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ON THE QUADRATIC EXTENSIONS AND THE EXTENDED WITT RING OF A COMMUTATIVE RING

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Let B be a ring and A a subring of B with the common identity element 1. If the residue A-module B/A is inversible as an A-A-bimodule, i.e. $B/A \otimes_A \mathrm{Hom}_A(B/A, A) \approx \mathrm{Hom}_A(B/A, A) \otimes_A B/A \approx A$, then B is called a quadratic extension of A. In the case where B and A are division rings, this definition coincides with in P. M. Cohn [2]. We can see easily that if B is a Galois extension of A with the Galois group G of order 2, in the sense of [3], and if $\mathrm{Tr}_G(B) = \{\sum_{\sigma \in G} \sigma(b) : b \in B\} = A$, B is a quadratic extension of A. A generalized crossed product $\Delta(f, A, \Phi, G)$ of a ring A and a group G of order 2, in [4], is also a quadratic extension of A.

In this note, we study the case of commutative quadratic extensions, where A is a commutative ring and B is an A-algebra. Let A be a commutative ring with the identity element 1. We shall say that B is a quadratic extension of A if B is a ring extension of A with the common identity element and B is a finitely generated projective A-module of rank 2 so that B is a commutative ring. We denote by $Q(A)$ (resp. $Q_s(A)$) the set of all A-algebra isomorphism classes of quadratic (resp. separable quadratic) extensions of A. It is known that $Q_s(A)$ forms a group under a certain product, and in [1], [6] and [7], the group $Q_s(A)$ is investigated. In this note, in § 1, we define a product in $Q(A)$, which coincides with the product defined in [1], [6] and [7] in the subset $Q_s(A)$. Then, $Q(A)$ forms an abelian semi-group containing the subsemi-group $Q_s(A)$ which is a group, and an element $[B]$ in $Q(A)$ is contained in $Q_s(A)$ if and only if $[B]^2 = [B][B]$ is the identity element of $Q(A)$. In § 2, we give a generalization of a quadratic module and define A-isomorphisms between them. Then, we can consider a category consisting of these

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extended quadratic modules and A-isomorphisms. From this category we can construct a commutative ring $W^*(A)$. In § 3, we shall show that $W^*(A)$ is a commutative ring with the identity element, and there exists a ring homomorphism of the Witt ring $W(A)$ to $W^*(A)$ for which the image is an ideal of $W^*(A)$. Especially, if 2 is inversible in A, then $W(A)$ and $W^*(A)$ are isomorphic. In § 4, we shall give a group homomorphism of $Q_s(A)$ to the unit group $U(W^*(A))$ of $W^*(A)$.

1. Quadratic extension.

Let A be an arbitrary commutative ring with the identity element 1. A commutative extension ring B of A is called a quadratic extension of A if B is a finitely generated projective A-module of rank 2 and B has the same identity element 1. If B is a quadratic extension of A, then there exist a finitely generated projective A-module U of rank 1 and quadratic forms $q: U \to A$ and $q': U \to U$[1)] such that $B = A \oplus U$ and $x^2 = q(x) + q'(x)$ for all x in U.

LEMMA 1. *Let U be a finitely generated projective A-module of rank 1, and $q': U \to U$ a quadratic form. Then there exists an A-homomorphism $f: U \to A$ such that $q'(x) = f(x)x$ for all x in U.*

Proof. For the quadratic form $q': U \to U$, there exists a bilinear form $B: U \times U \to U$ such that $q'(x) = B(x, x)$ for all x in U, (cf. (2.3) in [2]). We may consider that B is an element in $\mathrm{Hom}_A(U \otimes_A U, U)$. Then by the following natural isomorphisms; $\mathrm{Hom}_A(U \otimes_A U, U) \approx \mathrm{Hom}_A(U \otimes_A U, A) \otimes_A U \approx \mathrm{Hom}_A(U, A) \otimes_A \mathrm{Hom}_A(U, A) \otimes_A U \approx \mathrm{Hom}_A(U, A) \otimes_A A$, there exist f_i in $\mathrm{Hom}_A(U, A)$ and a_i in A, $i = 1, 2, \cdots n$ such that $B(x, y) = \sum_{i=1}^n f_i(x) a_i y$ for all x and y in U. Put $f = \sum_{i=1}^n a_i f_i$ in $\mathrm{Hom}_A(U, A)$, then we have $q'(x) = B(x, x) = f(x)x$ for all x in U.

LEMMA 2. *Let U be a finitely generated projective A-module of rank 1, and f and g elements in $\mathrm{Hom}_A(U, A)$. If $f(x)x = g(x)x$ for all x in U, then $f = g$.*

Proof. If $f(x)x = g(x)x$ for all x in U, then we have also $f \otimes I(x)x = g \otimes I(x)x$ for all x in $U_{\mathfrak{m}} = U \otimes A_{\mathfrak{m}}$ and for every maximal ideal $\mathfrak{m}$ of A. For the local ring A, this lemma is clear, therefore we get easily $f = g$.

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1) cf. p. 490 in [5].

Thus, for a given quadratic extension B of A there exist a finitely generated projective A-module U of rank 1, an A-homomorphism $f: U \to A$ and a quadratic form $q: U \to A$ such that $B = A \oplus U$ and $x^2 = f(x)x + q(x)$ for all x in U. Conversely, if a finitely generated projective A-module U of rank 1, A-homomorphism $f: U \to A$ and a quadratic form $q: U \to A$ are given, then a quadratic extension $B = A \oplus U$ of A is constracted by $x^2 = f(x)x + q(x)$ for x in U. We denote such a quadratic extension of A by $B = (U, f, q)$.

In general, we can define as follows:

DEFINITION. Let P be a finitely generated projective and faithful A-module, $f: P \to A$ an A-homomorphism and $q: P \to A$ a quadratic form. Let $T(P) = A \oplus P \oplus P \otimes_A P \oplus \cdots$ be the tensor algebra of P over A. We denote by (P, f, q) the residue ring $T(P)/(x \otimes x - f(x)x - q(x); x \in P)$ of $T(P)$ by the ideal generated from the set $\{x \otimes x - f(x)x - q(x); x \in P\}$.[2)]

PROPOSITION 1. *Let (U, f, q) and (U', f', q') be quadratic extensions of A. Then (U, f, q) and (U', f', q') are A-algebra-isomorphic if and only if there exist an A-isomorphism $\sigma_1: U \to U'$ and an A-homomorphism $g: U \to A$ satisfying the following identities;*

$$q' \circ \sigma_1 = fg + q - g^2$$
$$f' \circ \sigma_1 = f - 2g ,$$

where fg, g^2 and $q' \circ \sigma_1$ are defined by $fg(x) = f(x)g(x)$, $g^2(x) = g(x)^2$ and $q' \circ \sigma_1(x) = q'(\sigma_1(x))$ for x in U.

Proof. Let $\sigma: (U, f, q) = A \oplus U \to (U', f', q') = A \oplus U'$ be an A-algebra-isomorphism. Then there exist an A-isomorphism $\sigma_1: U \to U'$ and an A-homomorphism $g: U \to A$ such that $\sigma(x) = g(x) + \sigma_1(x)$ for x in U. Since σ satisfies $\sigma(x^2) = \sigma(x)^2$ for x in U, we get the following identity

$$f(x)g(x) + q(x) + f(x)\sigma_1(x) = g(x)^2 + q'(\sigma_1(x)) + (f'(\sigma_1(x)) + 2g(x))\sigma_1(x)$$

for all x in U. Therefore we have

$$f(x)g(x) + q(x) = g(x)^2 + q'(\sigma_1(x)) \tag{1}$$
$$f(x)\sigma_1(x) = (f'(\sigma_1(x)) + 2g(x))\sigma_1(x) \tag{2}$$

2) The composition of natural homomorphisms $A \oplus U \hookrightarrow T(U) \to T(U)/(x \otimes x - f(x)x - q(x); x \in U)$ is an A-isomorphism as A-modules. For any quadratic extension $C = A \oplus U$ satisfying $x^2 = f(x)x + q(x)$ for all $x \in U$, $C \approx T(U)/(x \otimes x - f(x)x - q(x); x \in U)$ as A-algebras.

for all x in U. From (2) we have $f(x)x = (f'(\sigma_1(x)) + 2g(x))x$ for x in U, and by Lemma 2, we get $f(x) = f'(\sigma_1(x)) + 2g(x)$ for x in U. Thus, we have the identities of this proposition. The converse is obvious.

