism φ_i of R onto R_i . Then we would have $\bigcap_{i=1}^n c'_i + n_i = \theta$ ⁶⁾, where n_i denotes the kernel of φ_i . Since every semi-local ring is a normal space and since each n_i is closed in R , there exists, for each i , an open set U_i in R such that $U_i \supseteq c'_i + n_i$ and $\bigcap_{i=1}^n U_i = \theta$. This contradicts to our assumption that c is the limit of the sequence (a_i) in R , and we have $c \in R$.

2. Completeness of a semi-local ring.

LEMMA 2.1. Let R be a semi-local ring and a a closed ideal in R . Then R is complete if both R/a and a are complete.

Proof. Let \bar{R} be the completion of R . Since a is complete, it follows that $a\bar{R} = a$ and a is closed in \bar{R} . Further, since R/a is complete, it follows $\bar{R}/a\bar{R} = \bar{R}/a = R/a$,⁷⁾ and this proves our assertion.

⁴⁾ Cf. [1, II, Proposition 13] or [7, Chapter II, Proposition 1].

⁵⁾ If R is complete, then R^* is complete without the assumption that R is a subspace of R^* .

⁶⁾ θ denotes the empty set.

⁷⁾ Cf. l.c. note 4).

LEMMA 2.2. Let R be a Noetherian semi-local ring. Let c be an element of R such that $c^2=0$. Then R is complete whenever R/cR is complete.

Proof. By virtue of preceding lemma, we have only to prove that cR is complete. Let (ca_n) ($n=1, 2, \dots$) ($a_n \in R$) be a convergent sequence in R such that $c(a_n - a_{n+1}) \in m^n$ ($n=1, 2, \dots$), where m denotes the J-radical of R . Set $q_n = m^n : cR$ ($n=1, 2, \dots$) and $\mathfrak{d} = (0) : cR$. Then we have $\bigcap_{n=1}^{\infty} q_n = \mathfrak{d}$ because $\bigcap_{n=1}^{\infty} m^n = (0)$. Since R/\mathfrak{d} is complete, $\{q_n/\mathfrak{d}; n=1, 2, \dots\}$ forms a system of neighbourhoods of zero in R/\mathfrak{d} , by virtue of Lemma 1.2, this shows that (a_n) is a convergent sequence in R/\mathfrak{d} . Let a be its limit, then ca is the limit of (ca_n) . This proves our assertion.

THEOREM 4. Let R be a Noetherian semi-local ring with p -radical⁸⁾ \mathfrak{n} . Then R is complete whenever R/\mathfrak{n} is complete.

This is an immediate consequence of Lemma 2.2.

THEOREM 5. Let R be a Noetherian semi-local ring. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_h$ be the totality of minimal prime divisors of zero ideal in R . Then R is complete whenever every R/\mathfrak{p}_i is complete.

This follows immediately from Corollary to Theorem 1 and Theorems 3, 4.

3. Subrings of the completion of a semi-local ring.

Let R be a ring and m its ideal. Suppose that $\bigcap_{n=1}^{\infty} m^n = (0)$. Then R is called an m -adic ring if R is topologized by taking $\{m^n; n=1, 2, \dots\}$ as a system of neighbourhoods of zero.

THEOREM 6. Let R be a semi-local ring and let \bar{R} be its completion. Let m and \bar{m} be the J-radicals of R and \bar{R} respectively. If R' is a subring of \bar{R} containing R and if we set $m' = \bar{m} \cap R'$, then we have $R \cap m'^k = m^k$ ($k=1, 2, \dots$). Consequently, R is a subspace of m' -adic ring R' .

Proof. Since \bar{R} is an \bar{m} -adic ring, R' becomes an m' -adic ring. Since clearly $m = m' \cap R$, we have $m^k \subseteq m'^k \cap R$. On the other hand, it follows from $\bar{m}^k \cap R = m^k$ that $m^k = (\bar{m}^k \cap R') \cap R \supseteq m'^k \cap R$, because $m' = \bar{m} \cap R$. These prove our assertion.

THEOREM 7. Let R be a semi-local ring and \bar{R} its completion. If a subring R' of \bar{R} containing R is finite with respect to R , then R' is a semi-local ring and R is a subspace of R' , but (the semi-local ring) R' is not a subspace of \bar{R} unless

⁸⁾ The p -radical of a ring R is the intersection of all prime ideals in R ; cf. [8]. If R is Noetherian, it is the largest nilpotent ideal.

R' coincides with R .

