

SOME CONSEQUENCES OF MARTIN'S AXIOM AND THE NEGATION OF THE CONTINUUM HYPOTHESIS

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§0. W. Sierpinski [3] demonstrated 82 propositions, called C_1 - C_{82} , with the aid of the continuum hypothesis. D. A. Martin and R. M. Solovay remarked in [2] that 48 of these propositions followed from Martin's axiom (MA), 23 were refuted by $MA + 2^{\aleph_0} > \aleph_1$ and three were independent of $MA + 2^{\aleph_0} > \aleph_1$. But the relation of the remaining eight propositions to $MA + 2^{\aleph_0} > \aleph_1$ has been unsettled.

In this paper, we shall show at least five of them ($C_8, C_{13}, C_{61}, C_{62}$ and C_{70}) are also refuted by $MA + 2^{\aleph_0} > \aleph_1$.

The following table gives the relation of C_1 - C_{82} to $MA + 2^{\aleph_0} > \aleph_1$.

	0	1	2	3	4	5	6	7	8	9
		×	○	○	○	○	×	○	×*	×
10	×	×	×	×*	○	×	○	○	○	○
20	○	○	○	○	○	○	×	×	×	×
30	○	○	×	×	×	○	○	○	○	○
40	○	○	○	○	○	○	○	?	?	○
50	×	×	△	○	○	○	○	○	○	○
60	○	×*	×*	○	○	×	×	×	×	×
70	×*	○	○	○	○	○	○	×	△	×
80	?	△	○							

By ○, we denote the propositions following from MA, by × the propositions refuted by $MA + 2^{\aleph_0} > \aleph_1$, by △ the propositions independent of $MA + 2^{\aleph_0} > \aleph_1$ and by ? the propositions whose relation to $MA + 2^{\aleph_0} > \aleph_1$ we do not know about at present.

Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set. A subset X of P is said to be dense in \mathcal{P} if, for every $p \in P$, there is $q \in X$ such that $p \leq q$. If \mathcal{F} is a collection of dense subsets of P , a subset G of P is said to be an \mathcal{F} -generic filter on \mathcal{P} if G has the following properties:

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- (1) if $p, q \in P$, $p \in G$ and $q \leq p$, then $q \in G$;
- (2) if $p, q \in G$, then there is $r \in G$ such that $p \leq r$ and $q \leq r$;
- (3) if $X \in \mathcal{F}$, then $X \cap G \neq \emptyset$.

If $p, q \in P$, then p and q are said to be compatible if there is $r \in P$ such that $p \leq r$ and $q \leq r$. \mathcal{P} is said to have the countable chain condition if every collection of pairwise incompatible elements of P is countable.

Martin's axiom (MA) is the following statement:

If $\mathcal{P} = \langle P, \leq \rangle$ is a partially ordered set having the countable chain condition and \mathcal{F} is a collection of dense open subsets of P of cardinality $< 2^{\aleph_0}$, then there exists an \mathcal{F} -generic filter on \mathcal{P} .

§1. In this section, we shall show C_3, C_9, C_{61} and C_{62} are refuted by $\text{MA} + \neg \text{CH}$. From [2], we quote the following lemma.

LEMMA 1. *Let A and B be collections of subsets of ω , each of cardinality $< 2^{\aleph_0}$, such that if $x \in B$ and K is a finite subset of A then $x - \cup K$ is infinite. If we assume MA, then there exists a subset t of ω such that $x \cap t$ is finite if $x \in A$ and infinite if $x \in B$.*

Let ${}^\omega\omega$ be the set of all functions from ω into ω , (more generally, x_y be the set of all functions from x into y). Following Sierpinski [3], we define a partial ordering $<$ on ${}^\omega\omega$ as follows:

$$f < g \leftrightarrow (\exists k \in \omega)(\forall n \geq k)[f(n) < g(n)].$$

The following lemma is due to K. Kunen [1].

LEMMA 2. *Let F be a subset of ${}^\omega\omega$ of cardinality $< 2^{\aleph_0}$. If we assume MA, then there exists $g \in {}^\omega\omega$ such that if $f \in F$ then $f < g$.*

From Lemma 2, we have the following proposition, which is the negation of C_9 .