LEMMA 3. *Let* (U_i, f_i, q_i) *and* (U'_i, f'_i, q'_i) *be A-algebra-isomorphic quadratic extensions of* A, $i = 1, 2$. *Then* $(U_1 \otimes_A U_2, f_1 \otimes f_2, f_1^2 \overline{\otimes} q_2 + q_1 \overline{\otimes} f_2^2 + 2q_1 \otimes q_2)$ *and* $(U'_1 \otimes_A U'_2, f'_1 \otimes f'_2, f_1'^2 \overline{\otimes} q'_2 + q'_1 \overline{\otimes} f_2'^2 + 2q'_1 \otimes q'_2)$ *are also A-algebra-isomorphic, where* $f_1^2 \overline{\otimes} q_2 + q_1 \overline{\otimes} f_2^2 + 2q_1 \otimes q_2\, (x \otimes y) = f_1(x)^2 q_2(y) + q_1(x) f_2(y)^2 + 4q_1(x)q_2(y)$, $f_1^2 \overline{\otimes} q_2 + q_1 \overline{\otimes} f_2^2 + 2q_1 \otimes q_2\, (\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n (f_1(x_i)^2 q_2(y_i) + q_1(x_i) f_2(y_i)^2 + 4q_1(x_i)q_2(y_i)) + \sum_{i<j}^n (f_1(x_i) f_1(x_j) B_{q_2}(y_i, y_j) + B_{q_1}(x_i, x_j) f_2(y_i) f_2(y_j) + 2B_{q_1}(x_i, x_j) B_{q_2}(y_i, y_j))$, $(n > 1)$, *and* $f_1 \otimes f_2 (\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n f_1(x_i) f_2(y_i)$ *for* $\sum_{i=1}^n x_i \otimes y_i$ *and* $x \otimes y$ *in* $U_1 \otimes_A U_2$, (*cf.* (2.8) *in* [5]).[3)]

Proof. By Proposition 1, there exist A-isomorphisms $\sigma_1 : U_1 \to U'_1$ and $\sigma_2 : U_2 \to U'_2$, and A-homomorphisms $g_1 : U_1 \to A$ and $g_2 : U_2 \to A$ such that $q'_1 \circ \sigma_1 = f_1 g_1 + q_1 - g_1{}^2$, $f'_1 \circ \sigma_1 = f_1 - 2g_1$, and $q'_2 \circ \sigma_2 = f_2 g_2 + q_2 - g_2{}^2$, $f'_2 \circ \sigma_2 = f_2 - 2g_2$. By the computation, we get the following:

For any element $x \otimes y$ in $U_1 \otimes_A U_2$, $(f_1'^2 \overline{\otimes} q'_2 + q'_1 \overline{\otimes} f_2'^2 + 2q'_1 \otimes q'_2) \circ (\sigma_1 \otimes \sigma_2)(x \otimes y) = (f_1(x) - 2g_1(x))^2 (f_2(y) g_2(y) + q_2(y) - g_2(y)^2) + (f_1(x) g_1(x) + q_1(x) - g_1(x)^2)(f_2(y) - 2g_2(y))^2 + 4(f_1(x) g_1(x) + q_1(x) - g_1(x)^2)(f_2(y) g_2(y) + q_2(y) - g_2(y)^2) = f_1(x) f_2(y)(f_1(x) g_2(y) + g_1(x) f_2(y) - 2g_1(x) g_2(y)) + (f_1(x)^2 q_2(y) + q_1(x) f_2(y)^2 + 4q_1(x) q_2(y)) - (f_1(x) g_2(y) + g_1(x) f_2(y) - 2g_1(x) g_2(y))^2 = [(f_1 \otimes f_2)(f_1 \otimes g_2 + g_1 \otimes f_2 - 2g_1 \otimes g_2) + (f_1^2 \overline{\otimes} q_2 + q_1 \overline{\otimes} f_2^2 + 2q_1 \otimes q_2) - (f_1 \otimes g_2 + g_1 \otimes f_2 - 2g_1 \otimes g_2)^2](x \otimes y)$. Using the identities

$B_{q'_1}(\sigma_1(x_i), \sigma_1(x_j)) = f_1(x_i) g_1(x_j) + f_1(x_j) g_1(x_i) + B_{q_1}(x_i, x_j) - 2g_1(x_i) g_1(x_j)$ and $B_{q'_2}(\sigma_2(y_i), \sigma_2(y_j)) = f_2(y_i) g_2(y_j) + f_2(y_j) g_2(y_i) + B_{q_2}(y_i, y_j) - 2g_2(y_i) g_2(y_j)$ for $x_i \otimes y_i$ and $x_j \otimes y_j$ in $U_1 \otimes_A U_2$, we get as follows; $f'_1(\sigma_1(x_i)) f'_1(\sigma_1(x_j)) B_{q'_2}(\sigma_2(y_i), \sigma_2(y_j)) + B_{q'_1}(\sigma_1(x_i), \sigma_1(x_j)) f'_2(\sigma_2(y_i)) f'_2(\sigma_2(y_j)) + 2B_{q'_1}(\sigma_1(x_i), \sigma_1(x_j)) B_{q'_2}(\sigma_2(y_i), \sigma_2(y_j)) = (f_1(x_i) - 2g_1(x_i))(f_1(x_j) - 2g_1(x_j))(f_2(y_i) g_2(y_j) + f_2(y_j) g_2(y_i) + B_{q_2}(y_i, y_j) - 2g_2(y_i) g_2(y_j)) + (f_1(x_i) g_1(x_j) + f_1(x_j) g_1(x_i) + B_{q_1}(x_i, x_j) - 2g_1(x_i) g_1(x_j))(f_2(y_i) - 2g_2(y_i))(f_2(y_j) - 2g_2(y_j)) + 2(f_1(x_i) g_1(x_j) + f_1(x_j) g_1(x_i) + B_{q_1}(x_i, x_j) - 2g_1(x_i) g_1(x_j))(f_2(y_i) g_2(y_j) + f_2(y_j) g_2(y_i) + B_{q_2}(y_i, y_j) - 2g_2(y_i) g_2(y_j)) = f_1(x_i) f_2(y_i)(f_1(x_j) g_2(y_j) + g_1(x_j) f_2(y_j) - 2g_1(x_j) g_2(y_j)) + f_1(x_j) f_2(y_j)(f_1(x_i) g_2(y_i) + g_1(x_i) f_2(y_i) - 2g_1(x_i) g_2(y_i)) + f_1(x_i) f_1(x_j) B_{q_2}(y_i, y_j) + B_{q_1}(x_i, x_j)$

3) $f^2 \overline{\otimes} q'$ and $f^2 \otimes q'$ are defined by $f^2 \overline{\otimes} q'(\sum x_i \otimes y_i) = \sum_i f(x_i)^2 q'(y_i) + \sum_{i<j} f(x_i) f(x_j) B_{q'}(y_i, y_j)$ and $f^2 \otimes q'(\sum x_i \otimes y_i) = \sum_i 2f(x_i)^2 q'(y_i) + \sum_{i<j} B_{f^2}(x_i, x_j) B_{q'}(y_i, y_j)$ for $\sum x_i \otimes y_i$ in $M \otimes_A M'$.