Proof. Let \mathfrak{m} , \mathfrak{m}' and $\overline{\mathfrak{m}}$ be the J-radicals of R , R' and \overline{R} respectively. Put further $\mathfrak{m}'' = \overline{\mathfrak{m}} \cap R'$. Then it follows from Theorem 5 that $(\mathfrak{m}'')^k \cap R = \mathfrak{m}^k$ ($k=1, 2, \dots$), while we have clearly that $\mathfrak{m}R' \subseteq \mathfrak{m}' \subseteq \mathfrak{m}''$, which shows that R is a subspace of R' . Since $R/\mathfrak{m} = \overline{R}/\overline{\mathfrak{m}}$, we have $R/\mathfrak{m} = R'/\mathfrak{m}''$. Suppose now that R' is a subspace of R , then $\mathfrak{m}R' = \mathfrak{m}\overline{R} \cap R' = \mathfrak{m}''$, because $\mathfrak{m}R'$ is (open whence) closed in R' . We have therefore $R + \mathfrak{m}R' = R'$, which implies $R = R'$ by virtue of [6, Appendix, Corollary to Proposition 4].

Remark. As was shown in the above proof, we have also that $\mathfrak{m}R' \not\cong \mathfrak{m}''$ if $R' \not\cong R$.

COROLLARY 1. Let R and \overline{R} be the same as in Theorem 6. Then \overline{R} is not finite with respect to R whenever $R \not\cong \overline{R}$.

COROLLARY 2. Let R be a Noetherian semi-local ring and let R' be a semi-local ring in which R is contained as a subring as well as a subspace. Then R is closed in R' whenever R' is finite with respect to R .

We prove, by the way, some properties of \mathfrak{m} -adic rings.

PROPOSITION 3.1. If an \mathfrak{m} -adic ring R is a subspace as well as a subring of an \mathfrak{m}' -adic ring R' and if both \mathfrak{m} and \mathfrak{m}' are semi-prime ideals⁹⁾ in R and R' respectively, then we have $\mathfrak{m}' \cap R = \mathfrak{m}$.

Proof. Since $\mathfrak{m}' \cap R$ is an open semi-prime ideal in R , we have $\mathfrak{m}' \cap R \supseteq \mathfrak{m}$. On the other hand, since we can find a natural number k so that $\mathfrak{m} \supseteq (\mathfrak{m}')^k \cap R \supseteq (\mathfrak{m}' \cap R)^k$ and since \mathfrak{m} is a semi-prime ideal, we have $\mathfrak{m} \supseteq \mathfrak{m}' \cap R$.

PROPOSITION 3.2. Let R be an \mathfrak{m} -adic ring, and suppose that \mathfrak{m} is a finite intersection of maximal ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_h$ of R . Let S be the complementary set of $\bigcup_{i=1}^h \mathfrak{p}_i$ with respect to R . Then the ring R_s of quotients of S with respect to R in the sense of H. Grell [4] is definable and is a semi-local ring. Further R is a dense subset of R_s .¹⁰⁾

Proof. S is clearly multiplicatively closed. S contains no zero divisor, because $\mathfrak{m} \cap S = \emptyset$, every \mathfrak{m}^n ($n=1, 2, \dots$) is an intersection of primary ideals and $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$.¹¹⁾ Therefore R_s is definable. Further, since $\mathfrak{m}^n R_s \cap R = \mathfrak{m}^n$, R_s is a semi-

⁹⁾ A semi-prime ideal in a ring R is an ideal which is an intersection of prime ideals in R ; cf. [8].

¹⁰⁾ Cf. [11, § 7].

¹¹⁾ Cf. [7, Chapter I, Lemma 3].

local ring and R is a subspace of R_s . Now we prove that R is dense in R_s . That $((b/a) + m^n R_s) \cap R \neq \emptyset$ ($b \in R, a \in S$) is equivalent to that $b - ac_n \in m^n$ for a suitable $c_n \in R$. Since $a \in S$, a is unit in R/m^n , and this shows the existence of such c_n (for each n). This completes our proof.

Remark. Set $S' = \{a \in R; a - 1 \in m\}$. Then R_s coincides with the ring of quotients of S' with respect to R , because every element of S is unit in R/m .¹⁰⁾

4. Supplementary remarks to [7, Chapter II, Proposition 8].

First we prove

PROPOSITION 4.1.¹²⁾ Let R be a ring in which every maximal ideal is principal. Then the following five conditions for R are equivalent to each other:

(1) R is a direct sum of a finite number of principal ideal rings each of which is einartig.¹³⁾

(2) R is a principal ideal ring.

(3) R is Noetherian.

(4) R is a subdirect sum of a finite number of einartig rings.

(5) Zero ideal of R is an intersection of a finite number of primary ideals q_1, \dots, q_s such that $\bigcap_{n=1}^{\infty} p^n \subseteq q_i$ for any maximal ideal p containing q_i (for each i).

