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# AN EXACT SEQUENCE ASSOCIATED WITH A GENERALIZED CROSSED PRODUCT

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## § 0. Introduction

The purpose of this paper is to generalize the seven terms exact sequence given by Chase, Harrison and Rosenberg [8]. Our work was motivated by Kanzaki [16] and, of course, [8], [9]. The main theorem holds for any generalized crossed product, which is a more general one than that in Kanzaki [16]. In § 1, we define a group $P(A/B)$ for any ring extension $A/B$, and prove some preliminary exact sequences. In § 2, we fix a group homomorphism $J$ from a group $G$ to the group of all invertible two-sided $B$-submodules of $A$. We put $\Delta/B = \oplus J_\sigma/B$ (direct sum), which is canonically a generalized crossed product of $B$ with $G$. And we define an abelian group $C(\Delta/B)$ for $\Delta/B$. The two groups $C(\Delta/B)$ and $P(A/B)$ are our main objects. $C(\Delta/B)$ may be considered as a generalization of the group of all central separable algebras split by a fixed Galois extension. The main theorem is Th. 2.12, which is a generalization of the seven terms exact sequence theorem in [8]. However it is proved that the exact sequence in Th. 2.12 is almost reduced to the one which is obtained from the homomorphism $G \to \mathrm{Aut}(K)$ induced by $J$, where $K$ is the center of $B$. This fact is proved in Th. 2.15. In § 3, we fix a group homomorphism $u\colon G \to \mathrm{Aut}(A/B)$. From $u$ we obtain a free crossed product $\oplus Au_\sigma/B$, where $u_\sigma u_\tau = u_{\sigma\tau}$, $u_\sigma a = \sigma(a)u_\sigma (a \in A)$. Therefore the results in § 2 is applicable for this case. In § 4 we prove the Morita invariance of the exact sequence in Th. 2.12. In § 5, we treat a kind of duality, which is based on a result obtained in [19]. In § 6 we study the splitting of $P(A/B)$ in particular cases.

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## § 1. The definition of $P(A/B)$, and related exact sequences.

As to notations and terminologies used in this paper we follow [19], unless otherwise expressed.

Let $G, G'$ be groups, and $f$ a homomorphism from $G$ to the group of all automorphisms of $G'$. Then $G$ operates on $G'$, by $f$. Then we call $G'$ a $G$-group. We denote by $G'^G$ the subgroup $\{g' \in G' \mid g(g') = g'$ for all $g \in G\}$.

Let $A \supseteq B$ be rings with common identity, and let $L, K$ be the centers of $A$ and $B$, respectively. We denote by $\mathfrak{G}(A/B)$ the group of all invertible two-sided $B$-submodules of $A$ (cf. [19]), where a two-sided $B$-submodule $X$ of $A$ is invertible in $A$ if and only if $XY = YX = B$ for some $B$-$B$-submodule $Y$ of $A$. We denote by $\mathrm{Aut}(A/B)$ the group of all $B$-automorphisms of a ring $A$, which operates on the left. Then it is evident that $\mathfrak{G}(A/B)$ is canonically a left $\mathrm{Aut}(A/B)$-group. On the other hand we have

PROPOSITION 1.1. *$\mathrm{Aut}(A/B)$ is a $\mathfrak{G}(A/B)$-group.*

*Proof.* Let $X$ be in $\mathfrak{G}(A/B)$. Then $A = XA = X \otimes_B A = AX^{-1} = A \otimes_B X^{-1}$ canonically (cf. [19; Prop. 1.1]), and hence $X \otimes_B A \otimes_B X^{-1} \to A, x \otimes a \otimes x' \mapsto xax'$ is an isomorphism. Therefore, for any $\sigma$ in $\mathrm{Aut}(A/B)$, the mapping $X(\sigma): x \otimes a \otimes x' \mapsto x \otimes \sigma(a) \otimes x'$ $(x \in X, x' \in X^{-1})$ from $A$ to $A$ is well defined. Then it is easily seen that $X(\sigma)$ is a $B$-automorphism of $A$, and this defines a $\mathfrak{G}(A/B)$-group $\mathrm{Aut}(A/B)$.

Here we continue the study of $X(\sigma)$ for the sequel. Since $XX^{-1} = B \ni 1$, 1 is written as $1 = \sum_i a_i a'_i (a_i \in X, a'_i \in X^{-1})$. Then $\sum_i \tau(a_i)\sigma(a'_i) \cdot \sum_i \sigma(a_i)\tau(a'_i) = 1$ for $\sigma, \tau$ in $\mathrm{Aut}(A/B)$. Since $\sum_i a_i \otimes a'_i \mapsto 1$ under the isomorphism $X \otimes_B X^{-1} \to B$, we know that $\sum_i ba_i \otimes a'_i = \sum_i a_i \otimes a'_i b$ for all $b$ in $B$, and so $b \sum_i \tau(a_i)\sigma(a'_i) = \sum_i \tau(a_i)\sigma(a'_i)b$. Thus $\sum_i \tau(a_i)\sigma(a'_i) \in U(V_A(B))$ (the group of all invertible elements of $V_A(B)$), and $(\sum_i \tau(a_i)\sigma(a'_i))^{-1} = \sum_i \sigma(a_i)\tau(a'_i)$. Put $u = \sum_i a_i \cdot \sigma(a'_i)$. Then, for any a in $A, u \cdot \sigma(a)u^{-1} = \sum_{i,j} a_i \cdot \sigma(a'_i)\sigma(a)\sigma(a_j)a'_j = \sum_{i,j} a_i \cdot \sigma(a'_i a a_j)a'_j = X(\sigma)(\sum_{i,j} a_i a'_i a a_j a'_j) = X(\sigma)(a)$. Hence $X(\sigma)$ differs from $\sigma$ by the inner automorphism induced by $u$. Therefore $X(\sigma) = \sigma$ is equivalent to that $u$ is in the center $L$ of $A$. To be easily seen, $u \cdot \sigma(x) = x$ for all $x$ in $X$, (and similarly $\sigma(x')u^{-1} = x'$ for all $x'$ in $X^{-1}$). Conversely, since the left annihilator of $X$ in $A$ is zero, this characterizes $u$, and hence $u$ is independent of the choice of

$a_i, a'_i$, and is denoted by $u(X, 1, \sigma)$, in the sequel. As $\sum_i \tau(a_i)\sigma(a'_i) = \tau(\sum_i a_i \cdot \tau^{-1}\sigma(a'_i))$, $\sum_i \tau(a_i)\sigma(a'_i)$ is also independent of the choice of $a_i, a'_i$, and is denoted by $u(X, \tau, \sigma)$.

LEMMA 1.2. *Let ${}_BP_{B'}$ and ${}_BP'_{B'}$ be Morita modules, $A$ and $A'$ are over rings of $B$ and $B'$, respectively. Let $f_0$ be a left $B$, right $B'$-isomorphism $P \to P'$, and $f: A \otimes_B P \xrightarrow{\approx} P' \otimes_{B'} A'$ is a $B$-$B'$-isomorphism such that $f(1 \otimes p) = f_0(p) \otimes 1$ for all $p \in P$. Assume that $xf^{-1}(f(a \otimes p)x') = f^{-1}(f(xa \otimes p)x')$ for all $x, a \in A$, $x' \in A'$. Then, if we define $(a \otimes p) * x' = f^{-1}(f(a \otimes p)x')$, then ${}_AA \otimes_B P_{A'}$ is a Morita module.* (cf. [19])

*Proof.* Put End$({}_AA \otimes_B P)/B' = A''/B'$. Then, by [19; Lemma 3.1], $P \otimes_{B'} A'' \to A \otimes_B P$, $p \otimes a'' \mapsto (1 \otimes p)a''$ is an isomorphism. On the other hand $f^{-1}: P' \otimes_{B'} A' \to A \otimes_B P$, $f_0(p) \otimes a' \mapsto (1 \otimes p) * a' (p \in P)$. By hypothesis, the image of $A'$ in the endomorphism ring is contained in $A''$. And, since $P_{B'}$ is a generator, the above two isomorphisms imply that the image of $A'$ is equal to $A''$.

Next we define a group $P(A/B)$. $P(A/B)$ consists of all isomorphic classes of left $B$, right $B$-homomorphism $\varphi$ from a Morita module ${}_BP_B$ to a Morita module ${}_AN_A$ such that the homomorphism $A \otimes_B P \to N$, $a \otimes p \mapsto a \cdot \varphi(p)$ is an isomorphism (cf. [19; §3]). An isomorphism from $\varphi: P \to N$ to $\varphi': P' \to N'$ is a pair $(f, g)$ of isomorphisms such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ P' & \xrightarrow{\varphi'} & N' \end{array}$$

is commutative, where $f$ is a left $B$, right $B$-isomorphism, and $g$ is a left $A$, right $A$-isomorphism. The isomorphism class of $\varphi$ is denoted by $[\varphi]$. The product of $\varphi: P \to N$ and $\psi: Q \to U$ is $\varphi \otimes \psi: P \otimes_B Q \to N \otimes_A U$, where $(\varphi \otimes \psi)(p \otimes q) = \varphi(p) \otimes \psi(q)$. We define $[\varphi][\psi] = [\varphi \otimes \psi]$. Then this is well-defined, and associative. The inclusion map $B \to A$ is evidently the identity element. Let $P^* = \mathrm{Hom}_r({}_BP, {}_BB)$ (cf. [19]), $N^* = \mathrm{Hom}_r({}_AN, {}_AA)$, and $\varphi^*: P^* \to N^*$ the homomorphism such that $\varphi^*(p^*) = (a \cdot \varphi(p) \to a \cdot p^{p^*})(p^* \in P^*, a \in A, p \in P)$ (cf. [19; Lemma 3.1]). Then it is obvious that $[\varphi^*]$ is the inverse element of $[\varphi]$ in $P(A/B)$. Thus we have proved

THEOREM 1.3. *$P(A/B)$ is a group.*

*Remark.* Similarly $P(A/B)$ can be defined for any ring homomorphism $B \to A$.