PROPOSITION 1 (Assume MA and $2^{\aleph_0} > \aleph_1$). *Let E be an uncountable subset of \mathbf{R} , the set of reals, and $\langle f_n : n \in \omega \rangle$ be a convergent sequence of functions from E to \mathbf{R} . Then there exists an uncountable subset N of E such that $\langle f_n : n \in \omega \rangle$ is uniformly convergent on N .*

Proof. We may assume E is of cardinality \aleph_1 . Let f be the limit of $\langle f_n : n \in \omega \rangle$. Then for any $x \in E$ and $m \in \omega$, there is $k \in \omega$ such that if $n \geq k$ then $|f_n(x) - f(x)| < 1/m + 1$. Take such $k \in \omega$ and denote it by $\varphi_x(m)$. Then we can define \aleph_1 functions φ_x from ω into ω . Using

Lemma 2, we can find $\varphi \in {}^\omega \omega$ such that $\varphi_x < \varphi$ for all $x \in E$. For each $x \in E$, let k_x denote the least $k \in \omega$ such that $\varphi_x(m) < \varphi(m)$ for all $m \geq k$. Since E is uncountable, there is $k \in \omega$ and an uncountable subset N of E such that if $x \in N$ then $k_x = k$. Then for any $x \in N$ and $m \geq k$, if $n \geq \varphi(m)$ then $|f_n(x) - f(x)| < 1/m + 1$. This means $\langle f_n : n \in \omega \rangle$ converges uniformly to f on N .

Since C_δ and C_γ are equivalent, C_δ is also refuted by $MA + 2^{\aleph_0} > \aleph_1$.

Recall that an F_σ -set is the union of a countable family of closed sets and a G_δ -set is the intersection of a countable family of open sets.

LEMMA 3.¹⁾ *Let X be a separable metric space of cardinality $< 2^{\aleph_0}$. If we assume MA, then every subset of X is F_σ and G_δ in X .*

Proof. Let D be any subset of X and $\{B_i : i \in \omega\}$ be a basis for open sets of X such that all B_i are non-empty. For each $x \in X$, let $s_x = \{i \in \omega : x \in B_i\}$. If we put $A = \{s_x : x \in X - D\}$ and $B = \{s_y : y \in D\}$, then A and B are of cardinality 2^{\aleph_0} . It is easily checked that if $y \in D$ and $x_1, \dots, x_n \in X - D$ then $s_y - (s_{x_1} \cup \dots \cup s_{x_n})$ is infinite. By Lemma 1, we can find a subset t of ω such that $s_x \cap t$ is finite if $x \in X - D$ and $s_y \cap t$ is infinite if $y \in D$. For each $n \in \omega$, let

$$K_n = \bigcup_{\substack{i > n \\ i \in t}} B_i .$$

And let $K = \bigcap_{n \in \omega} K_n$. Then K is a G_δ -set of X . In order to prove that D is a G_δ -set of X , it suffices to prove the following (1) and (2):

- (1) $D \subseteq K$
- (2) $(X - D) \cap K = \emptyset$.

Let y be an arbitrary element of D and $n \in \omega$. Since $t \cap s_y$ is infinite, there is $i \in t \cap s_y$ such that $i > n$. Then $y \in B_i$ and $B_i \subseteq K_n$, so $y \in K_n$. Since y and n are arbitrary, we have (1). Let x be any element of $X - D$. Since $t \cap s_x$ is finite, there is $n \in \omega$ such that if $i \in t$ and $i > n$ then $i \notin s_x$. For such $n \in \omega$, we have $x \notin K_n$, and so $x \notin K$. Thus we have (2).

Replacing D with $X - D$, we have that $X - D$ is a G_δ -set of X . Hence D is an F_σ -set of X . Therefore D is F_σ and G_δ in X .

¹⁾ This lemma is a slight generalization of that of J. Silver.

The following proposition is the negation of C_{62} .

PROPOSITION 2. (Suppose MA and $2^{\aleph_0} > \aleph_1$). Let E be any uncountable set of reals and f be any function from E into \mathbf{R} , the set of reals. Then there exists an uncountable subset N of E such that $f \upharpoonright N$, the restriction of f to N , is continuous on N .