$f_2(y_i)f_2(y_j) + 2B_{q_1}(x_i, x_j)B_{q_2}(y_i, y_j) - 2(f_1(x_i)g_2(y_i) + g_1(x_i)f_2(y_i) - 2g_1(x_i) g_2(y_i))(g_1(x_j)f_2(y_j) + f_1(x_j)g_2(y_j) - 2g_1(x_j)g_2(y_j))$. Accordingly, we get $(f_1'^2 \overline{\otimes} q_2' + q_1' \overline{\otimes} f_2'^2 + 2q_1' \otimes q_2') \circ (\sigma_1 \otimes \sigma_2)(\sum_{i=1}^n x_i \otimes y_i) = [(f_1 \otimes f_2)(f_1 \otimes g_2 + g_1 \otimes f_2 - 2g_1 \otimes g_2) + (f_1^2 \overline{\otimes} q_2 + q_1 \overline{\otimes} f_2^2 + 2q_1 \otimes q_2) + (f_1 \otimes g_2 - 2g_1 \otimes g_2)^2] (\sum_{i=1}^n x_i \otimes y_i)$ for all $\sum_{i=1}^n x_i \otimes y_i$ in $U_1 \otimes_A U_2$. Put $G = f_1 \otimes g_2 + g_1 \otimes f_2 - 2g_1 \otimes g_2$, then G is an A-homomorphism of $U_1 \otimes_A U_2$ to A, $\sigma_1 \otimes \sigma_2$ is an A-isomorphism of $U_1 \otimes_A U_2$ to $U_1' \otimes_A U_2'$, and these satisfy $(f_1'^2 \overline{\otimes} q_2' + q_1' \overline{\otimes} f_2'^2 + 2q_1' \otimes q_2') \circ (\sigma_1 \otimes \sigma_2) = (f_1 \otimes f_2)G + (f_1^2 \overline{\otimes} q_2 + q_1 \overline{\otimes} f_2^2 + 2q_1 \otimes q_2) + G^2$, and $(f_1' \otimes f_2') \circ (\sigma_1 \otimes \sigma_2) = f_1 \otimes f_2 - 2G$.

By Proposition 1, we have $(U_1 \otimes_A U_2, f_1 \otimes f_2, f_1^2 \overline{\otimes} q_2 + q_1 \overline{\otimes} f_2^2 + 2q_1 \otimes q_2)$ and $(U_1' \otimes_A U_2', f_1' \otimes f_2', f_1'^2 \overline{\otimes} q_2' + q_1' \overline{\otimes} f_2'^2 + 2q_1' \otimes q_2')$ are isomorphic as A-algebras.

DEFINITION. We denote by $Q(A)$ the set of all A-algebra-isomorphism classes $[U, f, q]$ of quadratic extensions (U, f, q) of A.

PROPOSITION 2. *$Q(A)$ forms an abelian semi-group with unit element* $[A, 1, 0]$ *by the product* $[U, f, q] \cdot [U', f', q'] = [U \otimes_A U', f \otimes f', f^2 \overline{\otimes} q' + q' \overline{\otimes} f'^2 + 2q \otimes q']$, *where (A, a, b) denotes a quadratic extension $A \oplus Av$ such that $v^2 = av + b$, a and b in A, i.e. $f(v) = a$, $q(v) = b$.*

Proof. By Lemma 3, the product in $Q(A)$ is well defined. The associative law is easily seen as follows; $([U, f, q][U', f', q'])[U'', f'', q''] = [U \otimes_A U' \otimes_A U'', f \otimes f' \otimes f'', f^2 \overline{\otimes} f'^2 \overline{\otimes} q'' + f^2 \overline{\otimes} q' \overline{\otimes} f''^2 + q \overline{\otimes} f'^2 \overline{\otimes} f''^2 + 2(q \otimes q' \overline{\otimes} f''^2 + q \overline{\otimes} f'^2 \otimes q'' + f^2 \overline{\otimes} q' \otimes q'') + 4q \otimes q' \otimes q''] = [U, f, q] ([U', f', q'][U'', f'', q''])$.[4]

DEFINITION. Let P be a finitely generated projective and faithful A-module, $f: P \to A$ an A-homomorphism and $q: P \to A$ a quadratic form. For the A-algebra $(P, f, q) = T(P)/(x \otimes x - f(x)x - q(x); x \in P)$, we consider a symmetric bilinear form $D_{f,q}: P \times P \to A$ defined by $D_{f,q}(x, y) = f(x)f(y) + 2B_q(x, y)$ for x, y in P, where $B_q(x, y) = q(x + y) - q(x) - q(y)$ for x, y in P. Then we shall call the bilinear A-module $(P, D_{f,q})$ the *discriminant* of (P, f, q).

Remark 1. If 2 is inversible in A, then we have that (P, f, q) is a separable algebra over A if and only if $(P, D_{f,q})$ is a non-degenerate bilinear A-module, i.e. $P \to \operatorname{Hom}_A(P, A)$; $x \rightsquigarrow D_{f,q}(x, -)$ is an isomorphism.

[4] $(q \overline{\otimes} f^2) \otimes q' = q \otimes (f^2 \overline{\otimes} q')$, $(f^2 \overline{\otimes} q') \otimes q'' = f^2 \overline{\otimes} (q' \otimes q'')$.

Proof. $d = f^2 + 4q$ is a quadratic form of P to A, and satisfies $d(x) = f(x)^2 + 2B_q(x, x) = D_{f,q}(x, x)$. In the tensor algebra $T(P)$, we put $P' = \{x - (1/2)f(x) \in A \oplus P \subset T(P)\,;\, x \in P\}$, then the map $P \to P'\,;\, x \rightsquigarrow x - (1/2)f(x)$ is an A-isomorphism. We denote by h the inverse isomorphism of it. For the ideal of $T(P)$ generated by the set $\{x \otimes x - f(x)x - q(x)\,;\, x \in P\} = \{x \otimes x - d(h(x/2))\,;\, x \in P'\}$, we have $(P, f, q) = T(P)/(x \otimes x - f(x)x - q(x)\,;\, x \in P) = T(P')/(x \otimes x - d(h(x)/2))\,;\, x \in P') = (P', 0, d \circ (1/2)h)$, since $T(P) = T(P')$. But, $(P', 0, d\circ(1/2)h)$ is a Clifford algebra $\mathrm{Cl}(P', d\circ(1/2)h)$ of a quadratic module $(P', d \circ (1/2)h)$. It is known that $\mathrm{Cl}\,(P', d \circ (1/2)h)$ is a separable algebra over A if and only if $(P', d \circ (1/2)h)$ is non-degenerated. Since (P, d) and $(P', d \circ (1/2)h)$ are isometric, we get this remark.

THEOREM 1. *Let U be a finitely generated projective A-module of rank* 1, *$f: U \to A$ an A-homomorphism, $q: U \to A$ a quadratic form, and (U, f, q) the quadratic extension of A. Then the following conditions are equivalent*:

1) *(U, f, q) is a separable algebra over A.*
2) *$(U, D_{f,q})$ is a non-degenerate bilinear A-module.*
3) $[U, f, q]^2 = [A, 1, 0]$.

Proof. 1) $\rightleftarrows$ 2): To prove the equivalence of the conditions 1) and 2), we may assume that A is a local ring. Let A be the local ring. Then $U = Au$ and $(U, f, q) \approx A[X]/(X^2 - aX - b)$, where $a = f(u)$, $b = q(u)$. Hence, (U, f, q) is separable over A if and only if $a^2 + 4b = f^2 + 4q(u) = D_{f,q}(u, u)$ is inversible in A. On the other hand, $(U, D_{f,q})$ is non-degenerated if and only if $D_{f,q}(u, u)$ is inversible in A. Therefore, we obtain the equivalence.