THEOREM 1.4. *There is an exact sequence*

$$1 \to U(L) \cap U(K) \to U(L) \to \mathfrak{S}(A/B) \to P(A/B) \to \operatorname{Pic}(A) ,$$

*where $U(*)$ is the group of invertible elements of a ring $*$, and* Pic$(A)$ *is the group of isomorphic classes of two-sided $A$-Morita modules.*

*Proof.* The mapping $U(L) \cap U(K) \to U(L)$ is the canonical one, and the mapping $U(L) \to \mathfrak{S}(A/B)$ is $c \mapsto Bc$. Then $1 \to U(L) \cap U(K) \to U(L) \to \mathfrak{S}(A/B)$ is evidently exact. For $X$ in $\mathfrak{S}(A/B)$, we correspond the canonical inclusion map $i_X: X \to A$. If $i_X$ is isomorphic to $i_B$, then there is a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{i_B} & A \\ \approx\big\downarrow & & \big\downarrow\approx \\ X & \xrightarrow[i_X]{} & A \end{array}$$

and hence there is an element $d$ in $U(L)$ such that $Bd = X$. Hence $U(L) \to \mathfrak{S}(A/B) \to P(A/B)$ is exact. For $\varphi: P \to M$ in $P(A/B)$, we correspond $[M]$ (the isomorphic class of $M$). If $M \xrightarrow{\approx} A$ as $A$-$A$-modules, then we may assume that $M = A$ and $P$ is a $B$-$B$-submodule of $A$ (cf. [19; Lemma 3.1 (4)]). Then, by [19; Prop. 1.1], we have $P \in \mathfrak{S}(A/B)$. This completes the proof.

On the other hand we have

THEOREM 1.5. *There is an exact sequence*

$$1 \to U(L) \cap U(K) \to U(K) \to \operatorname{Aut}(A/B) \to P(A/B) \to \operatorname{Pic}(B) .$$

*Proof.* The map $U(L) \cap U(K) \to U(K)$ is the canonical one, and the map $U(K) \to \operatorname{Aut}(A/B)$ is $d \mapsto \tilde{d}$, where $\tilde{d}(a) = dad^{-1}$ for all $a \in A$. Then $1 \to U(K) \cap U(L) \to U(K) \to \operatorname{Aut}(A/B)$ is evidently exact. For any $\sigma$ in Aut$(A/B)$, we correspond the map $i_\sigma: B \to Au_\sigma$, $b \mapsto bu_\sigma$ (cf. [19]). For $d$ in $U(K), d \mapsto \tilde{d} \mapsto i_{\tilde{d}}$. Put $\tilde{d} = \tau$. Then $A \xrightarrow{\approx} Au_\tau, a \mapsto ad^{-1}u_\tau$, as $A$-$A$-modules, and $B \xrightarrow{\approx} B$, as $B$-$B$-modules, by $b \mapsto bd^{-1}$, and we have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{i_B} & A \\ \approx \Big\downarrow d^{-1} & & \Big\downarrow \approx \\ B & \xrightarrow[i_\tau]{} & Au_\tau \end{array}$$

Let $\sigma$ be in $\mathrm{Aut}\,(A/B)$, and suppose that $i_\sigma$ is isomorphic to $i_B: B \to A$. Then there are isomorphisms $\alpha, \beta$ such that

$$\begin{array}{ccc} B & \xrightarrow{i_B} & A \\ \beta \Big\downarrow & & \Big\downarrow \alpha \\ B & \xrightarrow[i_\sigma]{} & Au_\sigma \end{array}$$

is commutative. Put $\alpha^{-1}(u_\sigma) = d$. Then, for any $a \in A, \sigma(a)d = \alpha^{-1}(\sigma(a)u_\sigma)$ $= \alpha^{-1}(u_\sigma a) = da$, and so $\sigma(a)d = da$. Since $\beta(d)u_\sigma = \alpha(d) = u_\sigma$, we have $\beta(d) = 1$, whence $d$ is in $U(K)$, because $\beta$ is a $B$-$B$-isomorphism. Finally, for $\varphi: P \to M$ in $P(A/B)$, we correspond $[P] \in \mathrm{Pic}\,(B)$. If ${}_BB_B \xrightarrow{\approx} {}_BP_B, 1 \mapsto u$, then $P = Bu$ and $M = A\cdot\varphi(u)$. Since $M \xrightarrow{\approx} A \otimes {}_BP$ as left $A$, right $B$-modules, $a\cdot\varphi(u) = 0$ $(a \in A)$ implies $a = 0$. Hence there is an automorphism $\sigma \in \mathrm{Aut}\,(A/B)$ such that $\varphi(u)a = \sigma(a)\varphi(u)$ for all $a \in A$. Then $\varphi$ is isomorphic to $i_\sigma$. This completes the proof.

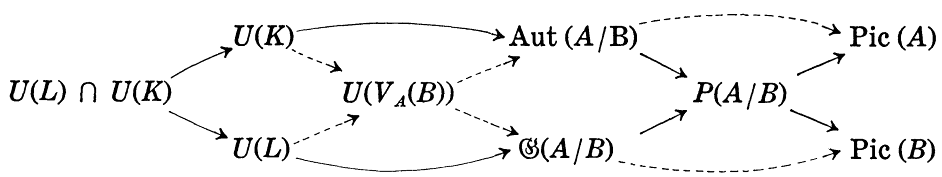
If we cut out $P(A/B)$, we have well known exact sequences.

PROPOSITION 1.6. *There are two exact sequences*

$$1 \longrightarrow U(K) \longrightarrow U(V_A(B)) \xrightarrow{\alpha} \mathfrak{G}(A/B) \longrightarrow \mathrm{Pic}\,(B)\ ,$$
$$1 \longrightarrow U(L) \longrightarrow U(V_A(B)) \xrightarrow{\beta} \mathrm{Aut}\,(A/B) \longrightarrow \mathrm{Pic}\,(A)\ ,$$

*where* $\alpha(d) = Bd$ *and* $\beta(d)(a) = dad^{-1}(d \in U(V_A(B)), a \in A)$.

Here we indicate Th. 1.4, Th. 1.5, and Prop. 1.6 by the following diagram:




If $A$ is an $R$-algebra, we define $\mathrm{Pic}_R\,(A) = \{[P] \in \mathrm{Pic}\,(A) \mid rp = pr$ for all $r \in R$ and all $p \in P\}$ and $P^R(A/B) = \{[\varphi] \in P(A/B) \mid \varphi: P \to N, [N] \in$

$\mathrm{Pic}_R(A)\}$. If $B$ is an $S$-algebra, we define $P_S(A/B) = \{[\varphi] \in P(A/B) \mid \varphi: P \to N, [P] \in \mathrm{Pic}_S(B)\}$.

## § 2. The definition of $C(\Delta/B)$, and an exact sequence associated with $\Delta/B$.

In this section, we fix a (finite or infinite) group $G$, rings $B \subseteq A$, and a group homomorphism $J: \sigma \mapsto J_\sigma$ from $G$ to $\mathfrak{G}(A/B)$. Then $J$ induces a group homomorphism $G \to \mathrm{Aut}(V_A(B)/L)$ (cf. [19; Prop. 3.3]), and further $G \to \mathrm{Aut}(K/K \cap L)$. A generalized crossed product $\oplus_{\sigma \in G} J_\sigma/B$ associated with $J$ is defined by $(x_\sigma)(y_\sigma) = (z_\sigma)$, where $z_\sigma = \sum_{\tau\rho=\sigma} x_\tau y_\rho$. We denote this by $\Delta/B$ in the sequel. $\mathrm{Pic}(B)$ is a left $G$-group defined by ${}^\sigma[P] = [J_\sigma \otimes_B P \otimes_B J_{\sigma^{-1}}]$ (conjugation). Then we define $\mathrm{Pic}(B)^G = \{[P] \in \mathrm{Pic}(B) \mid {}^\sigma[P] = [P]$ for all $\sigma \in G\}$, and $\mathrm{Pic}_K(B)^G = \mathrm{Pic}(B)^G \cap \mathrm{Pic}_K(B)$. The homomorphism $\mathfrak{G}(A/B) \to P(A/B)$ in Th. 1.4 induces a left $G$-group $P(A/B)$ defined by conjugation.

PROPOSITION 2.1. *The following exact sequences consist of G-homomorphisms*:

$$1 \longrightarrow U(L) \cap U(K) \longrightarrow U(K) \longrightarrow \mathrm{Aut}(A/B) \longrightarrow P(A/B) \longrightarrow \mathrm{Pic}(B)$$
$$1 \longrightarrow U(L) \longrightarrow U(V_A(B)) \longrightarrow \mathrm{Aut}(A/B) \longrightarrow \mathrm{Pic}(A)$$

*Proof.* Let $\sigma \in \mathrm{Aut}(A/B)$, and $X \in \mathfrak{G}(A/B)$, and let $\sum_i a_i a'_i = 1$ $(a_i \in X, a'_i \in X^{-1})$. Then $X(\sigma)(a) = \sum_i a_i \cdot \sigma(a'_i)\sigma(a) \sum_j \sigma(a_j)a'_j$ for all a in $A$ (cf. § 1), and so $Au_\sigma \xrightarrow{\approx} Au_{X(\sigma)}$ as $A$-$A$-modules, by the map $au_\sigma \to a \cdot \sum_i \sigma(a_i)a'_i u_{X(\sigma)}$. Then the following diagram is commutative:

$$\begin{array}{cccccc} X \otimes_B B \otimes_B X^{-1} & \longrightarrow & Au_\sigma\,, & x \otimes b \otimes x' & \longmapsto & xbu_\sigma x' = xb \cdot \sigma(x')u_\sigma\,. \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ B & \longrightarrow & Au_{X(\sigma)} & xbx' & \longmapsto & xbx'u_{X(\sigma)} \end{array}$$

Hence $\mathrm{Aut}(A/B) \to P(A/B)$ is a $G$-homomorphism. Let $c$ be in $U(V_A(B))$. Then, since $X$ induces an automorphism of $V_A(B)$, there is a $c' \in U(V_A(B))$ such that $xc = c'x$ for all $x \in X$ (i.e., $X(c) = c'$). Put $u = \sum_i a_i \cdot \tilde{c}(a'_i)$. Then $c'c^{-1} \cdot \tilde{c}(x) = c'c^{-1} \cdot cxc^{-1} = c'xc^{-1} = x$ for all $x$ in $X$. Hence we know that $c'c^{-1} = u$ (cf. § 1). For any a in $A$, $X(\tilde{c})(a) = u \cdot \tilde{c}(a)u^{-1} = c'c^{-1}cac^{-1} \cdot cc'^{-1} = c'ac'^{-1}$. Hence $X(\tilde{c}) = \tilde{c'} = \widetilde{X(c)}$. The remainder is obvious.