Proof. We may assume E is of cardinality \aleph_1 . Let F be an arbitrary closed set in \mathbf{R} . Then, by Lemma 3, $f^{-1}(F)$, the inverse image of F , is a G_δ -set of E . Thus f_a is Baire function of class ≤ 1 . As is well-known, every Baire function of class ≤ 1 whose range is a subset of \mathbf{R} is the limit of a sequence of continuous functions. Let $\langle f_n : n \in \omega \rangle$ be a sequence of continuous functions from E to \mathbf{R} which converges to f . Then, by Proposition 1, there exists an uncountable subset N of E such that $\langle f_n : n \in \omega \rangle$ converges uniformly to f on N . Since each $f_n \upharpoonright N$ is continuous on N , so is $f \upharpoonright N$.

This proposition implies the following proposition, which is the negation of C_{61} .

PROPOSITION 3. (Suppose MA and $2^{\aleph_0} > \aleph_1$). There is a subset F of ${}^{\mathbf{R}}\mathbf{R}$ of cardinality 2^{\aleph_0} such that if $g \in {}^{\mathbf{R}}\mathbf{R}$ then for some $f \in F$ the set $\{x \in \mathbf{R} : f(x) = g(x)\}$ is uncountable.

Proof. Let F be the set of Baire functions from \mathbf{R} into \mathbf{R} . Then clearly, F is of cardinality 2^{\aleph_0} . By Proposition 2, if $g \in {}^{\mathbf{R}}\mathbf{R}$, then there exists an uncountable subset N of \mathbf{R} such that $g \upharpoonright N$ is continuous on N . The following is a well-known theorem.

Let X be an arbitrary metric space, let Y be a complete separable space and A be a subset of X . Then every Baire function from A to Y can be extended to a Baire function from X into Y .

Since $f \upharpoonright N$ is a Baire function on N , by this theorem, there exists $f \in F$ such that $f \upharpoonright N = g \upharpoonright N$. Thus the set $\{x \in \mathbf{R} : f(x) = g(x)\}$ includes N , and is uncountable.

§ 2. Let $[\omega]^{\aleph_0}$ denote the set of all infinite subsets of ω . We define a relation \subseteq^* on $[\omega]^{\aleph_0}$ as follows:

$$a \subseteq^* b \leftrightarrow a - b \text{ is finite, where } a, b \in [\omega]^{\aleph_0}.$$

Intuitively $a \subseteq^* b$ iff $a \subseteq b$ almost everywhere.

LEMMA.¹⁾ Suppose MA. Let Θ be an ordinal such that $\Theta < 2^{\aleph_0}$, and let $\langle a_\alpha : \alpha < \Theta \rangle$ be a sequence of elements of $[\omega]^{\aleph_0}$ such that if $\alpha < \beta < \Theta$ then $a_\beta \subseteq^* a_\alpha$. Then there exists $a \in [\omega]^{\aleph_0}$ such that if $\alpha < \Theta$ then $a \subseteq^* a_\alpha$.

Proof. Let $A = \{\omega - a_\alpha : \alpha < \Theta\}$ and $B = \{a_\alpha : \alpha < \Theta\}$. Then clearly, A and B are of cardinality $< 2^{\aleph_0}$. If $\alpha, \alpha_1, \dots, \alpha_n < \Theta$, then

$$a_\alpha - \bigcup_{i=1}^n (\omega - a_{\alpha_i}) = a_\alpha \cap a_{\alpha_1} \cap \dots \cap a_{\alpha_n}.$$

It is easily checked the intersection of finite elements of B is an element of $[\omega]^{\aleph_0}$. Thus A and B satisfy the condition of Lemma 1 of § 1. Therefore there is a subset a of ω such that $a - a_\alpha$ is finite and $a \cap a_\alpha$ is infinite for any $\alpha < \Theta$. For such $a \subseteq \omega$, we have $a \in [\omega]^{\aleph_0}$ and $a \subseteq^* a_\alpha$.

From this lemma, we obtain the following proposition, which is the negation of C_{13} .