2) $\to$ 3): Assume that $(U, D_{f,q})$ is non-degenerate. Then the A-isomorphism $U \to \mathrm{Hom}_A(U, A)\,;\, x \rightsquigarrow D_{f,q}(x, -)$ induces an A-isomorphism $D_{f,q}: U \otimes_A U \to A\,;\, x \otimes y \rightsquigarrow D_{f,q}(x, y)$. Put $\sigma_1 = D_{f,q}$ and $g = -B_q$. Then we have $I \circ \sigma_1 = D_{f,q} = f \otimes f + 2B_q = f \otimes f - 2g$. Furthermore, we can prove the following identity:

$$(f \otimes f)g + (f^2 \mathbin{\overline{\otimes}} q + q \mathbin{\overline{\otimes}} f^2 + 2q \otimes q) - g^2 = 0\,.$$

Because, by the localizations of A and U by every maximal ideal $\mathfrak{m}$ of A, we can check that quadratic forms $f^2 \mathbin{\overline{\otimes}} q + q \mathbin{\overline{\otimes}} f^2 - B_q \cdot f \otimes f : U \otimes_A U \to A$, and $2q \otimes q - B_q^2: U \otimes_A U \to A$ are equal to 0. Thus, by Proposition 1 we get $[U, f, q]^2 = [U \otimes_A U,\, f \otimes f,\, f^2 \mathbin{\overline{\otimes}} q + q \mathbin{\overline{\otimes}} f^2 + 2q \otimes q] =$

$[A, 1, 0]$.

$3) \to 2)$: Let $[U, f, q]^2 = [A, 1, 0]$. To prove the condition 2) it is sufficient to show that for any maximal ideal $\mathfrak{m}$ of A, $D_{f,q}(u, u)$ is inversible in $A_{\mathfrak{m}}$, where $U_{\mathfrak{m}} = A_{\mathfrak{m}}u$. Now, we assume A is a local ring with maximal ideal $\mathfrak{m}$ and $U = Au$. We shall show $D_{f,q}(u, u) = f(u)^2 + 2B_q(u, u) = f(u)^2 + 4q(u) \notin \mathfrak{m}$. From $[U, f, q]^2 = [A, 1, 0]$, there exist an A-homomorphism $g: U \otimes_A U \to A$ and an A-isomorphism $\sigma_1: U \otimes_A U \to A$ such that $\sigma_1(x \otimes y) = f(x)f(y) - 2g(x \otimes y)$ and $0 = f(x)f(y)g(x \otimes y) + f(x)^2 q(y) + q(x)f(y)^2 + 4q(x)q(y) + g(x \otimes y)^2$ for all $x \otimes y \in U \otimes_A U$. Especially, taking $x = y = u$, we get

$$\sigma_1(u \otimes u) = f(u)^2 - 2g(u \otimes u) \qquad (3),$$

and

$$0 = f(u)^2 g(u \otimes u) + 2f(u)^2 q(u) + 4q(u)^2 - g(u \otimes u) \qquad (4).$$

Eliminating $f(u)^2$ from (3) and (4), we get $(\sigma_1(u \otimes u) + 2g(u \otimes u))g(u \otimes u) + 2(\sigma_1(u \otimes u) + 2g(u \otimes u))q(u) + 4q(u)^2 - g(u \otimes u)^2 = 0$, and so

$$(\sigma_1(u \otimes u) + g(u \otimes u) + 2q(u))(g(u \otimes u) + 2q(u)) = 0 .$$

If $g(u \otimes u) + 2q(u)$ is contained in $\mathfrak{m}$, then from $\sigma_1(u \otimes u) \notin \mathfrak{m}$, $\sigma_1(u \otimes u) + g(u \otimes u) + 2q(u)$ is inversible in A. Therefore, we have $g(u \otimes u) + 2q(u) = 0$, and $D_{f,q}(u, u) = f(u)^2 + 4q(u) = f(u)^2 - 2g(u \otimes u) = \sigma_1(u \otimes u)$ is inversible in A. If $g(u \otimes u) + 2q(u) \notin \mathfrak{m}$, then $\sigma_1(u \otimes u) + g(u \otimes u) + 2q(u) = 0$. From (3) and $2\sigma_1(u \otimes u) + 2g(u \otimes u) + 4q(u) = 0$, we get $\sigma_1(u \otimes u) + f(u)^2 + 4q(u) = 0$, accordingly, $D_{f,q}(u, u) = f(u)^2 + 4q(u) = -\sigma_1(u \otimes u)$ is inversible in A.

COROLLARY 1. *The set $Q_s(A)$ of A-algebra-isomorphism classes of the separable quadratic extensions of A forms an abelian group with exponent* 2.

PROPOSITION 3. *Let (U, f, q) be a quadratic extension of A. The map $\tau_f: (U, f, q) \to (U, f, q)\,;\, a + x \rightsquigarrow a + f(x) - x$ is an A-algebra-isomorphism such that $\tau_f^2 = I$. If (U, f, q) and (U', f', q') are quadratic extensions of A and $\sigma: (U, f, q) \to (U', f', q')$ is an A-algebra-isomorphim, then we have the following commutative diagram;*

$$\begin{array}{ccc} (U, f, q) & \xrightarrow{\ \sigma\ } & (U', f', q') \\ \downarrow{\scriptstyle \tau_f} & \circlearrowleft & \downarrow{\scriptstyle \tau'_{f'}} \\ (U, f, q) & \xrightarrow{\ \sigma\ } & (U', f', q') \,. \end{array}$$

Proof. From Proposition 1, there exist g in $\mathrm{Hom}_A(U, A)$ and A-isomorphism $\sigma_1 : U \to U'$ such that $\sigma(x) = g(x) + \sigma_1(x)$ and $f'(\sigma_1(x)) = f(x) - 2g(x)$ for all x in U. Therefore, $\tau'_{f'}(\sigma(x)) = g(x) + \tau'_{f'}(\sigma_1(x)) = g(x) + f'(\sigma_1(x)) - \sigma_1(x) = g(x) + f(x) - 2g(x) - \sigma_1(x) = f(x) - (g(x) + \sigma_1(x)) = f(x) - \sigma(x) = \sigma(f(x) - x) = \sigma(\tau_f(x))$, for all x in U.

Remark 2.

1) In Proposition 3, if we take $\sigma = I$, then $\tau_f = \tau'_{f'}$.

2) If (U, f, q) is a separable algebra over A, then τ_f is the unique A-algebra-automorphism of (U, f, q) which is not the identity.

Let $B = (U, f, q)$ and $B' = (U', f', q')$ be separable quadratic extensions of A. Then $G = \{\tau_f, I\}$ and $G' = \{\tau'_{f'}, I\}$ are the groups of automorphisms of B over A and B' over A, respectively. In [1], [3] and [4], the product $B * B'$ of quadratic extensions B and B' was defined as the fixed subalgebra $(B \otimes_A B')^{\tau_f \otimes \tau'_{f'}} = \{x \in B \otimes_A B' ; \tau_f \otimes \tau'_f(x) = x\}$ of $B \otimes_A B'$ by $\tau_f \otimes \tau'_{f'}$. But this product coincides with our one.