We define $P(A/B)^{(G)} = \{[\phi] \in P(A/B) \mid \phi: P \to M, J_\sigma \cdot \phi(P) = \phi(P) \cdot J_\sigma$ for all $\sigma \in G\}$. Then $P(A/B)^{(G)}$ is a subgroup of $P(A/B)^G$. In fact, for $\phi: P \to M$ in $P(A/B)$, $[\phi]$ belongs to $P(A/B)^{(G)}$ if and only if, for any $\sigma$

in $G$, there is a $B$-$B$-isomorphism $f_\sigma: P \to J_\sigma \otimes {}_B P \otimes {}_B J_{\sigma^{-1}}$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\quad \phi \quad} & M \\ {\scriptstyle f_\sigma}\downarrow & \nearrow {\scriptstyle {}^\sigma\phi} & \\ J_\sigma \otimes P \otimes J_{\sigma^{-1}} & & \end{array}$$

is commutative, where $({}^\sigma\phi)(x_\sigma \otimes p \otimes x'_\sigma) = x_\sigma \cdot \phi(p) x'_\sigma$. Here we shall check that $P(A/B)^{(G)}$ is closed with respect to inverse. We may assume that $P \subseteq M$ and $P^* \subseteq M^*$ (cf. [19; Lemma 3.1]). Then $P^* = \{g \in M^* \mid P^g \subseteq B\}$. In this sense, $(P)J_\sigma P^* J_{\sigma^{-1}} = (PJ_\sigma)P^* J_{\sigma^{-1}} = (J_\sigma P)P^* J_{\sigma^{-1}} = J_\sigma((P)P^*)J_{\sigma^{-1}} = J_\sigma J_{\sigma^{-1}} = B$, and so $J_\sigma P^* J_{\sigma^{-1}} \subseteq P^*$ for all $\sigma \in G$. Hence $J_\sigma P^* J_{\sigma^{-1}} = P^*$ for all $\sigma \in G$.

We put $P_K(A/B)^{(G)} = P_K(A/B) \cap P(A/B)^{(G)}$. Further we define $\mathrm{Aut}\,(A/B)^{(G)} = \{f \in \mathrm{Aut}\,(A/B) \mid f(J_\sigma) = J_\sigma$ for all $\sigma \in G\}$. Then we have

PROPOSITION 2.2. *There is an exact sequence*

$$\begin{aligned} 1 \longrightarrow U(L) \cap U(K) \longrightarrow U(K) \longrightarrow & \mathrm{Aut}\,(A/B)^{(G)} \\ & \longrightarrow P_K(A/B)^{(G)} \longrightarrow \mathrm{Pic}_K\,(B)^G\,. \end{aligned}$$

*Proof.* The above sequence is a subsequence of the one in Th. 1.5. Therefore it suffices to prove that, for $f$ in $\mathrm{Aut}\,(A/B)$, the image of $f$ is contained in $P_K(A/B)^{(G)}$ if and only if $f \in \mathrm{Aut}\,(A/B)^{(G)}$. However $J_\sigma \cdot Bu_f J_{\sigma^{-1}} = J_\sigma \cdot f(J_\sigma)^{-1} u_f$, so that $J_\sigma \cdot Bu_f J_{\sigma^{-1}} = Bu_f$ if and only if $J_\sigma \cdot f(J_\sigma)^{-1} = B$, or equivalently, $f(J_\sigma) = J_\sigma$. This completes the proof.

Next we state several lemmas (which are well known).

For any two-sided $B$-module $U$, we denote by $V_U(B)\,\{u \in U \mid bu = ub$ for all $b \in B\}$.

LEMMA 2.3. *Let $B$ be an $R$-algebra, and $P$ an $R$-module such that ${}_R P \mid {}_R R$ (i.e., finitely generated and projective). Then* $\mathrm{End}_r\,({}_B B \otimes {}_R P) \xrightarrow{\approx} B \otimes {}_R \mathrm{End}_r\,({}_R P)$ *canonically, and* ${}_B B \otimes {}_R P_B \mid {}_B B_B$ (cf. [19]). *And further* $V_{B\otimes P}(B) \xrightarrow{\approx} K \otimes {}_R P$ *canonically, where $K$ is the center of $B$. Therefore if* $\mathrm{End}\,({}_R P) = R$ *then* ${}_B B \otimes {}_R P_B$ *is a Morita module.*

*Proof.* The first assertion is well known. The remainder is evident, if ${}_R P$ is free. Hence it is true for any $P$ such that ${}_R P \mid {}_R R$.

LEMMA 2.4. *Let* ${}_B M_B \mid {}_B B_B$. *Then* $M = B \cdot V_M(B) \xrightarrow{\approx} B \otimes {}_K V_M(B)$

*canonically, and* ${}_KV_M(B)\,|\,{}_KK$. *Further* $\mathrm{End}_r\,({}_KV_M(B)) \xrightarrow{\approx} \mathrm{End}_r\,({}_BM_B)$ *and* $\mathrm{End}_r\,({}_BM) \xrightarrow{\approx} B \otimes {}_K\mathrm{End}_r\,({}_BM_B)$, *canonically.*

*Proof.* ${}_BM_B\,|\,{}_BB_B$ implies that $V_M(K) = M$, and hence $M$ may be considered as a left $B^e$-module, where $B^e = B \otimes {}_KB^{\circ p}$. Then ${}_{B^e}M\,|\,{}_{B^e}B$. Evidently $\mathrm{Hom}_r\,({}_{B^e}B, {}_{B^e}M) \xrightarrow{\approx} V_M(B)$ canonically. By [14; Th. 1.1], ${}_{B^e}M \xrightarrow{\approx} \mathrm{Hom}_r\,({}_{B^e}B^e,\ {}_{B^e}M) \xrightarrow{\approx} \mathrm{Hom}_r\,({}_{B^e}B^e,\ {}_{B^e}B) \otimes {}_K\mathrm{Hom}_r\,({}_{B^e}B,\ {}_{B^e}M) \xrightarrow{\approx} B \otimes {}_KV_M(B)$, ${}_KV_M\,|\,{}_KK$ and $\mathrm{End}_r\,({}_K\mathrm{Hom}_r\,({}_{B^e}B, {}_{B^e}M)) \xrightarrow{\approx} \mathrm{End}_r\,({}_BM_B)$. Combining this with Lemma 2.3, we obtain the last assertion.

COROLLARY 1. *Further assume that* $\mathrm{End}_r\,({}_BM_B) = K$, *Then* ${}_BM_B$ *is a Morita module.*

COROLLARY 2. *Let* ${}_BM_B\,|\,{}_BB_B$ *and* ${}_BM'_B\,|\,{}_BB_B$. *Then* ${}_BM_B \xrightarrow{\approx} {}_BM'_B$ *if and only if* ${}_KV_M(B) \xrightarrow{\approx} {}_KV_{M'}(B)$.

The following corollary is repeatedly used to check commutativity of diagrams.

COROLLARY 3. *Let* ${}_BM_B\,|\,{}_BB_B$ *and* ${}_BM'_B\,|\,{}_BB_B$. *Then* $V_{M\otimes M'}(B) \xrightarrow{\approx} V_M(B) \otimes {}_KV_{M'}(B)$ *canonically, and there is an isomorphism* ${}_BM \otimes M'_B \to {}_BM' \otimes M_B$, $m_0 \otimes m' \mapsto m' \otimes m_0$, $m \otimes m'_0 \mapsto m'_0 \otimes m$ $(m_0 \in V_M(B),\ m \in M,\ m'_0 \in V_{M'}(B), m' \in M')$, *where unadorned* $\otimes$ *means* $\otimes_B$. *We call this isomorphism the "transposition" of* $M$ *and* $M'$

*Proof.* By Lemma 2.4, $M = B \otimes {}_KV_M(B)$ and $M' = B \otimes {}_KV_{M'}(B)$. Consequently, $M \otimes M' = B \otimes {}_KV_M(B) \otimes {}_KV_{M'}(B)$. Then, by Lemma 2.3, $V_{M\otimes M'}(B) \xrightarrow{\approx} V_M(B) \otimes {}_KV_{M'}(B)$ canonically. Since $V_M(B) \otimes {}_KV_{M'}(B) \xrightarrow{\approx} V_{M'}(B) \otimes {}_KV_M(B)$ by transposition, we obtain the latter assertion.

*Remark.* We put $\{[M] \in \mathrm{Pic}(B)\,|\,{}_BM_B \sim {}_BB_B\} = \mathrm{Pic}_0(B)$ ([19]). Then, by Lemma 2.3, Lemma 2.4, and Cor. 3 to Lemma 2.4, $\mathrm{Pic}_K\,(K) \xrightarrow{\approx} \mathrm{Pic}_0\,(B)$, $[P] \mapsto [P \otimes {}_KB]$.

The following lemma is also used to check commutativity of diagrams

LEMMA 2.5. *Let* ${}_BU \otimes {}_BW_B \sim {}_BB_B \sim {}_BM_B$. *If* $x \in V_M(B)$ *and* $\sum_i u_i \otimes w_i \in V_{U\otimes W}(B)$, *then* $\sum_i u_i \otimes x \otimes w_i \in V_{U\otimes M\otimes W}(B)$.

*Proof.* For any $x$ in $V_M(B), U \otimes {}_BW \to U \otimes M \otimes W, u \otimes w \mapsto u \otimes x \otimes w$ is a $B$-$B$-homomorphism.