Before proving this, we state some lemmas:

LEMMA 4.1. If a ring R is a subdirect sum of a finite number of Noetherian rings, then R is Noetherian, too.

Proof. Let R be a subdirect sum of Noetherian rings R_1, \dots, R_n . Let a be an ideal in R . Let a_i be the natural image of a in R_i . Then there exists a finite basis (a'_1, \dots, a'_r) for a_i in R_i . Let a_i be, for each i , an element of a whose R_i -component is a'_i . Then clearly $a = (a_1, \dots, a_r) + a \cap (R_2 + \dots + R_n)$. Thus we can prove our assertion by induction on n .

LEMMA 4.2. Every local ring with principal maximal ideal is an einartig principal ideal ring. [7, Chapter II, Proposition 8.]

LEMMA 4.3. An einartig ring R is a principal ideal ring whenever every maximal ideal is principal.

For, R is Noetherian by virtue of [3, Theorem 2].

Proof of Proposition 4.1. It is clear that (2), (3), (4) and (5) follows from (1) and that (3) follows from (2). (3) follows from (4) by virtue of Lemmas

¹²⁾ As for the equivalence of (1), (2) and (3), cf. [5, Theorem 9].

¹³⁾ A ring R is said to be einartig if every proper prime ideal is maximal.

4.1 and 4.3. To prove that (1) follows from (3), let $q_1 \cap \dots \cap q_n$ be a shortest representation of zero ideal in R as an intersection of primary ideals. Let \mathfrak{p} be a maximal ideal in R , then the ring of quotients¹⁴⁾ of \mathfrak{p} with respect to R is a local ring, whence an einartig principal ideal ring. This shows that \mathfrak{p} contains only one q_i and R/q_i is einartig.¹⁵⁾ It follows from this that R is the direct sum of $R/q_1, \dots, R/q_n$ each of which is, by virtue of Lemma 4.3, an einartig principal ideal ring. That (4) follows from (5) is easy if we observe the following

LEMMA 4.4. If a principal ideal aR in a ring R contains properly a prime ideal \mathfrak{p} , then $\mathfrak{p} \subseteq \bigcap_{n=1}^{\infty} a^n R$.

Proof. Since $aR \supset \mathfrak{p}$, we have $\mathfrak{p} = a\mathfrak{p}'$ for an ideal \mathfrak{p}' in R . Since $a \notin \mathfrak{p}'$, we have $\mathfrak{p} = \mathfrak{p}'$. This shows that $\mathfrak{p} = a^n \mathfrak{p}$, which proves our assertion.

Next we construct a semi-local ring R which is not Noetherian, but every maximal ideal in R is principal:

EXAMPLE. Let K be a field and let x, y and z be indeterminates. Let R_1 be the subring of $K(x, y)$ generated by $K[x, y]$ and y/x . Then $R_1 \cong K(x, y)$ and xR_1 is a maximal ideal in R_1 . Let R_2 be the ring of quotients of xR with respect to R_1 . Let S be the intersection of complementary sets of $xR_2[z]$ and $zR_2[z]$ with respect to $R_2[z]$. Then the ring R of quotients of S with respect to $R_2[z]$ is a required ring.

In fact, R has only two maximal ideals xR and zR , while R is semi-local because $\bigcap_{n=1}^{\infty} z^n R = (0)$.

BIBLIOGRAPHY

- [1] C. Chevalley, On the theory of local rings, Ann. Math. **44**, pp. 690-708 (1943).
- [2] I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc. **59**, pp. 54-106 (1946).
- [3] I. S. Cohen, Commutative rings with restricted minimum condition, Duke Math. J. **17**, pp. 27-42 (1950).
- [4] H. Grell, Beziehungen zwischen den Idealen verschiedener Ringe, Math. Ann. **97**, pp. 409-523 (1927).
- [5] S. Mori, Allgemeine Z.P.I-Ringe, J. Sci. Hiroshima Univ. A, **10**, pp. 117-136 (1940).
- [6] M. Nagata, On the structure of complete local rings, Nagoya Math. J. **1**, pp. 63-70 (1950).
- [7] ———, On the theory of semi-local rings, forthcoming.

¹⁴⁾ In the sense of [10]; cf. also [7].

¹⁵⁾ Observe the correspondence between primary ideals of R and those of a ring of quotients.

- [8] —, On the theory of radicals in a ring, forthcoming.
- [9] —, Note on subdirect sums of rings, Nagoya Math. J. **2**, pp. 49-53 (1951).
- [10] A. I. Uzkov, On the rings of quotients of commutative rings, Mat. Sbornik N. S. **22** (64) (1948) [in Russian].
- [11] O. Zariski, Generalized semi-local rings, Sum. Bras. Math. **1**, pp. 169-195 (1946).

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