PROPOSITION 4. *Let (U, f, q) and (U', f', q') be separable quadratic extensions of A. Then we have $[(U, f, q) \otimes_A (U', f', q')^{\tau_f \otimes \tau'_{f'}}] = [U, f, q] \cdot [U', f', q']$ in $Q_s(A)$.*

Proof. For $B = (U, f, q)$ and $B' = (U', f', q')$, $B \otimes_A B'$ is expressed as a direct sum $B \otimes_A B' = A \oplus U \oplus U' \oplus U \otimes_A U'$. Put $V = \{\sum_i f(x_i)y_i + f'(y_i)x_i - 2x_i \otimes y_i \in U \oplus U' \oplus U \otimes_A U' ;$ for all $\sum_i x_i \otimes y_i$ in $U \otimes_A U'\}$. Then V is an A-submodule of $B \otimes_A B'$, which is A-isomorphic to $U \otimes_A U'$ by the isomorphism $\theta : U \otimes_A U' \to V ; x \otimes y \rightsquigarrow f(x)y + f'(y)x - 2x \otimes y$. It is easily seen that the A-submodule $C = A \oplus V$ of $B \otimes_A B'$ generated by V and A is contained in $B \otimes_A B'^{\tau_f \otimes \tau'_{f'}}$. To show $C = B \otimes_A B'^{\tau_f \otimes \tau'_{f'}}$, we shall prove first that the map $\theta' : (U \otimes_A U', f \otimes f', f^2 \overline{\otimes} q' + q \overline{\otimes} f'^2 + 2q \otimes q') = A \oplus U \otimes_A U' \to C = A \oplus V ; a + x \otimes y \rightsquigarrow a + \theta(x \otimes y)$ is an A-algebra-isomorphism. We can easily compute that for any $x \otimes y$ in $U \otimes_A U'$, $\theta'(x \otimes y)^2 = (f(x)y + f'(y)x - 2x \otimes y)^2 = f(x)^2y^2 + f'(y)^2x^2 + 4x^2 \otimes y^2 + 2f(x)f'(y)x \otimes y - 4f(x)x \otimes y^2 - 4f'(y)x^2 \otimes y = f(x)^2(f'(y)y + q'(y)) + f'(y)^2(f(x)x + q(x)) + 4(f(x) + q(x)) \otimes (f'(y)y + q'(y)) + 2f(x)f'(y)$

$x \otimes y - 4f(x)x \otimes (f'(y)y + q'(y)) - 4f'(y)(f(x)x + q(x)) \otimes y = f(x)f'(y)(f(x)y + f'(y)x - 2x \otimes y) + f(x)^2q'(y) + f'(y)^2q(x) + 4q(x)q'(y) = f \otimes f'(x \otimes y)\theta'(x \otimes y) + (f^2 \overline{\otimes} q' + q \overline{\otimes} f'^2 + 2q \otimes q')(x \otimes y) = \theta'[(f \otimes f'(x \otimes y)x \otimes y + (f^2 \overline{\otimes} q' + q \overline{\otimes} f'^2 + q \otimes q')(x \otimes y)] = \theta'((x \otimes y)^2)$, and for $x_i \otimes y_i$, $x_j \otimes y_j$ in $U \otimes_A U'$, $2\theta'(x_i \otimes y_i) \cdot \theta'(x_j \otimes y_j) = 2(f(x_i)y_i + f'(y_i)x_i - 2x_i \otimes y_i)(f(x_j)y_j + f'(y_j)x_j - 2x_j \otimes y_j) = f(x_i)f'(y_i)(f(x_j)y_j + f'(y_j)x_j - 2x_j \otimes y_j) + f(x_j)f'(y_j)(f(x_i)y_i + f'(y_i)x_i - 2x_i \otimes y_i) + f(x_i)f(x_j)B_{q'}(y_i, y_j) + f'(y_i)f'(y_j)B_q(x_i, x_j) + 2B_q(x_i, x_j)B_{q'}(y_i, y_j) = \theta'(f(x_i)f'(y_i)x_j \otimes y_j + f(x_j)f'(y_j)x_i \otimes y_i) + f(x_i)f(x_j)B_{q'}(y_i, y_j) + f'(y_i)f'(y_j)B_q(x_i, x_j) + 2B_q(x_i, x_j)B_{q'}(y_i, y_j)$.

Therefore, we have $\theta'(\sum_i x_i \otimes y_i)^2 = \theta'((\sum_i x_i \otimes y_i)^2)$ for any $\sum_i x_i \otimes y_i$ in $U \otimes_A U'$. Accordingly, θ' is an A-algebra isomorphism. Thus, C is also a separable algebra over A. Since $B \otimes_A B'$ is a finitely generated projective A-module, $B \otimes_A B'$ is also finitely generated projective over C. Therefore, C is a direct summand of $B \otimes_A B'$, and hence also a direct summand of $B \otimes_A B'^{\tau_f \otimes \tau'_{f'}}$ as C-module. But, $\operatorname{rank}(C: A) = \operatorname{rank}(B \otimes_A B'^{\tau_f \otimes \tau'_{f'}}: A) = 2$, hence we have $B \otimes_A B'^{\tau_f \otimes \tau'_{f'}} = C = A \oplus V \approx (U \otimes_A U', f \otimes f', f^2 \overline{\otimes} q' + q \overline{\otimes} f'^2 + 2q \otimes q')$ as A-algebra.

2. Extended quadratic module.

In this section, we give a generalization of quadratic module. Let A be an arbitrary commutative ring with unit element. Let M be an A-module, $f: M \to A$ an A-homomorphism, and $q: M \to A$ a quadratic form. Then, we call the triple $\langle M, f, q\rangle$ an *extended quadratic module*

DEFINITION. Let $\langle M, f, q\rangle$ and $\langle M', f', q'\rangle$ be extended quadratic modules. If there exist an A-isomorphism $\sigma: M \to M'$ and A-homomorphism $g: M \to A$ satisfying $q' \circ \sigma = q + 2fg - 2g^2$ and $f' \circ \sigma = f - 2g$, then we call that $\langle M, f, q\rangle$ and $\langle M', f', q'\rangle$ are A-isomorphic, and denote by $(\sigma, g): \langle M, f, q\rangle \to \langle M', f', q'\rangle$ the A-isomorphism of extended quadratic modules, or simply $\langle M, f, q\rangle \approx \langle M', f', q'\rangle$.

Then we have easily

1) $(I, 0)$ is identity,
2) $(\sigma', g')(\sigma, g) = (\sigma' \circ \sigma, g + g' \circ \sigma)$ and
3) $(\sigma, g)^{-1} = (\sigma^{-1}, -g \circ \sigma^{-1})$.

Thus, we can consider a category Qua* (A) in which objects are extended quadratic modules and morphisms are A-isomorphisms of extended quadratic modules. Then, Qua* (A) includes the category Qua (A) of the ordinaly quadratic modules as a sub-category. Because, $(\sigma, g): \langle M, 0, q\rangle \to \langle M', 0, q'\rangle$ is an A-isomorphism in Qua* (A) if and only if $\sigma: (M, q) \to (M', q')$ is an A-isomorphism in Qua (A), therefore we may regard as $\langle M, 0, q\rangle = (M, q)$ and $(\sigma, 0) = \sigma$ in Qua (A).

