INTEGRABILITY OF THE BACKWARD DIFFUSION EQUATION IN A COMPACT RIEMANNIAN SPACE

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1. Introduction. Let R be an orientable, compact Riemannian space with the metric $ds^2=g_{ij}(x)dx^i dx^j$, and consider the *backward diffusion equation*

$$(1) \qquad \frac{\partial f(t,x)}{\partial t}=A\cdot f(t,x), \qquad t\geqq 0,$$

$$(Af)(x)=b^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}+a^i(x)\frac{\partial f}{\partial x^i}.$$

Here $b^{ij}(x)$ is a contravariant tensor such that the quadratic form $b^{ij}(x)\xi_i\xi_j$ is >0 for $\sum_i \xi_i^2>0$, and $a^i(x)$ changes, by a coordinate transformation $x\to\bar{x}$, as follows:

$$(2) \qquad \bar{a}^i(\bar{x})=\frac{\partial \bar{x}^i}{\partial x^k}a^k(x)+\frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^s}b^{ks}(x).$$

These conditions for the coefficients $a^i(x)$ and $b^{ij}(x)$ are connected with the probabilistic interpretation of the equation (1).[1] In preceding notes,[2] the author treated the stochastic integrability of the *forward diffusion equation* (*Fokker-Planck's equation*)

$$(3) \qquad \frac{\partial f(t,x)}{\partial t}=A'\cdot f(t,x), \qquad t\geqq 0,$$

$$(A'f)(x)=\frac{1}{\sqrt{g(x)}}\frac{\partial^2}{\partial x^i \partial x^j}(\sqrt{g(x)}b^{ij}(x)f(x))$$
$$+\frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}(-\sqrt{g(x)}a^i(x)f(x)), \quad g(x)=\det(g_{ij}(x))$$

Received March 15, 1951.

1) A. Kolmogoroff: Zur Theorie der stetigen zufälligen Prozesse, Math. Ann., **108** (1933), 149-160. K. Yosida: An extension of Fokker-Planck's equation, Proc. Japan Acad., **25** (1949), (9), 1-3.

2) K. Yosida: Integration of Fokker-Planck's equation in a compact Riemannian space, Arkiv för Matematik, **1** (1949), 9, 71-75. K. Yosida: Integration of Foker-Planck's equation with boundary condition, Journ. Math. Soc. Japan, (1951), Takagi's Congratulation volume.

by the semi-group theory.[3] The purpose of the present note is to consider the *stochastic integrability* (to be explained below) of (1), also by the semi-group theory. The result[4] may be considered, in a certain sense, a dual of the result in the preceding notes referred to above.

2. The theorems. For the sake of simplicity, we assume that R is an analytic manifold and that $a^j(x)$ and $b^{ij}(x)$ are holomorphic functions of the coordinates $x=(x^1, x^2, \ldots, x^n)$. We consider A as an additive operator whose domain $D(A)$ is the totality of infinitely differentiable functions defined in R, with values in the Banach space $C(R)$ of the totality of continuous functions $f(x)$ defined in R and metrized by the norm $\|f\|=\max_{x\in R}|f(x)|$. The following two simple lemmas are essential for our arguments.

LEMMA 1. For any $f\in D(A)$ and for any positive number m, we have

$$\max_{x\in R} h(x)\geqq f(x)\geqq \min_{x\in R} h(x),\quad h(x)=f(x)-m^{-1}(Af)(x). \tag{4}$$

Proof. Let $f(x)$ reach its maximum (minimum) at $x_0(x_1)$. Then we have

$$h(x_0)=f(x_0)-m^{-1}(Af)(x_0)\geqq f(x_0)\quad (h(x_1)=f(x_1)-m^{-1}(Af)(x_1)\leqq f(x_1)).$$

COROLLARY. For any $f\in D(A)$ and for any positive number, we have

$$\|f-m^{-1}Af\|\geqq\|f\|. \tag{5}$$

LEMMA 2. Let $\{f_n\}\subseteqq D(A)$ and $\{Af_n\}$ converge, as $n\to\infty$, strongly[5] to 0 and h respectively. Then we have $h(x)\equiv 0$.

Proof. For any $k(x)\in D(A)$, we have $(dx=\sqrt{g(x)}dx^1dx^2\ldots dx^n,\ g(x)=\det(g_{ij}(x))$

$$\int_R h(x)k(x)dx=\lim_{n\to\infty}\int_R(Af_n)k(x)dx=\lim_{n\to\infty}\int_R f_n(x)(A'k)(x)dx=0,$$

and thus $h(x)$ must $\equiv 0$.

COROLLARY. The smallest closed extension $\bar{A}$ of the operator A exists. $\bar{A}$ is defined as follows:

(6) $\bar{A}f$ is defined and $=h$ if there exists $\{f_n\}\subseteqq D(A)$ such that $\{f_n\}$ and $\{Af_n\}$ converge, as $n\to\infty$, strongly to f and h.