PROPOSITION. (Assume MA and $2^{\aleph_0} > \aleph_1$). Let $\langle f_n : n \in \omega \rangle$ be a sequence of functions from \mathbf{R} to \mathbf{R} . Then there exists a sequence $\langle m_k : k \in \omega \rangle$ of natural numbers such that $m_0 < m_1 < \dots < m_k < \dots$ and the set $\{x \in \mathbf{R} : \langle f_{m_k}(x) : k \in \omega \rangle \text{ converges to a finite or infinite value}\}$ is uncountable.

*Proof.*²⁾ For each $a \in [\omega]^{\aleph_0}$, let a' denote the sequence $\langle n_k : k \in \omega \rangle$ such that $n_0 < n_1 < \dots < n_k < \dots < \dots$ and $a = \{n_k : k \in \omega\}$. By the limit of the sequence $\langle f_n(x) : n \in a \rangle$, we mean the limit of the sequence $\langle f_{n_k}(x) : k \in \omega \rangle$ in the usual sense, where $\langle n_k : k \in \omega \rangle = a'$. Let E be a subset of \mathbf{R} of cardinality \aleph_1 . Order E into a transfinite sequence of type ω_1 as follows:

$$x_0, x_1, \dots, x_\alpha, \dots \quad (\alpha < \omega_1)$$

By transfinite induction on α , we define a sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ of elements of $[\omega]^{\aleph_0}$ such that $a_\beta \subseteq^* a_\alpha$ if $\alpha < \beta < \omega_1$ and the sequences $\langle f_n(x_\alpha) : n \in a_\alpha \rangle$ with $\alpha \in \omega_1$ are convergent. The sequence $\langle f_n(x_0) : n \in \omega \rangle$ includes a convergent subsequence $\langle f_{n_k}(x_0) : k \in \omega \rangle$, whose limit is finite or infinite. So, we define a_0 to be $\{n_k : k \in \omega\}$. Assume that a_β with $\beta < \alpha$ are defined and $a_\gamma \subseteq^* a_\beta$ if $\beta < \gamma < \alpha$. Then, by the above lemma, we can find $a \in [\omega]^{\aleph_0}$ such that $a \subseteq^* a_\beta$ for all $\beta < \alpha$. The sequence $\langle f_\alpha(x_\alpha) :$

¹⁾ It was pointed out by the referee that this lemma could be proved from Lemma 2 of § 1.

²⁾ This proof was suggested to the author by Professor Kanji Namba.

$i \in a$ includes a convergent subsequence $\langle f_{i_k}(x_\alpha) : k \in \omega \rangle$. So, we define a_α to be $\{i_k : k \in \omega\}$.

By the lemma of this section, let b be an element of $[\omega]^{\aleph_0}$ such that $b \subseteq^* a_\alpha$ for all $\alpha < \omega_1$. For every $\alpha < \omega_1$, since $b \subseteq^* a_\alpha$, the sequence $\langle f_m(x_\alpha) : m \in b \rangle$ is convergent. If we put $\langle m_k : k \in \omega \rangle = b'$, then the set $\{x : \langle f_{m_k}(x) : k \in \omega \rangle \text{ is convergent}\}$ includes E , and is uncountable.

§ 3. Let E be a subset of \mathbf{R} and $a \in \mathbf{R}$. By $E(a)$ we denote the set $\{x + a : x \in E\}$.

Without MA, we can prove the following proposition.

PROPOSITION. (Suppose $2^{\aleph_0} > \aleph_1$). If E is an uncountable subset of \mathbf{R} such that its complement is of cardinality 2^{\aleph_0} , then there exists $a \in \mathbf{R}$ such that $E(a) \Delta E$, the symmetric difference of $E(a)$ and E , is uncountable.