DEFINITION. Let $\langle M, f, q\rangle$ be an extended quadratic module, and let $B_{f,q}: M \times M \to A$ be a symmetric bilinear form defined by $B_{f,q}(x, y) = f(x)f(y) + B_q(x, y)$ for x and y in M. Then, we call the bilinear module $(M, B_{f,q})$ the associated bilinear module with $\langle M, f, q\rangle$. If $(M, B_{f,q})$ is a non-degenerate bilinear module, then $\langle M, f, q\rangle$ is called a *non-degenerate extended quadratic module.*

LEMMA 4. *If* $(\sigma, g): \langle M, f, q\rangle \to \langle M', f', q'\rangle$ *is an A-isomorphism in* Qua* (A), *then we have* $B_{f',q'}(\sigma(x), \sigma(y)) = B_{f,q}(x, y)$ *for all x and y in M, that is,* $\sigma: (M, B_{f,q}) \to (M', B_{f',q'})$ *is an A-isomorphism of bilinear modules.*

Proof. Since the A-isomorphism $\sigma: M \to M'$ and the A-homomorphism $g: M \to A$ satisfy $f' \circ \sigma = f - 2g$ and $q' \circ \sigma = q + 2fg - 2g^2$, we have $B_{f',q'}(\sigma(x), \sigma(y)) = f'(\sigma(x))f'(\sigma(y)) + B_{q'}(\sigma(x), \sigma(y)) = (f(x) - 2g(x))(f(y) - 2g(y)) + B_q(x, y) + 2(f(x)g(y) + f(y)g(x)) - 4g(x)g(y) = f(x)f(y) + B_q(x, y) = B_{f,q}(x, y)$.

COROLLARY 2. *If* $\langle M, f, q\rangle \approx \langle M', f', q'\rangle$ *and* $\langle M, f, q\rangle$ *is non-degenerate, then* $\langle M', f', q'\rangle$ *is also non-degenerate.*

DEFINITION. Let $\langle M_1, f_1, q_1\rangle$ and $\langle M_2, f_2, q_2\rangle$ be extended quadratic modules. We define the orthogonal sum $\perp$ and the tensor product $\otimes$ of extended quadratic modules as follows:

$$\langle M_1, f_1, q_1\rangle \perp \langle M_2, f_2, q_2\rangle = \langle M_1 \oplus M_2, f_1 \perp f_2, q_1 \perp q_2 - f_1 \times f_2\rangle \qquad (5),$$

$$\langle M_1, f_1, q_1\rangle \otimes \langle M_2, f_2, q_2\rangle = \langle M_1 \otimes M_2, f_1 \otimes f_2, f_1^2 \overline{\otimes}\, q_2 + q_1 \overline{\otimes}\, f_2^2 + q_1 \otimes q_2\rangle \qquad (6),$$

where $f_1 \perp f_2$ is defined by the A-homomorphism $M_1 \oplus M_2 \to A$; $x_1 \oplus x_2 \rightsquigarrow f_1(x_1) + f_2(x_2)$, and $f_1 \times f_2$ the quadratic form $M_1 \oplus M_2 \to A$; $x_1 \oplus x_2 \rightsquigarrow f_1(x_1) \cdot f_2(x_2)$.

LEMMA 5. *Let* $\langle M_i, f_i, q_i\rangle$ *and* $\langle M'_i, f'_i, q'_i\rangle$ *be extended quadratic modules, and* $(\sigma_i, g_i): \langle M_i, f_i, q_i\rangle \to \langle M'_i, f'_i, q'_i\rangle$ *an A-isomorphism in* Qua* (A) for $i = 1, 2$. *Then we have the following A-isomorphisms in* Qua* (A);

$$(\sigma_1 \oplus \sigma_2, g_1 \perp g_2): \langle M_1, f_1, q_1\rangle \perp \langle M_2, f_2, q_2\rangle \to \langle M_1', f_1', q_1'\rangle \perp \langle M_2', f_2', q_2'\rangle \quad (7),$$

$$(\sigma_1 \otimes \sigma_2, f_1 \otimes g_2 + g_1 \otimes f_2 - 2g_1 \otimes g_2): \langle M_1, f_1, q_1\rangle \otimes \langle M_2, f_2, q_2\rangle \to \langle M_1', f_1', q_1'\rangle \otimes \langle M_2', f_2', q_2'\rangle \quad (8).$$

Proof. The proof of (7). We shall show that $(\sigma_1 \oplus \sigma_2, g_1 \perp g_2)$: $\langle M_1 \oplus M_2, f_1 \perp f_2, q_1 \perp q_2 - f_1 \times f_2\rangle \to \langle M_1' \oplus M_2', f_1' \perp f_2', q_1' \perp q_2' - f_1' \times f_2'\rangle$ is an A-isomorphism in Qua* (A). For $x_1 \oplus x_2$ in $M_1 \oplus M_2$, we have

$(q_1' \perp q_2' - f_1' \times f_2') \circ (\sigma_1 \oplus \sigma_2)(x_1 \oplus x_2) = q_1'(\sigma_1(x_1)) + q_2'(\sigma_2(x_2)) - f_1'(\sigma_1(x_1)) f_2'(\sigma_2(x_2))$ $= q_1(x_1) + 2f_1(x_1)g_1(x_1) - 2g_1(x_1)^2 + q_2(x_2) + 2f_2(x_2)g_2(x_2) - 2g_2(x_2)^2 - (f_1(x_1) - 2g_1(x_1))(f_2(x_2) - 2g_2(x_2)) = (q_1 \perp q_2 - f_1 \times f_2)(x_1 \oplus x_2) + 2(f_1 \perp f_2)(g_1 \perp g_2)(x_1 \oplus x_2) - 2(g_1 \perp g_2)^2(x_1 \oplus x_2)$, and

$$(f_1' \perp f_2') \circ (\sigma_1 \oplus \sigma_2) = f_1' \circ \sigma_1 \perp f_2' \circ \sigma_2 = (f_1 - 2g_1) \perp (f_2 - 2g_2) = (f_1 \perp f_2) - 2(g_1 \perp g_2).$$

The proof of (8) is obtained by similar computations the proof of Lemma 3. We omit this proof.

DEFINITION. We denote by $B_{f,q} \perp B_{f',q'}$ the associated bilinear form with $\langle M, f, q\rangle \perp \langle M', f', q'\rangle$, and by $B_{f,q} \otimes B_{f',q'}$ the associated bilinear form with $\langle M, f, q\rangle \otimes \langle M', f', q'\rangle$, that is, $B_{f,q} \perp B_{f',q'} = B_{f\perp f',(q\perp q')-(f\times f')}$ and $B_{f,q} \otimes B_{f',q'} = B_{f\otimes f', f^2 \bar{\otimes} q' + q \bar{\otimes} f'^2 + q\otimes q'}$.

PROPOSITION 5. *The orthogonal sum and the tensor product of extended quadratic modules $\langle M, f, q\rangle$ and $\langle M', f', q'\rangle$ induce the following identities between the associated bilinear modules with them;*

$$(M \oplus M', B_{f,q} \perp B_{f',q'}) = (M, B_{f,q}) \perp (M', B_{f',q'}) \quad (9),$$

i.e. $B_{f,q} \perp B_{f',q'}(x \oplus x', y \oplus y') = B_{f,q}(x, y) + B_{f',q'}(x', y')$ *for* $x \oplus x'$ *and* $y \oplus y'$ *in* $M \oplus M'$, *and*

$$(M \otimes M', B_{f,q} \otimes B_{f',q'}) = (M, B_{f,q}) \otimes (M', B_{f',q'}) \quad (10),$$

i.e. $B_{f,q} \otimes B_{f',q'}(\sum_i x_i \otimes x_i', \sum_j y_j \otimes y_j') = \sum_{i,j} B_{f,q}(x_i, y_j) \cdot B_{f',q'}(x_i', y_j')$ *for* $\sum_i x_i \otimes x_i'$ *and* $\sum_j y_j \otimes y_j'$ *in* $M \otimes M'$.

Proof. The proof of (9): $B_{f,q} \perp B_{f',q'}(x \oplus x', y \oplus y') = (f \perp f'(x \otimes x'))(f \perp f'(y \oplus y')) + B_{(q\perp q')-(f\times f')}(x \oplus x', y \oplus y') = (f(x) + f'(x'))(f(y) + f'(y')) + B_{q\perp q'}(x \oplus x', y \oplus y') - B_{f\times f'}(x \oplus x', y \oplus y') = f(x)f(y) + f'(x')f'(y') + f'(x')f(y) + f(x)f'(y') + B_q(x, y) + B_{q'}(x', y') - (f(x)f'(y') + f(y)f'(x')) = B_{f,q}(x, y) + B_{f',q'}(x', y')$, for any $x \oplus x'$ and $y \oplus y'$ in $M \oplus M'$.