Next we shall define an abelian group $C(\Delta/B)$, which is the main object in the present paper. In the rest of this section, unadorned $\otimes$

always means $\otimes_B$. $C(\Delta/B)$ consists of all isomorphic classes of generalized crossed products $\oplus_{\sigma\in G} V_\sigma/B$ of $B$ with $G$ such that ${}_BV_{\sigma B} \sim {}_BJ_{\sigma B}$ for all $\sigma \in G$ (cf. [19]). Let $\oplus V_\sigma/B$ and $\oplus W_\sigma/B$ be generalized crossed products of $B$ with $G$, and let $f$ be a $B$-ring isomorphism from $\oplus V_\sigma/B$ to $\oplus W_\sigma/B$. If $f(V_\sigma) = W_\sigma$ for all $\sigma \in G$, we call $f$ an isomorphism as generalized crossed products. Precisely a generalized crossed product $\oplus V_\sigma/B$ is written as $(\oplus V_\sigma/B, f_{\sigma,\tau})$, and its isomorphic class is denoted by $[\oplus V_\sigma/B, f_{\sigma,\tau}]$, where $f_{\sigma,\tau}\colon V_\sigma \otimes V_\tau \to V_{\sigma\tau}$ is the multiplication. In particular, the multiplication of $\Delta$ is denoted by $\phi_{\sigma,\tau}$. However we denote often $(\oplus J_\sigma/B, \phi_{\sigma,\tau})$ by $\oplus J_\sigma/B$, simply. Let $(\oplus V_\sigma/B, f_{\sigma,\tau})$ and $(\oplus W_\sigma/B, g_{\sigma,\tau})$ be generalized crossed products in $C(\Delta/B)$. Then the $\sigma$-component of the product of $(\oplus V_\sigma/B, f_{\sigma,\tau})$ and $(\oplus W_\sigma/B, g_{\sigma,\tau})$ is defined as $V_\sigma \otimes J_{\sigma^{-1}} \otimes W_\sigma$. The multiplication is defined by $h_{\sigma,\tau}\colon V_\sigma \otimes J_{\sigma^{-1}} \otimes W_\sigma \otimes V_\tau \otimes J_{\tau^{-1}} \otimes W_\tau \xrightarrow{t} V_\sigma \otimes V_\tau \otimes J_{\tau^{-1}} \otimes J_{\sigma^{-1}} \otimes W_\sigma \otimes W_\tau \xrightarrow{*} V_{\sigma\tau} \otimes J_{(\sigma\tau)^{-1}} \otimes W_{\sigma\tau}$, where $t$ is the transposition of $J_{\sigma^{-1}} \otimes W_\sigma$ and $V_\tau \otimes J_{\tau^{-1}}$, and $* = f_{\sigma,\tau} \otimes \phi_{\sigma,\tau} \otimes g_{\sigma,\tau}$. The associativity of the above multiplication is proved by making use of Cor. 3 to Lemma 2.4. If we identify the canonical isomorphism $B \otimes B \otimes B \to B$, then we have a generalized crossed product $(\oplus (V_\sigma \otimes J_{\sigma^{-1}} \otimes W_\sigma)/B,\ h_{\sigma,\tau})$. The associativity of this composition in $C(\Delta/B)$ is proved by using Cor. 3 to Lemma 2.4, too. Evidently $[\oplus J_\sigma/B, \phi_{\sigma,\tau}]$ is the identity element of $C(\Delta/B)$. The $\sigma$-component of the inverse of $(\oplus V_\sigma/B, f_{\sigma,\tau})$ is $J_\sigma \otimes V_\sigma^* \otimes J_\sigma$, where $V_\sigma^* = \mathrm{Hom}_r({}_BV_\sigma, {}_BB)$. The multiplication is defined by $f^*_{\sigma,\tau}\colon J_\sigma \otimes (V_\sigma^* \otimes J_\sigma) \otimes (J_\tau \otimes V_\tau^*) \otimes J_\tau \xrightarrow{t} J_\sigma \otimes (J_\tau \otimes V_\tau^*) \otimes (V_\sigma^* \otimes J_\sigma) \otimes J_\tau \xrightarrow{\phi \otimes * \otimes \phi} J_{\sigma\tau} \otimes V_{\sigma\tau}^* \otimes J_{\sigma\tau}$, where $*\colon V_\tau^* \otimes V_\sigma^* \to (V_\sigma \otimes V_\tau)^* \to V_{\sigma\tau}^*$ is the canonical isomorphism induced by $f_{\sigma,\tau}$. We identify the canonical isomorphism $B \otimes B^* \otimes B \to B$, and we have a generalized crossed product $(\oplus (J_\sigma \otimes V_\sigma^* \otimes J_\sigma)/B,\ f^*_{\sigma,\tau})$. By the isomorphism $V_\sigma \otimes (J_{\sigma^{-1}} \otimes J_\sigma) \otimes V_\sigma^* \otimes J_\sigma \to (V_\sigma \otimes V_\sigma^*) \otimes J_\sigma \to J_\sigma$, the product of $(\oplus V_\sigma/B, f_{\sigma,\tau})$ and $(\oplus (J_\sigma \otimes V_\sigma^* \otimes J_\sigma)/B, f^*_{\sigma,\tau})$ is isomorphic to $\Delta$, as generalized crossed products. Hence $C(\Delta/B)$ is a group. Finally $C(\Delta/B)$ is an abelian group, because the isomorphism $V_\sigma \otimes J_{\sigma^{-1}} \otimes W_\sigma \to V_\sigma \otimes J_{\sigma^{-1}} \otimes W_\sigma \otimes (J_{\sigma^{-1}} \otimes J_\sigma) \xrightarrow{t} W_\sigma \otimes J_{\sigma^{-1}} \otimes V_\sigma \otimes (J_{\sigma^{-1}} \otimes J_\sigma) \to W_\sigma \otimes J_{\sigma^{-1}} \otimes V_\sigma$ is an isomorphism as generalized crossed products, where $t$ is the transposition of $V_\sigma \otimes J_{\sigma^{-1}}$ and $W_\sigma \otimes J_{\sigma^{-1}}$. By $C_0(\Delta/B)$, we denote the subgroup of all generalized crossed products $[\oplus V_\sigma/B, f_{\sigma,\tau}]$ such that ${}_BV_{\sigma B} \xrightarrow{\approx} {}_BJ_{\sigma B}$ for all $\sigma \in G$. We put $\mathrm{Pic}_K(B)^{[G]} = \{[P] \in \mathrm{Pic}_K(B) \mid {}_BP \otimes J_\sigma \otimes {}^*P_B \sim {}_BJ_{\sigma B}$ for all $\sigma$ in $G\}$, where ${}^*P = \mathrm{Hom}_l(P_B, B_B)$, and "$\sim$" means "similar" (cf. [19]). Then $\mathrm{Pic}_K(B)^{[G]}$ is evidently a subgroup of $\mathrm{Pic}_K(B)$. Then the canonical isomorphism $*P \otimes P \to B$ induces an isomorphism $P \otimes J_\sigma \otimes (*P \otimes P) \otimes J_\tau \otimes *P \to P \otimes J_\sigma \otimes J_\tau \otimes *P$, and we obtain ${}^P\phi_{\sigma,\tau}: (P \otimes J_\sigma \otimes *P) \otimes (P \otimes J_\tau \otimes *P) \to P \otimes J_\sigma \otimes J_\tau \otimes *P \xrightarrow{1 \otimes \phi \otimes 1} P \otimes J_{\sigma\tau} \otimes *P$. Then $(\oplus (P \otimes J_\sigma \otimes *P)/B, {}^P\phi_{\sigma,\tau})$ is a generalized crossed product, and $[P] \mapsto [\oplus (P \otimes J_\sigma \otimes *P)/B, {}^P\phi_{\sigma,\tau}]$ is a group homomorphism from $\mathrm{Pic}_K(B)^{[G]}$ to $C(\Delta/B)$. Thus we have proved the following theorem

THEOREM 2.6. *$C(\Delta/B)$ is an abelian group with identity $\Delta/B$, and $C_0(\Delta/B)$ is a subgroup of $C(\Delta/B)$. There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}_K(B)^G & \longrightarrow & C_0(\Delta/B) \\ \downarrow & & \downarrow \\ \mathrm{Pic}_K(B)^{[G]} & \longrightarrow & C(\Delta/B) \end{array}$$

*Remark.* $C_0(\Delta/B)$ is isomorphic to $H^2(G, U(K))$. The isomorphism is defined as follows: Let $[\oplus J_\sigma/B, f_{\sigma,\tau}]$ be in $C_0(\Delta/B)$. Then, for any $\sigma, \tau$ in $G$, there exists uniquely $a_{\sigma,\tau} \in U(K)$ such that $f_{\sigma,\tau}(x_\sigma \otimes x_\tau) = a_{\sigma,\tau} \cdot \phi_{\sigma,\tau}(x_\sigma \otimes x_\tau)$ for all $x_\sigma \in J_\sigma, x_\tau \in J_\tau$. Then $\{a_{\sigma,\tau} \mid \sigma, \tau \in G\}$ is a (normalized) factor set, and $[\oplus J_\sigma/B, f_{\sigma,\tau}] \mapsto$ class $\{a_{\sigma,\tau}\}$ is an isomorphism. $(\oplus J_\sigma/B, f_{\sigma,\tau})$ may be written as $(\oplus J_\sigma/B, a_{\sigma,\tau})$ when $\Delta$ is fixed.

PROPOSITION 2.7. *There is an exact sequence*

$$P_K(\Delta/B)^{(G)} \longrightarrow \mathrm{Pic}_K(B)^G \longrightarrow C_0(\Delta/B)\ .$$

*Proof.* The semi-exactness follows from the definition of $P_K(\Delta/B)^{(G)}$ ([19; § 3]). Let $[P] \in \mathrm{Pic}_K(B)^G$ be in the kernel. Then $(\oplus (P \otimes J_\sigma \otimes *P), {}^P\phi_{\sigma,\tau})$ is isomorphic to $(\oplus J_\sigma, \phi_{\sigma,\tau}) = \Delta$. However, by [19; p. 116], $(\oplus P \otimes J_\sigma \otimes *P), {}^P\phi_{\sigma,\tau})/B$ is isomorphic to $\mathrm{End}_l(P \otimes {}_B\Delta_\Delta)/B$, as rings, and so we have a Morita module ${}_\Delta P \otimes {}_B\Delta_\Delta$. Then the canonical homomorphism $P$ to $P \otimes \Delta, p \mapsto p \otimes 1$ is in $P_K(\Delta/B)^{(G)}$.

An abelian group $B(\Delta/B)$ is defined by the following exact sequence:

$$\mathrm{Pic}_K(B)^{[G]} \longrightarrow C(\Delta/B) \longrightarrow B(\Delta/B) \longrightarrow 1$$

Then we have

PROPOSITION 2.8. *There is an exact sequence*

$$\mathrm{Pic}_K(B)^G \longrightarrow C_0(\Delta/B) \longrightarrow B(\Delta/B)$$

*Proof.* The semi-exactness is trivial. If $[\oplus J_\sigma, f_{\sigma,\tau}]$ is in the kernel of $C_0(\Delta/B) \to B(\Delta/B)$, then there is $[P]$ in $\mathrm{Pic}_K(B)^{[G]}$ such that $[P] \mapsto [\oplus J_\sigma, f_{\sigma,\tau}]$ under the homomorphism $\mathrm{Pic}_K(B)^{[G]} \to C(\Delta/B)$. Then it is evident that $[P]$ is in $\mathrm{Pic}_K(B)^G$.

By Remark to Cor. 3 to-Lemma 2.4, $\mathrm{Pic}_K(K) \to \mathrm{Pic}_0(B)$, $[P_0] \mapsto [P_0 \otimes {}_K B]$ is an isomorphism, and $[P] \mapsto [V_P(B)]$ is its inverse.