Proof. Suppose, on the contrary, that for any $a \in \mathbf{R}$, $E(a) \Delta E$ is countable. Let N be a subset E of cardinality \aleph_1 . Then we show $\bigcap_{x \in N} [\mathbf{R} - E(-x)] \neq \emptyset$. If $\bigcap_{x \in N} [\mathbf{R} - E(-x)] = \emptyset$, then $\mathbf{R} = \bigcup_{x \in N} E(-x)$. On the other hand

$$\bigcup_{x \in N} E(-x) = \bigcup_{x, y \in N} [E(-x) \Delta E(-y)] \cup \bigcap_{x \in N} E(-x).$$

Therefore,

$$A \cup \bigcap_{x \in N} E(-x) = \mathbf{R}, \quad \text{where } A = \bigcup_{x, y \in N} [E(-x) \Delta E(-y)].$$

Since A and $\bigcap_{x \in N} E(-x)$ are disjoint, we have $\mathbf{R} - \bigcap_{x \in N} E(-x) = A$. Let x be an arbitrary element of N . Then we have $\mathbf{R} - E(-x) \subseteq A$. Note that each $E(a) \Delta E(b)$ is countable because $E(a) \Delta E(b) = J(a) \cup K(b)$, where $J = E(b - a) \Delta E$, $K = E(a - b) \Delta E$. Therefore A is of cardinality $\leq \aleph_1$. This contradicts the hypothesis that the complement of E is of cardinality 2^{\aleph_0} . Thus $\bigcap_{x \in N} [\mathbf{R} - E(-x)] \neq \emptyset$.

Let $a \in \bigcap_{x \in N} [\mathbf{R} - E(-x)]$, then $N \subseteq \mathbf{R} - E(-a)$ because $a \notin E(-x)$ iff $x \notin E(-a)$. Therefore $E(-a) \Delta E$ includes N , and is uncountable.

The following corollary is the negation of C_{70} .

COROLLARY. (suppose MA and $2^{\aleph_0} > \aleph_1$). Let E be a non-measurable set of reals. Then for some $a \in \mathbf{R}$, $E(a) \Delta E$ is uncountable.

Proof. If we assume MA, then every set of reals of cardinality

$< 2^{\aleph_0}$ is of Lebesgue measure 0 ([2, § 4]). Hence, if E is non-measurable, the E and its complement are of cardinality 2^{\aleph_0} . Thus E satisfies the condition of the proposition.

§ 4. A set E of reals is said to have the property $(M)^{1)}$ if, for any collection \mathcal{U} of open sets satisfying the condition

$$(*) \quad (\forall x \in E)(\forall \varepsilon > 0)(\exists U \in \mathcal{U})[\delta(U) < \varepsilon \wedge x \in U]$$

where $\delta(U)$ is the diameter of U , there is a sequence $\langle U_n : n \in \omega \rangle$ of elements of \mathcal{U} such that $E \subseteq \bigcup_{n \in \omega} U_n$ and $\lim_{n \rightarrow \infty} \delta(U_n) = 0$.

As a direct application of MA, we have the following proposition.

PROPOSITION. (Suppose MA). Every set of reals of cardinality $< 2^{\aleph_0}$ has the property (M) .

Proof. Let E be a set of reals of cardinality $< 2^{\aleph_0}$, and \mathcal{U} be a collection of open sets satisfying the condition $(*)$. For each $n \in \omega$, there is a sequence $\langle U_{nm} : m \in \omega \rangle$ of elements of \mathcal{U} such that $E \subseteq \bigcup_{m \in \omega} U_{nm}$ and $\delta(U_{nm}) < 1/n + 1$ for all $m \in \omega$. We define a partially ordered set $\mathcal{P} = \langle P, \leq \rangle$ as follows:

$$P = \{p : p \text{ is a finite function with } \text{dom}(p) \cup \text{rang}(p) \subseteq \omega\},$$

$$p \leq q \leftrightarrow p \subseteq q.$$

Then clearly, \mathcal{P} satisfies the countable chain condition. For each $x \in E$, if we put $X_x = \{p \in P : x \in \bigcup_{n \in \text{dom}(p)} U_{np(n)}\}$, then X_x is dense in \mathcal{P} . Let $\mathcal{F} = \{X_x : x \in E\}$. Then \mathcal{F} is of cardinality $< 2^{\aleph_0}$, so there is an \mathcal{F} -generic filter G on \mathcal{P} . If we put $f = \bigcup G$, then f is a function with $\text{dom}(f) \subseteq \omega$ and $\text{rang}(f) \subseteq \omega$. We define U_n as follows:

$$U_n = \begin{cases} U_{nf(n)} & \text{if } n \in \text{dom}(f) \\ U_{n0} & \text{otherwise} \end{cases}$$