The proof of (10): $B_{f,q}\otimes B_{f'q'}(\sum_i x_i\otimes x_i', \sum_j y_j\otimes y_j')=B_{f\otimes f', f^2\overline{\otimes}q'+q\overline{\otimes}f'^2+q\otimes q'}(\sum_i x_i\otimes x_i', \sum_j y_j\otimes y_j')=f\otimes f'(\sum_i x_i\otimes x_i')f\otimes f'(\sum_j y_j\otimes y_j')+B_{f^2\overline{\otimes}q'+q\overline{\otimes}f'^2+q\otimes q'}(\sum_i x_i\otimes x_i', \sum_j y_j\otimes y_j') = (\sum f(x_i)f'(x_i'))(\sum f(y_j)f'(y_j')) + B_{f^2\overline{\otimes}q'}(\sum x_i\otimes x_i', \sum y_j\otimes y_j') + B_{q\overline{\otimes}f'^2}(\sum x_i\otimes x_i', \sum y_j\otimes y_j') + B_{q\otimes q'}(\sum x_i\otimes x_i', \sum y_j\otimes y_j') = \sum_{i,j}(f(x_i)f(y_j)f'(x_i')f'(y_j') + f(x_i)f(y_j)B_{q'}(x_i', y_j') + B_q(x_i, y_j)f'(x_i')f'(y_j') + B_q(x_i, y_j)B_{q'}(x_i', y_j')) = \sum_{i,j}B_{f,q}(x_i, y_j)B_{f',q'}(x_i', y_j')$, for $\sum x_i\otimes x_i'$ and $\sum y_j\otimes y_j'$ in $M\otimes M'$.

COROLLARY 3. *If $\langle M,f,q\rangle$ and $\langle M',f',q'\rangle$ are non-degenerate extended quadratic modules, then $\langle M,f,q\rangle\perp\langle M',f',q'\rangle$ is also non-degenerate. Furthermore, if M and M' are finitely generated projective A-modules, then $\langle M,f,q\rangle\otimes\langle M',f',q'\rangle$ is non-degenerate.*

Remark 3. If 2 is inversible in the ring A, then the category Qua* (A) is equivalent to the category Qua (A), i.e. for any object $\langle M,f,q\rangle$ in Qua* (A), $\langle M,f,q\rangle\approx\langle M,0,q+(1/2)f^2\rangle$.

Remark 4. Let $\langle M_1,f_1,q_1\rangle$, $\langle M_2,f_2,q_2\rangle$ and $\langle M_3,f_3,q_3\rangle$ be extended quadratic modules. Then we get the following natural isomorphisms in Qua* (A);

1) $\langle M_1,f_1,q_1\rangle\perp\langle M_2,f_2,q_2\rangle\approx\langle M_2,f_2,q_2\rangle\perp\langle M_1,f_1,q_1\rangle$,

2) $\langle M_1,f_1,q_1\rangle\otimes\langle M_2,f_2,q_2\rangle\approx\langle M_2,f_2,q_2\rangle\otimes\langle M_1,f_1,q_1\rangle$,

3) $(\langle M_1,f_1,q_1\rangle\perp\langle M_2,f_2,q_2\rangle)\perp\langle M_3,f_3,q_3\rangle\approx\langle M_1,f_1,q_1\rangle\perp(\langle M_2,f_2,q_2\rangle\perp\langle M_3,f_3,q_3\rangle)$,

4) $(\langle M_1,f_1,q_1\rangle\otimes\langle M_2,f_2,q_2\rangle)\otimes\langle M_3,f_3,q_3\rangle\approx\langle M_1,f_1,q_1\rangle\otimes(\langle M_2,f_2,q_2\rangle\otimes\langle M_3,f_3,q_3\rangle)$,

5) $(\langle M_1,f_1,q_1\rangle\perp\langle M_2,f_2,q_2\rangle)\otimes\langle M_3,f_3,q_3\rangle\approx(\langle M_1,f_1,q_1\rangle\otimes\langle M_3,f_3,q_3\rangle)\perp(\langle M_2,f_2,q_2\rangle\otimes\langle M_3,f_3,q_3\rangle)$,

6) $\langle M_1,f_1,q_1\rangle\otimes\langle A,I,0\rangle\approx\langle M_1,f_1,q_1\rangle$.

Proof. We shall show only 5). For the other isomorphisms, we can see easily. To prove it, it is enough to show the identity

$(f_1\perp f_2)^2\overline{\otimes}q_3+(q_1\perp q_2-f_1\times f_2)\overline{\otimes}f_3^2+(q_1\perp q_2-f_1\times f_2)\overline{\otimes}q_3=(f_1^2\overline{\otimes}q_3+q_1\overline{\otimes}f_3^2+q_1\otimes q_3)\perp(f_2^2\overline{\otimes}q_3+q_2\overline{\otimes}f_3^2+q_2\otimes q_3)-(f_1\otimes f_3)\times(f_2\otimes f_3)$. For any $\sum_i(x_i+y_i)\otimes z_i$ in $(M_1\oplus M_2)\otimes M_3$, $(f_1\perp f_2)^2\overline{\otimes}q_3+(q_1\perp q_2-f_1\times f_2)\overline{\otimes}f_3^2+(q_1\perp q_2-f_1\times f_2)\otimes q_3(\sum_i(x_i\oplus y_i)\otimes z_i)=\sum_i[(f_1(x_i)+f_2(y_i))^2q_3(z_i)+(q_1(x_i)+q_2(y_i)-f_1(x_i)f_2(y_i))f_3(z_i)^2+2(q_1(x_i)+q_2(y_i)-f_1(x_i)f_2(y_i))q_3(z_i)]+\sum_{i<j}[(f_1(x_i)+f_2(y_i))(f_1(x_j)+f_2(y_j))B_{q_3}(z_i,z_j)+(B_{q_1}(x_i,x_j)+B_{q_2}(y_i,y_j)-f_1(x_i)f_2(y_j)-f_1(x_j)f_2(y_i))f_3(z_i)f_3(z_j)+(B_{q_1}(x_i,x_j)+B_{q_2}(y_i,y_j)-f_1(x_i)f_2(y_j)$

$- f_1(x_j)f_2(y_i))B_{q_3}(z_i, z_j)] = \sum_i [(f_1(x_i)^2 q_3(z_i) + q_1(x_i) f_3(z_i)^2 + 2q_1(x_i)q_3(z_i)) + (f_2(y_i)^2 q_3(z_i) q_2(y_i) f_3(z_i)^2 + 2q_2(y_i)q_3(z_i)) - f_1(x_i)f_3(z_i)f_2(y_i)f_3(z_i)] + \sum_{i<j} [(f_1(x_i) f_1(x_j)B_{q_3}(z_i, z_j) + B_{q_1}(x_i, x_j)f_3(z_i)f_3(z_j) + B_{q_1}(x_i, x_j)B_{q_3}(z_i, z_j)) + (f_2(y_i)f_2(y_j) B_{q_3}(z_i, z_j) + B_{q_2}(y_i, y_j)f_3(z_i)f_3(z_j) + B_{q_2}(y_i, y_j)B_{q_3}(z_i, z_j)) - (f_1(x_i)f_3(z_i)f_2(y_j) f_3(z_j) + f_1(x_j)f_3(z_j)f_2(y_i)f_3(z_i))] = (f_1^2 \overline{\otimes} q_3 + q_1 \overline{\otimes} f_3^2 + q_1 \otimes q_3) \perp (f_2^2 \overline{\otimes} q_3 + q_2 \overline{\otimes} f_3^2 + q_2 \otimes q_3) - (f_1 \otimes f_3) \times (f_2 \otimes f_3)(\sum_i (x_i \oplus y_i) \otimes z_i).$

DEFINITION. An extended quadratic module $\langle M, f, q\rangle$ is called *hyperbolic* if the associated bilinear module $(M, B_{f,q})$ with $\langle M, f, q\rangle$ is hyperbolic, i.e. there exists an A-module N such that $M = N \oplus N'$ for some A-submodule N', $f(N) = q(N) = 0$ and $N = N^{\perp}(= \{x \in M ; B_{f,q}(x, N) = 0\})$.