PROPOSITION 2.9. *The above isomorphism is a G-isomorphism.*

*Proof.* Let $[P]$ be in $\mathrm{Pic}_0(B)$. Then $P = B \otimes {}_K V_P(B)$, and $J_\sigma \otimes P \otimes J_{\sigma^{-1}} \xrightarrow{\approx} J_\sigma \otimes (B \otimes {}_K V_P(B)) \otimes J_{\sigma^{-1}} \xrightarrow{\approx} (J_\sigma \otimes {}_K V_P(B)) \otimes J_{\sigma^{-1}}$ as two-sided $B$-modules. It is easily seen that $J_\sigma \otimes {}_K V_P(B) \to Ku_\sigma \otimes {}_K V_P(B) \otimes {}_K Ku_{\sigma^{-1}} \otimes {}_K J_\sigma$, $x_\sigma \otimes p_0 \mapsto u_\sigma \otimes p_0 \otimes u_{\sigma^{-1}} \otimes x_\sigma$ is a $B$-$B$-isomorphism, where $\sigma$ denotes the automorphism induced by $J_\sigma$. Therefore $J_\sigma \otimes P \otimes J_{\sigma^{-1}} \xrightarrow{\approx} Ku_\sigma \otimes {}_K V_P(B) \otimes {}_K Ku_{\sigma^{-1}} \otimes {}_K B$, $x_\sigma \otimes p_0 \otimes x_{\sigma^{-1}} \mapsto u_\sigma \otimes p_0 \otimes u_{\sigma^{-1}} \otimes x_\sigma x_{\sigma^{-1}}$ ($x_\sigma \in J_\sigma$, $x_{\sigma^{-1}} \in J_{\sigma^{-1}}$, $p_0 \in V_P(B)$) is a $B$-$B$-isomorphism. Hence, by Lemma 2.3, $V_{J_\sigma \otimes P \otimes J_{\sigma^{-1}}}(B) \xrightarrow{\approx} Ku_\sigma \otimes {}_K V_P(B) \otimes {}_K Ku_{\sigma^{-1}}$, as $K$-modules. This completes the proof.

COROLLARY. $Z^1(G, \mathrm{Pic}_K(K)) \xrightarrow{\approx} Z^1(G, \mathrm{Pic}_0(B))$.

There is a group homomorphism $[\oplus V_\sigma, f_{\sigma,\tau}] \mapsto (\sigma \to [V_\sigma][J_\sigma]^{-1})$ $(\sigma \in G)$ from $C(\Delta/B)$ to $Z^1(G, \mathrm{Pic}_0(B))$. Then the following sequence is exact:

$$1 \longrightarrow C_0(\Delta/B) \longrightarrow C(\Delta/B) \longrightarrow Z^1(G, \mathrm{Pic}_0(B))$$

$\bar{H}^1(G, \mathrm{Pic}_0(B))$ is defined by the exactness of the following row:

$$\begin{array}{ccccccc} \mathrm{Pic}_K(B)^{[G]} & \longrightarrow & Z^1(G, \mathrm{Pic}_0(B)) & \longrightarrow & \bar{H}^1(G, \mathrm{Pic}_0(B)) & \longrightarrow & 1 \\ & \searrow & \nearrow & & & & \\ & & C(\Delta/B) & & & & \end{array}$$

PROPOSITION 2.10. $C_0(\Delta/B) \to B(\Delta/B) \to \bar{H}^1(G, \mathrm{Pic}_0(B))$ *is exact.*

*Proof.* Evidently the above sequence is semi-exact. Let $[[\oplus V_\sigma, f_{\sigma,\tau}]]$ (the class of $[\oplus V_\sigma, f_{\sigma,\tau}]$ in $B(\Delta/B)$) be in the kernel. Then there is a $[P] \in \mathrm{Pic}_K(B)^{[G]}$ such that $P \otimes J_\sigma \otimes {}^*P \xrightarrow{\approx} V_\sigma$ for all $\sigma \in G$, where ${}^*P = \mathrm{Hom}_l(P_B, B_B)$. For any $\sigma \in G$, we fix an isomorphism $h_\sigma: P \otimes J_\sigma \otimes {}^*P \to V_\sigma \cdot f'_{\sigma,\tau}$ is defined by the commutativity of the diagram

$$\begin{array}{ccc} P\otimes J_\sigma\otimes {}^*P\otimes P\otimes J_\tau\otimes {}^*P & \xrightarrow{h_\sigma\otimes h_\tau} & V_\sigma\otimes V_\tau \\ {\scriptstyle *}\downarrow & & \approx\downarrow f'_{\sigma,\tau} \\ P\otimes J_{\sigma\tau}\otimes {}^*P & \xrightarrow[h_{\sigma,\tau}]{} & V_{\sigma\tau} \end{array}$$

where $*$ is defined by ${}^*P\otimes P \xrightarrow{\approx} B$ (canonical) and $\phi_{\sigma,\tau}$. Then $(\oplus V_\sigma, f'_{\sigma,\tau})$ differs from $(\oplus V_\sigma, f_{\sigma,\tau})$ by some factor set $\{a_{\sigma,\tau}\}$, i.e., $f'_{\sigma,\tau} = a_{\sigma,\tau}f_{\sigma,\tau}$ (cf. Remark to Th. 2.6.). Then, by the canonical isomorphism $J_\sigma\otimes J_{\sigma^{-1}}\otimes V_\sigma \xrightarrow{\approx} V_\sigma$, $(\oplus J_\sigma, a_{\sigma,\tau})\times(\oplus V_\sigma, f_{\sigma,\tau})$ is isomorphic to $(\oplus V_\sigma, f'_{\sigma,\tau})$. Since $(\oplus V_\sigma, f'_{\sigma,\tau})$ is isomorphic to $(\oplus(P\otimes J_\sigma\otimes {}^*P), {}^P\phi_{\sigma,\tau})$, this completes the proof.

PROPOSITION 2.11. *There is an exact sequence*

$$B(\Delta/B) \longrightarrow \bar{H}^1(G, \mathrm{Pic}_0(B)) \longrightarrow H^3(G, U(K)) .$$

*Proof.* For $\phi$ in $Z^1(G, \mathrm{Pic}_0(B))$, a homomorphism $\Phi$ from $G$ to $\mathrm{Pic}(B)$ is defined by $\Phi(\sigma) = \phi(\sigma)[J_\sigma]$. Let $\Phi(\sigma) = [U_\sigma]$ and $U_1 = B$. Then $U_\sigma \sim J_\sigma$, as $B$-$B$-modules, for all $\sigma\in G$. For $\sigma,\tau$ in $G$, we take a $B$-$B$-isomorphism $f_{\sigma,\tau}\colon U_\sigma\otimes U_\tau \to U_{\sigma\tau}$. If $\sigma = 1$ or $\tau = 1$ then we take $f_{\sigma,\tau}$ as a canonical one. Then, for any $\sigma,\tau,\gamma$ in $G$, there exists uniquely $u(\sigma,\tau,\gamma)\in U(K)$ such that $u(\sigma,\tau,\gamma)f_{\sigma,\tau\gamma}(I_\sigma\otimes f_{\tau,\gamma})(x) = f_{\sigma\tau,\gamma}(f_{\sigma,\tau}\otimes I_\gamma)(x)$ for all $x$ in $J_{\sigma\tau\gamma}$, where $I_\sigma$ is the identity of $U_\sigma$.

$$\begin{array}{ccccc} & & U_\sigma\otimes U_\tau\otimes U_\gamma & & \\ & {\scriptstyle I_\sigma\otimes f_{\tau,\gamma}}\swarrow & & \searrow{\scriptstyle f_{\sigma,\tau}\otimes I_\gamma} & \\ U_\sigma\otimes U_{\tau\gamma} & & & & U_{\sigma\tau}\otimes U_\gamma \\ {\scriptstyle f_{\sigma,\tau\gamma}}\downarrow & & & & \downarrow{\scriptstyle f_{\sigma\tau,\gamma}} \\ U_{\sigma\tau\gamma} & & \xrightarrow{u(\sigma,\tau,\gamma)} & & U_{\sigma\tau\gamma} \end{array}$$

If $\sigma = 1$ or $\tau = 1$ or $\gamma = 1$, then $u(\sigma,\tau,\gamma) = 1$. Let $f'_{\sigma,\tau}$ be another isomorphism from $U_\sigma\otimes U_\tau$ to $U_{\sigma\tau}$, and let $u'(\sigma,\tau,\gamma)$ be the one determined by $f'_{\sigma,\tau}$. Then, for any $\sigma,\tau$ in $G$, there exists a unique $u(\sigma,\tau)\in U(K)$ such that $u(\sigma,\tau)f_{\sigma,\tau} = f'_{\sigma,\tau}$. If $\sigma = 1$ or $\tau = 1$, then $u(\sigma,\tau) = 1$. It is easily seen that $u'(\sigma,\tau,\gamma) = u(\sigma\tau,\gamma)u(\sigma,\tau)\cdot{}^\sigma u(\tau,\gamma)^{-1}u(\sigma,\tau\gamma)^{-1}u(\sigma,\tau,\gamma)$. Let $H$ be the group of all functions $u$ from $G\times G\times G$ to $U(K)$. Then $Z^1(G, \mathrm{Pic}_0(B)) \to H/B^3(G, U(K)), \phi\mapsto$ class $\{u(\sigma,\tau,\gamma)\}$ is well defined, and this induces $\alpha\colon \bar{H}^1(G, \mathrm{Pic}_0(B)) \to H/B^3(G, U(K))$, where $B^3(G, U(K))$ consists of all $u(-,-,-)\in H$ such that $u(\sigma,\tau,\gamma) = u(\sigma\tau,\gamma)u(\sigma,\tau)\cdot{}^\sigma u(\tau,\gamma)^{-1}u(\sigma,\tau\gamma)^{-1}$ for some mapping $u(-,-): G \times G \to U(K)$ such that $u(\sigma,\tau) = 1$ provided $\sigma = 1$ or $\tau = 1$. If class $\{u(\sigma,\tau,\gamma)\} = 1$ then, for a suitable choice of $f_{\sigma,\tau}$, we can take $u(\sigma,\tau,\gamma) = 1$ for all $\sigma,\tau,\gamma \in G$. Next we shall show that $\alpha$ is a homomorphism from $\bar{H}^1(G, \mathrm{Pic}_0(B))$ to $H/B^3(G, U(K))$. We take another $\psi \in Z^1(G, \mathrm{Pic}_0(B))$, and put $\Psi(\sigma) = \psi(\sigma)[J_\sigma] = [W_\sigma]$. And let each $g_{\sigma,\tau}: W_\sigma \otimes W_\tau \to W_{\sigma\tau}$ be a $B$-$B$-isomorphism, and $u_1(\sigma,\tau,\gamma)$ be the one determined by $g_{\sigma,\tau}$. Put $\phi\psi = \pi$. Then $\Pi(\sigma) = \phi(\sigma)\psi(\sigma)[J_\sigma] = \phi(\sigma)[J_\sigma][J_\sigma]^{-1} \cdot \psi(\sigma)[J_\sigma] = \Phi(\sigma)[J_\sigma]^{-1}\psi(\sigma) = [U_\sigma \otimes J_{\sigma^{-1}} \otimes W_\sigma]$. We take an isomorphism $k_{\sigma,\tau}: U_\sigma \otimes J_{\sigma^{-1}} \otimes W_\sigma \otimes U_\tau \otimes J_{\tau^{-1}} \otimes W_\tau \xrightarrow{t} U_\sigma \otimes U_\tau \otimes J_{\tau^{-1}} \otimes J_{\sigma^{-1}} \otimes W_\sigma \otimes W_\tau \xrightarrow{*} U_{\sigma\tau} \otimes J_{(\sigma\tau)^{-1}} \otimes W_{\sigma\tau}$, where $t$ is the transposition of $J_{\sigma^{-1}} \otimes W_\sigma$ and $U_\tau \otimes J_{\tau^{-1}}$, and $* = f_{\sigma,\tau} \otimes \phi_{\tau^{-1},\sigma^{-1}} \otimes g_{\sigma,\tau}$. Then, by using of Cor. 3 to Lemma 2.4, it is easily seen that $u(\sigma,\tau,\gamma)u_1(\sigma,\tau,\gamma)k_{\sigma,\tau\gamma}(I_\sigma \otimes k_{\tau,\gamma}) = k_{\sigma\tau,\gamma}(k_{\sigma,\tau} \otimes I_\gamma)$. The fact that $\mathrm{Im}\,\alpha$ is contained in $H^3(G, U(K))$ will be proved later. Thus we have obtained the following theorem, which may be considered as a generalization of Chase, Harrison, Resenberg [8; Cor. 5.5].

