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ON THE EXISTENCE OF VARIOUS BOUNDED HARMONIC FUNCTIONS WITH GIVEN PERIODS

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1. Consider a pair (R,Γ) of a Riemann surface R and a period Γ. By a period Γ we mean a real-valued function $\Gamma(\gamma)$ on one-dimensional cycles $\{\gamma\}$ of the Riemann surface R. Let O_X^* be the class of pairs (R,Γ) such that there is no harmonic function on the Riemann surface R which satisfies a boundedness property X and

$$\int_\gamma {}^*du = \Gamma(\gamma)$$

for every cycle γ. As for X we let B stand for boundedness, D for the finiteness of the Dirichlet integral, BD for B and D. The relations to standard notations O_{AX} in the classification theory of Riemann surfaces (cf. [1]) should be clear. For example, $R \in O_{AD}$ means that $(R,\Gamma_0) \in O_D^*$, where $\Gamma_0(\gamma) = 0$ for every cycle γ, and $R \in O_{ABD}$ means that $(R,\Gamma_0) \in O_{BD}^*$. From our standpoint H. Widom's articles [3] and [4] may be considered as the study of the class O_B^*. Our study may be also be considered as being in the frame work of that of Riemann matrices.

The well known Virtanen identity $O_{HD} = O_{HBD}$ is one of the beautiful results in the classification theory; what's more, the space $HBD(R)$ is dense in $HD(R)$ in the CD-topology (cf. [1, p. 178]). Therefore there exists a sequence $\{u_n\}$ in $HBD(R)$ convergent to a given $u \in HD(R)$ so that $\int_\gamma {}^*du_n$ converges to $\int_\gamma {}^*du$ for every cycle γ. In this connection one naturally asks whether $O_D^* = O_{BD}^*$. The question also relates to the unsettled strictness question $O_{AD} \subset O_{ABD}$. The main result of this paper is the following strict inclusion:

THEOREM. $$O_D^* < O_{BD}^*\,.$$

We will show that there exists a planar region Ω^* such that there

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exist HD-functions on Ω^* which have the same period as the given HB-functions on Ω^* but there exists no HB-function on Ω^* which has the same period as some HD-function on Ω^*.

2. Let Ω denote the right half plane of the complex plane and $\Omega[a\, b]$ the right half plane less the interval $[a\, b]$ on the real axis. The function

$$g(z, z_0) = \log\left|\frac{z + \bar{z}_0}{z - \bar{z}_0}\right|$$

is the Green's function for the region Ω with pole at z_0. The function

$$u[a\, b](z) = \int_a^b \log\left|\frac{z + t}{z - t}\right| dt \qquad (0 < a < b)$$

is the potential whose support is the interval $[a\, b]$. Therefore $u[a\, b](z)$ is positive and harmonic on the region $\Omega[a\, b]$ and vanishes on the imaginary axis, and furthermore has the following properties:

LEMMA 1. *Let β be a simple curve oriented clockwise enclosing the interval $[a\, b]$. Then $u[a\, b]$ is continuous on the region Ω and*

$$(1) \qquad \int_\beta {}^*du[a\, b] = 2\pi(b - a);$$

$$(2) \qquad D(u[a\, b]) = \pi\{(2b)^2 \log 2b - 2(a + b)^2 \log (a + b) + (2a)^2 \log 2a\} + 2\pi(b - a)^2 \log \frac{1}{b - a}.$$

Proof. Put $u = u[a\, b]$. For $a \leq x \leq b$,

$$\begin{aligned} u(x) &= \int_a^b \log\left|\frac{x + t}{x - t}\right| dt \\ &= \int_a^b \log (x + t)dt - \int_a^x \log (x - t)dt - \int_x^b \log (t - x)dt \\ &= (x + b) \log (x + b) - (x + a) \log (x + a) \\ &\quad - (x - a) \log (x - a) - (b - x) \log (b - x)\,. \end{aligned}$$

Thus $u(x)$ is continuous on the interval $[a\, b]$ which is the support of potential u, and therefore it follows from the continuity principle (cf. [2, p. 54]) that u is continuous on the region Ω.

Fix x, $a < x < b$, and consider

$$f(z) = \int_a^b \log \frac{z+t}{z-t} dt$$

on the upper plane. Observe that

$$f'(z) = \int_a^b \left(\frac{1}{z+t} - \frac{1}{z-t} \right) dt .$$

Since

$$\lim_{z \to x} \operatorname{Im} \left(\int_a^b \frac{1}{z+t} dt \right) = 0$$

and

$$\int_a^b \frac{1}{t-z} dt = \log (b-z) - \log (a-z) ,$$

whose imaginary part is the angle formed by the lines $\overline{za}$ and $\overline{zb}$, we conclude that

$$\lim_{z \to x} \operatorname{Im} (f'(z)) = \pi.$$

From this it follows that $*du = \pi$ on the interval $(a\ b)$ considered as the degenerate closed curve traced in the negative direction.

Therefore (1) is trivially true. By

$$D(u) = 2\pi \int_a^b u(t) dt$$

and direct calculations, we obtain (2).

COROLLARY. *For $a \geq e$,*

$$\int_\beta *du[a\ a+1] = 2\pi ; \tag{3}$$

$$D(u[a\ a+1]) \leq 10\pi \log a . \tag{4}$$

Proof. The relation (3) is trivial and (4) is seen by direct calculations.