From Proposition 5 and the well known properties on bilinear modules, we get the following proposition.

PROPOSITION 6.

1) *If $\langle M, f, q\rangle$ and $\langle M', f', q'\rangle$ are hyperbolic, then so is also $\langle M, f, q\rangle \perp \langle M', f', q'\rangle$.*

2) *If M is a finitely generated projective A-module and $\langle M, f, q\rangle$ is hyperbolic then $\langle M, f, q\rangle$ is non-degenerate.*

3) *If $\langle M, f, q\rangle$ and $\langle M', f', q'\rangle$ are non-degenerate and $\langle M, f, q\rangle$ is hyperbolic, then $\langle M, f, q\rangle \otimes \langle M', f', q'\rangle$ is also hyperbolic.*

3. Extended Witt ring $W^*(A)$.

From the argument in § 2, we can construct a commutative ring $W^*(A)$. Let $\mathrm{Qua}_p^*(A)$ be a full subcategory of $\mathrm{Qua}^*(A)$ consisting of non-degenerate extended quadratic modules with finitely generated projective modules. In the category $\mathrm{Qua}_p^*(A)$, as well as the construction of the Witt ring $W(A)$, we consider the full subcategory $\mathrm{HQua}_P^*(A)$ consisting of hyperbolic extended quadratic modules. And, using the notation of K-theory in [1], we define the extended Witt ring $W^*(A)$ by $W^*(A) = \mathrm{Coker}(K_0(\mathrm{HQua}_p^*(A)) \to K_0(\mathrm{Qua}_p^*(A)))$. Thus, it can be easily checked that $W^*(A)$ is a commutative ring with sum and product induced by orthogonal sum $\perp$ and tensor product $\otimes$. We denote by $[\langle P, f, q\rangle]$ the class of $\langle P, f, q\rangle$ in $W^*(A)$.

THEOREM 2. *The extended Witt ring $W^*(A)$ has always the identity element $[\langle A, I, 0\rangle]$, and there exists a ring homomorphism of the Witt ring $W(A)$ to $W^*(A)$. Then, the image of $W(A)$ becomes an ideal of*

$W^*(A)$. *If* 2 *is inversible in* A, *then it is an isomorphism*; $W(A) \xrightarrow[\approx]{} W^*(A)$.

Proof. Let $\mathrm{Qua}_p(A)$ be the full subcategory of $\mathrm{Qua}(A)$ consisting of non-degenerate quadratic modules (P, q) with finitely generated projective A-module P, and $\mathrm{HQua}_p(A)$ the full subcategory of $\mathrm{Qua}_p(A)$ whose objects are hyperbolic in $\mathrm{Qua}_p(A)$. Consider the functor $\Phi\colon \mathrm{Qua}_p(A) \to \mathrm{Qua}_p{}^*(A)$; $(P, q) \rightsquigarrow \langle P, 0, q\rangle$, then we have the following commutative diagram

$$\begin{array}{ccccccc} K_0(\mathrm{HQua}_p{}^*(A)) & \to & K_0(\mathrm{Qua}_p{}^*(A)) & \to & W^*(A) & \to & 0 \\ \uparrow{\scriptstyle K_0(\Phi)} & & \uparrow{\scriptstyle K_0(\Phi)} & & & & \\ K_0(\mathrm{HQua}_p(A)) & \to & K_0(\mathrm{Qua}_p(A)) & \to & W(A) & \to & 0 \end{array}$$

where two rows are exact.

Thus, the ring homomorphism $K_0(\Phi)$ induces a ring homomorphism $\omega\colon W(A) \to W^*(A)$. Then, $\mathrm{Im}\,\omega$ becomes an ideal of $W^*(A)$, for $[\langle P, f, q\rangle][\langle P', 0, q'\rangle] = [\langle P \otimes P', 0, q \otimes q' + f^2 \overline{\otimes} q'\rangle]$ in $W^*(A)$. If 2 is inversible in A, by Remark 3, $K_0(\Phi)$ is an isomorphism, therefore, so is also $\omega\colon W(A) \xrightarrow[\approx]{} W^*(A)$.

4. The unit group of $W^*(A)$ and $Q_s(A)$.

In this section, we consider a relation between the separable quadratic extension group $Q_s(A)$ and the unit group $U(W^*(A))$ of the extended Witt ring $W^*(A)$.

THEOREM 3. *There exists a group homomorphism of* $Q_s(A)$ *to* $U(W^*(A))$;

$$\Theta\colon Q_s(A) \longrightarrow U(W^*(A))\,;\, [U, f, q] \rightsquigarrow [\langle U, f, 2q\rangle]\,.$$

Proof. Let $[U, f, q]$ be an element in $Q_s(A)$. By Theorem 1, the bilinear module $(U, D_{f,q})$, called the discriminant of $[U, f, q]$, is non-degenerate. Since $D_{f,q}(x, y) = f(x)f(y) + 2B_q(x, y) = f(x)f(y) + B_{2q}(x, y) = B_{f,2q}(x, y)$ for any x and y in U, we have $D_{f,q} = B_{f,2q}$. Therefore, $\langle U, f, 2q\rangle$ is in $\mathrm{Qua}_p{}^*(A)$. Now, we shall show that Θ is well defined: If $[U, f, q] = [U', f', q']$ is in $Q_s(A)$, then there exist an A-isomorphism $\sigma\colon U \to U'$ and an A-homomorphism $g\colon U \to A$ such that $q' \circ \sigma = q + fg - g^2$ and $f' \circ \sigma = f - 2g$. Then, we get $2q' \circ \sigma = 2q + 2fg - 2g^2$ and $f' \circ \sigma =$

$f-2g$, that is, $\langle U,f,2q\rangle \approx \langle U',f',2q'\rangle$ in $\mathrm{Qua}_p{}^*(A)$. Thus, the map $\Theta: Q_s(A) \to W^*(A)\,;\,[U,f,q] \rightsquigarrow [\langle U,f,2q\rangle]$ is well defined. Furthermore, we have

$\Theta([U,f,q][U',f',q']) = \Theta([U\otimes U', f\otimes f', f^2 \overline{\otimes} q' + q\overline{\otimes} f'^2 + 2q\otimes q']) = [\langle U\otimes U', f\otimes f', f^2\overline{\otimes} 2q' + 2q\overline{\otimes} f'^2 + 2q\otimes 2q'\rangle] = [\langle U,f,2q\rangle]\cdot[\langle U',f',2q'\rangle]$, and $\Theta([A,I,0]) = [\langle A,I,0\rangle]$. Accordingly, $\mathrm{Im}\,\Theta$ is contained in $U(W^*(A))$ and $\Theta: Q_s(A) \to U(W^*(A))$ is a group homomorphism.

Remark 5.

1) if K is a field with the characteristic $\neq 2$, then $U(W^*(A)) = U(W(A)) \approx U(K)/U(K)^2$, $Q_s(K) \approx U(K)/U(K)^2$ and Θ is an isomorphism.

2) If K is a field with characteristic 2, then Θ is a zero homomorphism.

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