Then, clearly, $U_n \in \mathcal{U}$ and $\lim_{n \rightarrow \infty} \delta(U_n) = 0$. Let x be an arbitrary element of E . Since $X_x \cap G \neq \emptyset$, there is $p \in G$ such that $x \in \bigcup_{n \in \text{dom}(p)} U_{np(n)}$. Since $P \in G$, we have $\bigcup_{n \in \text{dom}(p)} U_{np(n)} \subseteq \bigcup_{n \in \omega} U_n$, so $x \in \bigcup_{n \in \omega} U_n$. Therefore E has the property (M) .

¹⁾ See [3, p. 48]

§5. A set E of reals is said to have the property $(\lambda)^1$ if every countable subset of E is a G_δ -set of E .

In this section, we shall show there is a non-measurable set of reals of cardinality 2^{\aleph_0} with the property (λ) .

A set E of reals is said to have the property $(S^*)^2$ if, for every set N of Lebesgue measure 0, $E \cap N$ is of cardinality $< 2^{\aleph_0}$. If a set E is measurable and has positive measure, then E includes a set of measure 0 and cardinality 2^{\aleph_0} . If we assume MA, then every set of reals of cardinality $< 2^{\aleph_0}$ is of Lebesgue measure 0. Therefore every set of reals of cardinality 2^{\aleph_0} with the property (S^*) is non-measurable. The existence of a non-measurable set of reals of cardinality 2^{\aleph_0} with the property (λ) follows from the following proposition.

PROPOSITION. (Suppose MA). *There is a set E of reals of cardinality 2^{\aleph_0} with the property (S^*) such that every subset of E of cardinality $< 2^{\aleph_0}$ is G_δ in E .*

Proof. Order the set of all G_δ -sets of measure 0 into a transfinite sequence of type 2^{\aleph_0} as follows:

$$N_0, N_1, \dots, N_\xi, \dots, (\xi < 2^{\aleph_0}).$$

By transfinite induction on α , we define a sequence $\langle x_\alpha : \alpha < 2^{\aleph_0} \rangle$ of reals and a sequence $\langle K_\alpha : \alpha < 2^{\aleph_0} \rangle$ of G_δ -sets of measure 0. Let $K_0 = N_0$ and x_0 be an arbitrary element of \mathbf{R} . Suppose x_β and K_β with $\beta < \alpha$ are defined, and let

$$S_\alpha = \bigcup_{\beta < \alpha} K_\beta \cup \{x_\beta : \beta < \alpha\} \cup N_\alpha.$$

Then, by MA, S_α is of measure 0, so $\mathbf{R} - S_\alpha \neq \emptyset$. Let x_α be an arbitrary element of $\mathbf{R} - S_\alpha$ and K_α be the first N_ξ such that $S_\alpha \cup \{x_\alpha\} \subseteq N_\xi$.

Let E be the set $\{x_\alpha : \alpha < 2^{\aleph_0}\}$. Then we have

- (1) E is of cardinality 2^{\aleph_0} ;
- (2) for each $\alpha < 2^{\aleph_0}$, $E \cap N_\alpha$ is of cardinality $< 2^{\aleph_0}$;
- (3) $K_\alpha \subseteq K_\beta$ if $\alpha < \beta < 2^{\aleph_0}$.

From (1) and (2), E is a set of cardinality 2^{\aleph_0} with the property (S^*) .

Let D be an arbitrary subset of E of cardinality $< 2^{\aleph_0}$. Since 2^{\aleph_0} is a regular cardinal, there is $\alpha < 2^{\aleph_0}$ such that $D \subseteq \{x_\beta : \beta \leq \alpha\}$. Put

¹⁾ See [3, p. 94]

²⁾ Cf. [3, p. 81]

$X = \{x_\beta : \beta \leq \alpha\}$. Then, by Lemma 3 of §1, D is a G_δ -set in X . Since $X = E \cap K_\alpha$ and K_α is G_δ in R , X is G_δ in E . Therefore D is a G_δ -set in E .

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