THEOREM 2.12. *Let $G$ be a group, and $\Delta/B = (\oplus J_\sigma, \phi_{\sigma,\tau})$ be a generalized crossed product of $B$ with $G$. Let $C$ and $K$ be the centers of $\Delta$ and $B$, respectively. Then there is an exact sequence*

$$\begin{aligned} 1 &\longrightarrow U(C) \cap U(K) \longrightarrow U(K) \longrightarrow \mathrm{Aut}\,(\Delta/B)^{(G)} \\ &\longrightarrow P_K(\Delta/B)^{(G)} \longrightarrow \mathrm{Pic}_K(B)^G \longrightarrow C_0(\Delta/B) \\ &\longrightarrow B(\Delta/B) \longrightarrow \bar{H}^1(G, \mathrm{Pic}_0(B)) \longrightarrow H^3(G, U(K))\ . \end{aligned}$$

*Proof.* This follows from Propositions 2.2, 2.7, 2.8, 2.10 and 2.11.

*Remark.* The above sequence can be expressed as a seven term exact sequence:

$$\begin{aligned} 1 &\longrightarrow H^1(G, U(K)) \longrightarrow P_K(\Delta/B)^{(G)} \longrightarrow \mathrm{Pic}_K(B)^G \longrightarrow H^2(G, U(K)) \\ &\longrightarrow B(\Delta/B) \longrightarrow \bar{H}^1(G, \mathrm{Pic}_0(B)) \longrightarrow H^3(G, U(K))\ . \end{aligned}$$

In fact, for any $f \in \mathrm{Aut}\,(\Delta/B)^{(G)}$ and any $\sigma \in G$, there exists uniquely $c_0 \in U(K)$ such that $f(x_\sigma) = c_\sigma x_\sigma$ for all $x_\sigma \in J_\sigma$. Then it is easily seen that $c_{\sigma\tau} = c_\sigma \cdot {}^\sigma c_\tau$ for all $\sigma, \tau \in G$, and we have an isomorphism $\mathrm{Aut}\,(\Delta/B)^{(G)} \xrightarrow{\approx} Z^1(G, U(K))$. Evidently the image of $U(K)$ in $\mathrm{Aut}\,(\Delta/B)^{(G)}$ corresponds to $B^1(G, U(K))$.

Let $P_\sigma (\sigma \in G)$ be a family of Morita $B$-$B$-modules such that ${}_BP_{\sigma B} \sim {}_BB_B, P_1 = B$. Then ${}_BP_\sigma \otimes J_{\sigma B} \sim {}_BJ_{\sigma B}$. Put $V_{P_\sigma}(B) = P_{0,\sigma}$. Then ${}_KP_{0,\sigma}$

$\sim {}_K K$, and so ${}_K P_{0,\sigma} \otimes {}_K K u_{\sigma_K} \sim {}_K K u_{\sigma_K}$. It was noted in the proof of Prop. 2.9 that $Ku_\sigma \otimes {}_K P_{0,\tau} \otimes {}_K K u_{\sigma^{-1}} \xrightarrow{\approx} V_{J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}}}(B)$, as $K$-$K$-modules, $u_\sigma \otimes p_\tau \otimes u_{\sigma^{-1}} \mapsto \sum_i a_i \otimes p_\tau \otimes a_i'$, where $a_i \in J_\sigma$, $a_i' \in J_{\sigma^{-1}}$, $\sum_i a_i a_i' = 1$. Let $f^*_{\sigma,\tau}\colon P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}} \to P_{\sigma\tau} (\sigma, \tau \in G)$ be a family of $B$-$B$-isomorphisms. Then, since $V_{J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}}}(B) \xrightarrow{\approx} Ku_\sigma \otimes {}_K P_{0,\tau} \otimes {}_K K u_{\sigma^{-1}}$, each $f^*_{\sigma,\tau}$ induces a $K$-$K$-isomorphism $f^*_{0,\sigma,\tau}\colon P_{0,\sigma} \otimes {}_K K u_\sigma \otimes {}_K P_{0,\tau} \otimes {}_K K u_{\sigma^{-1}} \to P_{0,\sigma\tau}$ (cf. Cor. 3 to Lemma 2.4), and conversely, and it is evident that $\{f^*_{\sigma,\tau} \mid \sigma, \tau \in G\} \mapsto \{f^*_{0,\sigma,\tau} \mid \sigma, \tau \in G\}$ is a one to one mapping between them. This is nothing but an isomorphism in Cor. to Prop. 2.9, and we can prove the commutativity of the following diagram:

$$\begin{array}{ccc} Z^1(G, \mathrm{Pic}_K(K)) & \longrightarrow & Z^1(G, \mathrm{Pic}_0(B)) \\ & \searrow \quad \nearrow & \\ & H/B^3(G, U(K)) & \end{array}$$

Then, by the same way as in [16; Lemma 8], the image of $Z^1(G, \mathrm{Pic}_K(K))$ in $H/B^3(G, U(K))$ is contained in $H^3(G, U(K))$, and this completes the proof of Th. 2.12. On the other hand, $f^*_{\sigma,\tau}\colon P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}} \xrightarrow{f^*_{\sigma,\tau} \otimes \phi_{\sigma,\tau}} P_{\sigma\tau} (\sigma, \tau \in G)$ induces $f_{\sigma,\tau}\colon P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_\tau \to (P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}}) \otimes (J_\sigma \otimes J_\tau) \to P_{\sigma\tau} \otimes J_{\sigma\tau} (\sigma, \tau \in G)$ and conversely, and $\{f^*_{\sigma,\tau} \mid \sigma, \tau \in G\} \mapsto \{f_{\sigma,\tau} \mid \sigma, \tau \in G\}$ is a $1-1$ mapping. A similar fact holds with respect to $P_{0,\sigma} (\sigma \in G)$ and a crossed product $\oplus Ku_\sigma$ with trivial factor set: $\{f^*_{0,\sigma,\tau} \mid \sigma, \tau \in G\} \mapsto \{f_{0,\sigma,\tau} \mid \sigma, \tau \in G\}$. Let $\{f_{\sigma,\tau}\} \leftrightarrow \{f^*_{\sigma,\tau}\} \leftrightarrow \{f^*_{0,\sigma,\tau}\} \leftrightarrow \{f_{0,\sigma,\tau}\}$. Then $\{f_{\sigma,\tau}\}$ defines a generalized crossed product if and only if so is $\{f_{0,\sigma,\tau}\}$. Its proof is easy, but it is tedious, so we omit it. Next we shall show that $\{f_{\sigma,\tau}\} \mapsto \{f_{0,\sigma,\tau}\}$ is an isomorphism from $C(\Delta/B)$ to $C(\oplus Ku_\sigma/K)$. To this end, let $[\oplus (Q_\sigma \otimes J_\sigma), g_{\sigma,\tau}]$ be another element in $C(\Delta/B)$, and let $[\oplus (P_\sigma \otimes Q_\sigma \otimes J_\sigma), h_{\sigma,\tau}]$ be the product of $[\oplus (P_\sigma \otimes J_\sigma), f_{\sigma,\tau}]$ and $[\oplus (Q_\sigma \otimes J_\sigma), g_{\sigma,\tau}]$ (cf. the proof of Th. 2.6). Then $f^*_{\sigma,\tau}\colon P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}} \xrightarrow{\approx} P_{\sigma\tau}$ and $g^*_{\sigma,\tau}\colon Q_\sigma \otimes J_\sigma \otimes Q_\tau \otimes J_{\sigma^{-1}} \xrightarrow{\approx} Q_{\sigma\tau}$ induce $f^*_{\sigma,\tau} \otimes g^*_{\sigma,\tau}\colon P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}} \otimes Q_\sigma \otimes J_\sigma \otimes Q_\tau \otimes J_{\sigma^{-1}} \xrightarrow{\approx} P_{\sigma\tau} \otimes Q_{\sigma\tau}$. Similarly $f^*_{0,\sigma,\tau}$ and $g^*_{0,\sigma,\tau}$ induce $f^*_{0,\sigma,\tau} \otimes g^*_{0,\sigma,\tau}$. On the other hand there are isomorphisms $P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}} \otimes Q_\sigma \otimes J_\sigma \otimes Q_\tau \otimes J_{\sigma^{-1}} \xrightarrow{t} P_\sigma \otimes Q_\sigma \otimes J_\sigma \otimes P_\tau \otimes (J_{\sigma^{-1}} \otimes J_\sigma) \otimes Q_\tau \otimes J_{\sigma^{-1}} \xrightarrow{*} P_\sigma \otimes Q_\sigma \otimes J_\sigma \otimes P_\tau \otimes Q_\tau \otimes J_{\sigma^{-1}}$, where $t$ is the transposition of $J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}}$ and $Q_\sigma$. Similarly we have an isomorphism $P_{0,\sigma} \otimes Ku_\sigma \otimes P_{0,\tau} \otimes Ku_{\sigma^{-1}} \otimes Q_{0,\sigma} \otimes Ku_\sigma \otimes Q_{0,\tau} \otimes Ku_{\sigma^{-1}} \to P_{0,\sigma} \otimes Q_{0,\sigma} \otimes Ku_\sigma \otimes P_{0,\tau} \otimes Q_{0,\tau} \otimes Ku_{\sigma^{-1}}$ for all $\sigma, \tau \in G$. Then the following two diagrams are commutative:

$$\begin{array}{ccc}
P_\sigma \otimes J_\sigma \otimes P_\tau \otimes J_{\sigma^{-1}} \otimes Q_\sigma \otimes J_\sigma \otimes Q_\tau \otimes J_{\sigma^{-1}} & \xrightarrow{f^*_{\sigma,\tau} \otimes g^*_{\sigma,\tau}} & P_{\sigma\tau} \otimes Q_{\sigma\tau} \\
\downarrow {\scriptstyle *\circ t} & \nearrow {\scriptstyle h^*_{\sigma,\tau}} & \\
P_\sigma \otimes Q_\sigma \otimes J_\sigma \otimes P_\tau \otimes Q_\tau \otimes J_{\sigma^{-1}} & &
\end{array}$$

$$\begin{array}{ccc}
P_{0,\sigma} \otimes_K Ku_\sigma \otimes_K P_{0,\tau} \otimes_K Ku_{\sigma^{-1}} \otimes Q_{0,\sigma} \otimes Ku_\sigma \otimes Q_{0,\tau} \otimes_K Ku_{\sigma^{-1}} & \xrightarrow{f^*_{0,\sigma,\tau} \otimes g^*_{0,\sigma,\tau}} & P_{0,\sigma\tau} \otimes_K Q_{0,\sigma\tau} \\
\downarrow {\scriptstyle *\circ t} & \nearrow {\scriptstyle h^*_{0,\sigma,\tau}} & \\
P_{0,\sigma} \otimes_K Q_{0,\sigma} \otimes_K Ku_\sigma \otimes_K P_{0,\tau} \otimes_K Q_{0,\tau} \otimes_K Ku_{\sigma^{-1}} & &
\end{array}$$

where $[\oplus (P_{0,\sigma} \otimes_K Q_{0,\sigma} \otimes_K Ku_\sigma), h_{0,\sigma,\tau}]$ is the product of $[\oplus (P_{0,\sigma} \otimes_K Ku_\sigma), f_{0,\sigma,\tau}]$ and $[\oplus (Q_{0,\sigma} \otimes_K Ku_\sigma), g_{0,\sigma,\tau}]$. Then, since $\{f^*_{\sigma,\tau} \otimes g^*_{\sigma,\tau}\} \leftrightarrow \{f^*_{0,\sigma,\tau} \otimes g^*_{0,\sigma,\tau}\}$ is evident, we know that $\{h_{\sigma,\tau}\} \leftrightarrow \{h_{0,\sigma,\tau}\}$. Thus we have proved that $C(\Delta/B) \to C(\oplus Ku_\sigma/K), \{f_{\sigma,\tau}\} \mapsto \{f_{0,\sigma,\tau}\}$ is an isomorphism. It is easily seen that $C_0(\Delta/B) \xrightarrow{\approx} C_0(\oplus Ku_\sigma/K)$ under the above isomorphism. Thus we have proved

THEOREM 2.13. *There are commutative diagrams:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C_0(\Delta/B) & \longrightarrow & C(\Delta/B) & \longrightarrow & Z^1(G, \mathrm{Pic}_0(B)) & & \text{(exact)} \\
 & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow & & \\
1 & \longrightarrow & C_0(\oplus Ku_\sigma/K) & \longrightarrow & C(\oplus Ku_\sigma/K) & \longrightarrow & Z^1(G, \mathrm{Pic}_K(K)) & & \text{(exact)}
\end{array}$$

$$\begin{array}{ccc}
Z^1(G, \mathrm{Pic}_0(B)) & \searrow & \\
\approx\downarrow & & H^3(G, U(K)) \\
Z^1(G, \mathrm{Pic}_K(K)) & \nearrow &
\end{array}$$

We shall further continue the study of the relation between $\Delta/B$ and $\oplus Ku_\sigma/K$ (with trivial factor set).

PROPOSITION 2.14. *There exists a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Pic}_K(K) & \longrightarrow & C(\oplus Ku_\sigma/K) \\
\downarrow & & \approx\downarrow \\
\mathrm{Pic}_K(B)^{[G]} & \longrightarrow & C(\Delta/B)
\end{array}$$

*Proof.* Let $[P_0] \in \mathrm{Pic}_K(k)$. It is necessary to prove that $(\oplus (P_0 \otimes_K Ku_\sigma \otimes_K {}^*P_0), {}^{P_0}\phi_{0,\sigma,\tau})$ corresponds to $(\oplus ((B \otimes_K P_0) \otimes J_\sigma \otimes (B \otimes_K {}^*P_0)), {}^P\phi_{\sigma,\tau})$ under the isomorphism $C(\oplus Ku_\sigma/K) \to C(\Delta/B)$, where $\phi_{0,\sigma,\tau}$ is the canonical isomorphism $Ku_\sigma \otimes_K Ku_\tau \to Ku_{\sigma\tau}$, $u_\sigma \otimes u_\tau \mapsto u_{\sigma\tau}$, $P = B \otimes_K P_0$, and ${}^*P_0 = \mathrm{Hom}_l(P_{0K}, K_K)$ (cf. the proof of Th. 2.6). However this is done by using

$Ku_\sigma \otimes_K {}^*P_0 \otimes_K Ku_{\sigma^{-1}} \xrightarrow{\approx} V_{J_\sigma \otimes {}^*P \otimes J_{\sigma^{-1}}}(B)$ and ${}^*P \xrightarrow{\approx} B \otimes_K {}^*P_0$ canonically (cf. the proof of Th. 2.13).

Next we define a homomorphism from $P_K(\oplus Ku_\sigma/K)^{(G)}$ to $P_K(\Delta/B)^{(G)}$. Let $\phi_0: P_0 \to M_0$ be in $P_K(\oplus Ku_\sigma/K)^{(G)}$. Then $Ku_\sigma \otimes_K P_0 \otimes_K Ku_{\sigma^{-1}} \xrightarrow{\approx} V_{J_\sigma \otimes P \otimes J_{\sigma^{-1}}}(B)$, as $K$-$K$-modules, $u_\sigma \otimes p_0 \otimes u_{\sigma^{-1}} \mapsto \sum_i a_{\sigma,i} \otimes (1 \otimes p_0) \otimes a'_{\sigma,i}$, where $P = B \otimes_K P_0$, $a_{\sigma,i} \in J_\sigma$, $a'_{\sigma,i} \in J_{\sigma^{-1}}$, $\sum_i a_{\sigma,i} a'_{\sigma,i} = 1$. Therefore $Ku_\sigma \otimes_K P_0 \otimes_K Ku_{\sigma^{-1}} \otimes_K J_\sigma \xrightarrow{\approx} J_\sigma \otimes P$, as $B$-$B$-modules, $u_\sigma \otimes p_0 \otimes u_{\sigma^{-1}} \otimes x_\sigma \mapsto x_\sigma \otimes (1 \otimes p_0)$ (cf. the proof of Prop. 2.9). Now, for the sake of simplicity, we may assume that $P_0 \subseteq M_0$. Then $u_\sigma P_0 u_{\sigma^{-1}} = P_0$ for all $\sigma \in G$. Then $P_0 \otimes_K J_\sigma \xrightarrow{\approx} J_\sigma \otimes_K P_0$, as $B$-$B$-modules, $u_\sigma p_0 u_{\sigma^{-1}} \otimes x_\sigma \mapsto x_\sigma \otimes p_0$, and this induces a $B$-$B$-isomorphism $P_0 \otimes_K \Delta \ (\xrightarrow{\approx} P \otimes \Delta) \xrightarrow{\approx} \Delta \otimes_K P_0 \ (\xrightarrow{\approx} \Delta \otimes P)$. Then, by Lemma 1.2, we have a Morita module ${}_\Delta \Delta \otimes_K P_{0\Delta}$, where $(x_\sigma \otimes p_0)x_\tau = x_\sigma x_\tau \otimes u_{\tau^{-1}} p_0 u_\tau$ $(x_\sigma \in J_\sigma, p_0 \in P_0, x_\tau \in J_\tau)$. Hence the canonical homomorphism $\phi: B \otimes_K P_0 = P \to \Delta \otimes_K P_0$ is in $P_K(\Delta/B)^{(G)}$. Let $\psi_0: Q_0 \to U_0$ be another element of $P_K(\oplus Ku_\sigma/K)^{(G)}$. Then $[\phi_0][\psi_0]: P_0 \otimes_K Q_0 \to M_0 \otimes' U_0$, $p_0 \otimes q_0 \mapsto \phi_0(p_0) \otimes \psi_0(q_0)$, where $\otimes'$ means the tensor product over $\oplus Ku_\sigma$. On the other hand, $[\phi][\psi]: (B \otimes_K P_0) \otimes (B \otimes_K Q_0) \to (\Delta \otimes_K P_0) \otimes_\Delta (\Delta \otimes_K Q_0)$ is the canonical map. Then it is easily seen that the canonical isomorphism $\Delta \otimes_K P_0 \otimes_K Q_0 \to (\Delta \otimes_K P_0) \otimes_\Delta (\Delta \otimes_K Q_0)$ is a $\Delta$-$\Delta$-isomorphism such that the diagram

$$\begin{array}{ccc} B \otimes_K P_0 \otimes_K Q_0 & \longrightarrow & \Delta \otimes_K P_0 \otimes_K Q_0 \\ \downarrow & & \downarrow \\ (B \otimes_K P_0) \otimes_B (B \otimes_K Q_0) & \longrightarrow & (\Delta \otimes_K P_0) \otimes_\Delta (\Delta \otimes_K Q_0) \end{array}$$

is commutative. Hence $\beta: [\phi_0] \mapsto [\phi]$ is a homomorphism from $P_K(\oplus Ku_\sigma/K)^{(G)}$ to $P_K(\Delta/B)^{(G)}$.

