ON THE DEGREES OF IRREDUCIBLE REPRESENTATIONS OF A FINITE GROUP

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In 1896 G. Frobenius proved: the degree of any (absolutely) irreducible representation of a finite group divides its order.[1] This theorem was improved by I. Schur in 1904 as follows: the degree of any irreducible representation of a finite group divides the index of its centre.[2] Using the results of H. Blichfeld[5] and those of I. Schur[3,4] we shall prove more precisely the following

THEOREM. *The degree of any irreducible representation of a finite group divides the index of each of its maximal abelian normal subgroups.*

LEMMA. Let $Z(\mathfrak{G})=\{Z(G)=(z_{\kappa\lambda}(G)),\ G\in\mathfrak{G}\}$ be an irreducible representation of degree z of a finite group $\mathfrak{G}$ of order g such that $z_{11}(G)$ is an algebraic integer for each $G\in\mathfrak{G}$. Let $\mathfrak{A}$ be a subgroup of $\mathfrak{G}$ of order a such that $Z(A)=\begin{pmatrix} z_{11}(A) & 0 \\ 0 & * \end{pmatrix}$ for each $A\in\mathfrak{A}$. Then z divides g/a.

Proof of the Lemma. By a fundamental relation of I. Schur,[3] we have

$$\sum_{G\in\mathfrak{G}} z_{11}(G)z_{11}(G^{-1})=g/z.$$

Since $Z(GA)=\begin{pmatrix} z_{11}(G)z_{11}(A) & * \\ * & * \end{pmatrix}$ and $Z(A^{-1}G^{-1})=\begin{pmatrix} z_{11}^{-1}(A)z_{11}(G^{-1}) & * \\ * & * \end{pmatrix}$, we have $z_{11}(G_1)z_{11}(G_1^{-1})=z_{11}(G_2)z_{11}(G_2^{-1})$, if $G_1^{-1}G_2$ belongs to $\mathfrak{A}$. Therefore $a\sum'_{G \bmod \mathfrak{A}} z_{11}(G)\times z_{11}(G^{-1})=g/z$. Since $\sum'_{G \bmod \mathfrak{A}} z_{11}(G)z_{11}(G)^{-1}$ is an algebraic integer, z divides g/a.

Proof of the Theorem. Let $Z(\mathfrak{G})=\{Z(G)=(z_{\kappa\lambda}(G)),\ G\in\mathfrak{G}\}$ be an irreducible representation of $\mathfrak{G}$ and $\mathfrak{A}$ an abelian normal subgroup of $\mathfrak{G}$. First we

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1) Über die Primfaktoren der Gruppendeterminante, Sitzb. Berlin, S. 1343.

2) Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. für Math. **127**, S. 20.

3) Neue Begründung der Theorie der Gruppencharaktere, Sitzb. Berlin (1905), S. 406.

4) Über Gruppen linearer Substitutionen mit Koeffizienten aus einem algebraischen Zahlkörper, Math. Ann. **71** (1912) S. 355.

5) Finite Collineation Groups, Chapter IV, Chicago (1917).

suppose that $Z(\mathfrak{A})$ is contained in the centre of $Z(\mathfrak{G})$. By a theorem of I. Schur[4] we can assume that $z_{11}(G)$ is an algebraic integer for each $G \in \mathfrak{G}$. By Schur's Lemma $Z(A)$ is a scalar for each $A \in \mathfrak{A}$. Therefore the theorem follows from the LEMMA. Secondly we suppose that $Z(\mathfrak{A})$ is not contained in the centre of $Z(\mathfrak{G})$. Then $Z(A)$ is not a scalar for some $A \in \mathfrak{A}$. Therefore $Z(\mathfrak{G})$ is imprimitive by a theorem of H. Blichfeld.[5] Let $\mathfrak{M} = \mathfrak{M}_1 + \ldots$ be the primitive decomposition of $Z(\mathfrak{G})$-space $\mathfrak{M}$ by $Z(\mathfrak{A})$. Let $\mathfrak{G}_1$ be the subgroup of $\mathfrak{G}$, which consists of all the elements G_1 of $\mathfrak{G}$ such that $Z(G_1)\mathfrak{M}_1 = \mathfrak{M}_1$. Obviously $\mathfrak{G}_1$ contains $\mathfrak{A}$. Let $[\mathfrak{G}_1]$ be the representation of $\mathfrak{G}_1$ induced by $Z(\mathfrak{G}_1)$ in $\mathfrak{M}_1$. Then $[\mathfrak{G}_1]$ is primitive. By a theorem of I. Schur[4] we can assume that all the coefficients of $[\mathfrak{G}_1]$ are algebraic integers. By a theorem of H. Blichfeld[5] $[\mathfrak{A}]$ is contained in the centre of $[\mathfrak{G}_1]$. Thus $z_{11}(\mathfrak{G}) = 0$ for each $G \notin \mathfrak{G}_1$, $z_{11}(G_1)$ is an algebraic integer for each $G_1 \in \mathfrak{G}_1$ and $Z(A) = \begin{pmatrix} z_{11}(A) & 0 \\ 0 & * \end{pmatrix}$ for each $A \in \mathfrak{A}$. Therefore the theorem follows from the LEMMA.

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