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[3] E. Hille: Functional Analysis and Semi-groups, New York (1948). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, Journ. Math. Soc. Japan, **1** (1949), 1, 15-21, and K. Yosida: An operator-theoretical treatment of temporally homogeneous Markoff process, ibid., **1** (1949), 1, 224-235.

[4] Cf. another approach by K. Itô: Stochastic differential equations on a differentiable manifold, Nagoya Math. J., **1** (1950), 35-48.

[5] By the topology defined by the norm $\|f\|$, viz. by the uniform convergence on R.

From these two lemmas we have the

THEOREM 1. *The inverse* $I_m=(-m^{-1}\bar{A})^{-1}$ *exists* (I=*the identity operator*) *for* $m>0$ *and* I_m *is a positive, contraction operator, leaving the constant functions invariant:*

(7) if $h(x)$ in the domain $D(\bar{A})$ of $\bar{A}$ be non-negative, then $(I_m h)(x)$ is also non-negative and $\|I_m h\| \leqq \|h\|$; $I_m \cdot 1 = 1$.

By the semi-group theory, the coincidence of the domains $D(I_m)$ of I_m with $C(R)$ is the necessary and sufficient condition for the existence of the one-parameter semi-group of linear operators in $C(R)$ with the properties:

(8) $T_t T_s = T_{t+s}$ $(t, s \geqq 0)$, $T_0 = I$;
T_t are positive, contraction operators, leaving constant functions invariant; strong $\lim_{t \to t_0} T_t f = T_{t_0} f$, $f \in C(R)$;
strong $\lim_{\delta \to 0} \frac{T_{t+\delta} - T_t}{\delta} f = \bar{A} T_t f = T_t \bar{A} f$ for f in $D(\bar{A})$, which is surely dense in $C(R)$.

This T_t is, in fact, defined by

(9) $T_t f = \text{strong} \lim_{m \to \infty} (I - t\, m^{-1}\bar{A})^{-m} f$, $f \in C(R)$.

The existance of this semi-group T_t may be considered as the *stochastic integrability* of (1). The coincidence of the domains $D(I_m)$ with $C(R)$ is equivalent to the denseness of the ranges $R(I-m^{-1}A) = \{f - m^{-1}Af;\ f \in D(A)\}$, $m>0$, in $C(R)$. Hence we have the

COROLLARY. (1) is stochastically integrable if and only if positive numbers m do not belong to the residual spectra of the operator A.

When the dimension n of the space R is $\geqq 2$, we have the

THEOREM 2. *The backward diffusion equation* (1) *is stochastically integrable if the compact space* R *is of dimension* $\geqq 2$.

Proof. Let the range $R(I-m^{-1}A)$ be not dense in $C(R)$. Then there exists a measure φ, countably additive for Borel sets of R such that

(10) $0 <$ total variation of φ in $R < \infty$,

(11) $\int_R (f(x) - m^{-1}(Af)(x))\varphi(dx) = 0$ for $f \in D(A)$,

since the conjugate space of $C(R)$ is the space of measures, countably additive for Borel sets and of bounded total variations. If we define the *distribution* (in the sense of L. Schwartz[6]) by

(12) $H(f) = \int_R f(x)\varphi(dx)$, $f \in D(A)$,

[6] L. Schwartz: Théorie des distributions, 1, Paris (1950).

H satisfies, by (11), the *differential equation* (*in the sense of the distribution*)

(13) $A'H=mH.$

By the *elliptic character* of the differential operator A', there must exist[7] an infinitely differentiable function $h(x)$ such that

(14) $(A'h)(x)=mh(x), \qquad H(f)=\int_R f(x)h(x)dx.$

By (10), we have

(15) $h(x)\not\equiv 0.$

Let $k(x)$ be $=1$, -1 or $=0$ according as $h(x)>0$, <0 or $=0$. Then we have

(16) $$0=\int_R |h(x)-m^{-1}(A'h)(x)|dx \geqq \int_R (h(x)-m^{-1}(A'h)(x))k(x)dx$$
$$=\int_R |h(x)|dx-m^{-1}\sum_i\int_{P_i}(A'h)(x)dx+m^{-1}\sum_j\int_{N_j}(A'h)(x)dx,$$

where $P(N)$ are connected domains in which $h(x)>0$ (<0) such that $h(x)=0$ on their boundaries $\partial P\,(\partial N)$. By the integral theorem of Green's type we have

(17) $$\int_P (A'h)(x)dx=\int_{\partial P}\frac{\partial h}{\partial n}dS,$$

where n and dS respectively denote outer normal and positive measure on ∂P. Hence we have $\int_P (A'h)(x)dx\leqq 0$, and similarly $\int_N (A'h)(x)dx\geqq 0$. Therefore we obtain, from (14)–(15), a contradiction $0\geqq\int_R |h(x)|dx>0$.

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[7] L. Schwartz: loc. cif., p. 136,