3. We denote by D_c the interior of the ellipse, whose horizontal axis is of length $\frac{1}{2}((1/r) + r) = c$ and vertical axis $\frac{1}{2}((1/r) - r)(0 < r < 1)$, less the interval with length 1 in the center on the horizontal axis. Let

v_c denote the harmonic measure of the interval with respect to the region D_c.

LEMMA 2. *Let β be a simple curve oriented clockwise enclosing the interval. Then*

$$\int_\beta {}^*dv_c \leq 2\pi (\log c)^{-1} .$$

Proof. Suppose that the center of the ellipse is the origin. The function $z = \frac{1}{4}((1/w) + w)$ maps the annulus $\{r < |w| < 1\}$ conformally onto D_c, the circle $|w| = r$ onto the ellipse and the circle $|w| = 1$ onto the interval. The harmonic measure of the circle $\{|w| = 1\}$ with respect to the annulus $\{r < |w| < 1\}$ is the function

$$\log \frac{|w|}{r} \Big/ \log \frac{1}{r}$$

whose flux is $2\pi(\log 1/r)^{-1}$. Therefore

$$\int_\beta {}^*dv_c = 2\pi \left(\log \frac{1}{r}\right)^{-1} = 2\pi (\log (c + (c^2 - 1)^{\frac{1}{2}}))^{-1} \leq 2\pi (\log 2c)^{-1} .$$

4. Put

$$a_n = \exp\left(\sum_{k=0}^{n} 2^k\right)$$

and

$$\Omega^* = \bigcap_{n=1}^{\infty} \Omega[a_n\, a_n + 1]$$

and $u_n = u[a_n\, a_n + 1]$ and $u = \sum_{n=1}^{\infty} n 2^{-n} u_n$. Let γ_n be a simple curve oriented clockwise enclosing $[a_n\, a_n + 1]$ so that γ_m and γ_n are disjoint if $m \neq n$. Then $\{\gamma_n\}_{n=1}^{\infty}$ is a homology basis of Ω^*.

In order to prove our theorem it is sufficient to show the following lemma:

LEMMA 3. *The region Ω^* has the following properties:*

(i) *The function u belongs to $HD(\Omega^*)$;*

(ii) *No function belong to $HB(\Omega^*)$ has the same period as the function u;*

(iii) *Give any function v belonging to $HB(\Omega^*)$,*

$$v^* = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left(\int_{\gamma_n} {}^*dv \right) u_n$$

belongs to $HD(\Omega^*)$ and has the same period as the function v.

Proof. Since

$$D(u_n) \leq 10\pi \log a_n = 10\pi \sum_{k=0}^{n} 2^k \leq 20\pi 2^n ,$$

$$\sum_{n=0}^{\infty} n 2^{-n} (D(u_n))^{\frac{1}{2}} \leq (20\pi)^{\frac{1}{2}} n (2^{-\frac{1}{2}})^n < \infty .$$

Noticing this and using properties of CD-topology [1, p. 149], the function u belongs to the class $HD(\Omega^*)$, i.e. (i) is true.

To prove (ii) it suffices to show that

$$\lim_{n\to\infty} \frac{\int_{\gamma_n} {}^*dv}{\int_{\gamma_n} {}^*du} = 0$$

for every $v \in HB(\Omega^*)$. We may, without loss of generality, assume that $M - 1 > v > 1$. Let D_n denote the region D_c, $c = a_n - a_{n-1} - \frac{1}{2}$, whose outer boundary is an ellipse having the center at $a_n + \frac{1}{2}$ and passing $a_{n-1} + 1$, and let v_n denote $2Mv_c$. For $\frac{1}{2} < t < 1$, the set $\{z \in D_n ; tv_n > v\}$ contains a neighbourhood of the interval $[a_n\, a_n + 1]$ and does not contain a neighbourhood of the ellipse. By the maximum principle, this set is a region and we can choose some t so that the set $\{z \in D_n \,|\, tv_n = v\}$ is a simple regular closed curve, which is denoted by δ_n, homologous to γ_n. Since

$$\int_{\delta_n} {}^*dtv_n > \int_{\delta_n} {}^*dv ,$$

and

$$\int_{\gamma_n} {}^*dv_n = \int_{\delta_n} {}^*dv_n > t \int_{\delta_n} {}^*dv_n = \int_{\delta_n} {}^*dtv_n > \int_{\delta_n} {}^*dv = \int_{\gamma_n} {}^*dv .$$

By Lemma 2,

$$0 < \int {}^*dv_n \leq 2\pi M (\log ((a_n - a_{n-1}) - 1/2))^{-1}$$

$$\leq 2\pi M \left(\log \frac{a_n}{a_{n-1}} \right)^{-1} = 2\pi M 2^{-n} .$$

From

$$\int_{\gamma_n} {}^*du = n2^{-n}\int_{\gamma_n} {}^*du_n = 2\pi n2^{-n} ,$$

it follows that

$$\int_{\gamma_n} {}^*du > \frac{1}{M} n \int_{\gamma_n} {}^*dv_n > \frac{1}{M} n \int_{\gamma_n} {}^*dv .$$

Since $M-1 > M-v > 1$, by the same arguments,

$$\int_{\gamma_n} {}^*du \geq -\frac{1}{M} n \int_{\gamma_n} {}^*dv .$$

The proof of (ii) is herewith complete.

Since $\int_{\gamma_n} {}^*dv = o\ (n2^{-n})$, by the same argument as for the function u, we can show that the function v^* belongs to $HD(\Omega^*)$. It is trivial that the function v^* has the same period as the function v, and (iii) is obtained.

References

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