THEOREM 2.15. *There is a commutative diagram with exact rows:*

$$\begin{array}{ccccccc} U(K) & \longrightarrow & \mathrm{Aut}\,(\oplus Ku_\sigma/K)^{(G)} & \longrightarrow & P_K(\oplus Ku_\sigma/K)^{(G)} & \longrightarrow & \mathrm{Pic}_K(K)^G \\ \| & (1) & \alpha \downarrow \approx & (2) & \beta \downarrow & & \gamma \downarrow \\ U(K) & \longrightarrow & \mathrm{Aut}\,(\Delta/B)^{(G)} & \longrightarrow & P_K(\Delta/B)^{(G)} & \longrightarrow & \mathrm{Pic}_K(B)^G \end{array}$$

$$\begin{array}{ccccccccc} \longrightarrow & C_0(\oplus Ku_\sigma/K) & \longrightarrow & B(\oplus Ku_\sigma/K) & \longrightarrow & H^1(G, \mathrm{Pic}_K(K)) & \longrightarrow & H^3(G, U(K)) \\ & \downarrow \approx & & \delta \downarrow & & \varepsilon \downarrow & & \| \\ \longrightarrow & C_0(\Delta/B) & \longrightarrow & B(\Delta/B) & \longrightarrow & \bar{H}^1(G, \mathrm{Pic}_0(B)) & \longrightarrow & H^3(G, U(K)) \end{array}$$

*where $\alpha$ is* $\mathrm{Aut}\,(\oplus Ku_\sigma/K)^{(G)} \xrightarrow{\approx} Z^1(G, U(K)) \xrightarrow{\approx} \mathrm{Aut}\,(\Delta/B)^{(G)}$ *(cf. Remark to Th.* 2.12). *and $\beta$ is the homomorphism defined above.*

*Proof.* By Cor. to Prop. 2.9 and the definition of $\bar{H}^1(G, \mathrm{Pic}_0(B))$, $\varepsilon$ is surjective, and hence so is $\delta$. As $\gamma$ is injective, so is $\beta$, if (1) and (2) are commutative. Therefore it suffices to prove that (1) and (2) are commutative. However the commutativity of (1) is evident. To prove the commutativity of (2), let $\alpha(f_0) = f$. Then, for any $\sigma \in G$, there exists uniquely $c_\sigma \in U(K)$ such that $f(x_\sigma) = c_\sigma x_\sigma$ for all $x_\sigma \in J_\sigma$. Then $f_0(u_\sigma) = c_\sigma u_\sigma$ for all $\sigma \in G$, and so $(x_\sigma \otimes u_{f_0})x_\tau = x_\sigma x_\tau \otimes u_{\tau^{-1}} u_{f_0} u_\tau = x_\sigma x_\tau \otimes u_{\tau^{-1}} c_\tau u_\tau u_{f_0}$ $= x_\sigma x_\tau \otimes \tau^{-1}(c_\tau) u_{f_0} = x_\sigma \cdot f(x_\tau) \otimes u_{f_0}$ in $\Delta \otimes_K K u_{f_0}$, where $x_\sigma \in J_\sigma, x_\tau \in J_\tau$ (cf. the definition of $\beta$). This means that (2) is commutative.

THEOREM 2.16. *There exists a commutative diagram*

$$\begin{array}{ccccccc}
U(K) & \longrightarrow & \mathrm{Aut}\,(A/B)^{(G)} & \longrightarrow & P_K(A/B)^{(G)} & \longrightarrow & \mathrm{Pic}_K\,(B)^G \\
\Vert & (1) & \downarrow & (2) & \downarrow & (3) & \Vert \\
U(K) & \longrightarrow & \mathrm{Aut}\,(\Delta/B)^{(G)} & \longrightarrow & P_K(\Delta/B)^{(G)} & \longrightarrow & \mathrm{Pic}_K\,(B)^G \,.
\end{array}$$

*Proof.* Let $f$ be in $\mathrm{Aut}\,(A/B)^{(G)}$. Then $f(J_\sigma) = J_\sigma$ for all $\sigma \in G$, so $f$ induces canonically an automorphism of $\Delta/B = \oplus J_\sigma/B$. Then the commutativity of (1) is evident. Next we define a homomorphism $P_K(A/B)^{(G)} \to P_K(\Delta/B)^{(G)}$. Let $\phi: P \to M$ be in $P_K(A/B)^{(G)}$. For the sake of simplicity, we may assume that $P$ is a submodule of $M$. Then $J_\sigma P = J_\sigma \otimes {}_B P = PJ_\sigma = P \otimes {}_B J_\sigma$ in $M$ for all $\sigma \in G$. We construct $\oplus J_\sigma P$, formally. Then this is isomorphic to $\Delta \otimes {}_B P$ canonically, as $B$-$B$-modules. Similarly $\oplus PJ_\sigma \xrightarrow{\approx} P \otimes {}_B \Delta$. Since $J_\sigma P = PJ_\sigma$, we have an isomorphism $\Delta \otimes {}_B P \xrightarrow{\approx} P \otimes {}_B \Delta$, as $B$-$B$-modules. It is easily seen that this isomorphism satisfies the condition of Lemma 1.2. Thus $\bar{\phi}: P \to \Delta \otimes {}_B P$, $p \mapsto 1 \otimes p$ is in $P_K(\Delta/B)^{(G)}$. Let $\psi: Q \to U$ be another element in $P_K(A/B)^{(G)}$. Then $[\phi][\psi]: P \otimes {}_B Q \to M \otimes {}_A U$. On the other hand, we have $[\bar{\phi}][\bar{\psi}]: P \otimes {}_B Q \to (\Delta \otimes {}_B P) \otimes {}_\Delta (\Delta \otimes {}_B Q)$. Then it is easily seen that the canonical isomorphism $\Delta \otimes {}_B P \otimes {}_B Q \to (\Delta \otimes {}_B P) \otimes {}_\Delta (\Delta \otimes {}_B Q)$ is a $\Delta$-$\Delta$-isomorphism such that the diagram

$$\begin{array}{ccc}
 & \nearrow & \Delta \otimes {}_B P \otimes {}_B Q \\
P \otimes {}_B Q & & \downarrow \\
 & \searrow & (\Delta \otimes {}_B P) \otimes_\Delta (\Delta \otimes {}_B Q)
\end{array}$$

is commutative. Hence the mapping $[\phi] \mapsto [\bar{\phi}]$ is a group homomorphism. Finally, the commutativity of (2) is evident from the definition of the homomorphism $P_K(A/B)^{(G)} \to P_K(\Delta/B)^{(G)}$.

Evidently $1 \to \mathrm{Aut}(A/\Sigma J_\sigma) \to \mathrm{Aut}(A/B)^{(G)} \to \mathrm{Aut}(\Delta/B)^{(G)}$ is exact. Then the commutativity of Th. 2.16 implies that

$$\mathrm{Aut}(A/\Sigma J_\sigma) \longrightarrow P_K(A/B)^{(G)} \longrightarrow P_K(\Delta/B)^{(G)}$$

is exact. Thus we have

COROLLARY. *The following diagram is commutative, and two rows are exact*:

$$\begin{array}{ccccccccccc}
 & & & & & & & & & \nearrow & \mathrm{Aut}(\Delta/B)^{(G)} \\
 & & & & & & & & \mathrm{Aut}(A/B)^{(G)} & & \\
 & & & & & & & \nearrow & \downarrow & & \downarrow \\
1 & \to & U(L) \cap U(K) & \to & U(K^G) & \to & \mathrm{Aut}(A/\Sigma J_\sigma) & \to & P_K(A/B)^{(G)} & \to & P_K(\Delta/B)^{(G)} \\
 & & & & & \nearrow & & & & & \\
 & & & & 1 & & & & & &
\end{array}$$

*Remark*. If $L \subseteq K$ then $\mathrm{Aut}(A/B)^G$ is a subgroup of $\mathrm{Aut}(A/B)^{(G)}$. On the other hand, if $V_\Delta(B) = K$ then $\mathrm{Aut}(\Delta/B)^{(G)} = \mathrm{Aut}(\Delta/B)$, because $\mathrm{Hom}({}_BJ_{\sigma B}, {}_BJ_{\tau B}) = 0$ for any $\sigma \neq \tau$ (cf. [17; § 6]).

**§ 3.** In this section, $G$ is a group, and $B \supseteq T$ are rings with a common identity. We fix a group homomorphism $G \to \mathrm{Aut}_l(B/T)$ (the group of all $T$-automorphisms of $B/T$), $\sigma \mapsto \bar{\sigma}$, and we consider $B$ as a $G$-group. $K$ and $F$ are centers of $B$ and $T$, respectively. We put $\Delta_1 = \oplus_{\sigma \in G} Bu_\sigma/B$, which is a crossed product of $B$ and $G$ with trivial factor set: $u_\sigma u_\tau = u_{\sigma\tau}, u_\sigma b = \sigma(b)u_\sigma$. We denote by $C_1$ the center of $\Delta_1$. Then, applying Th. 2.12 in § 2 to this generalized crossed product, we obtain an exact sequence

$$\begin{aligned}
1 \longrightarrow U(C_1) \cap U(K) \longrightarrow U(K) \longrightarrow \mathrm{Aut}(\Delta_1/B)^{(G)} \longrightarrow P_K(\Delta_1/B)^{(G)} \\
\longrightarrow \mathrm{Pic}_K(B)^G \longrightarrow C_0(\Delta_1/B) \longrightarrow B(\Delta_1/B) \\
\longrightarrow \overline{H}^1(G, \mathrm{Pic}_0(B)) \longrightarrow H^3(G, U(K)) ,
\end{aligned}$$

where $\mathrm{Aut}(\Delta_1/B)^{(G)} \xrightarrow{\approx} Z^1(G, U(K))$ and $C_0(\Delta_1/B) \xrightarrow{\approx} H^2(G, U(K))$.

We begin this section with the following

PROPOSITION 3.1. *The following two exact sequences consist of $G$-homomorphisms*:

$$1 \longrightarrow U(K) \cap U(F) \longrightarrow U(K) \longrightarrow \mathfrak{G}(B/T) \longrightarrow P(B/T) \longrightarrow \operatorname{Pic}(B)\ ,$$
$$1 \longrightarrow U(F) \longrightarrow U(V_B(T)) \longrightarrow \mathfrak{G}(B/T) \longrightarrow \operatorname{Pic}(T)\ .$$

*Proof.* The exactness was proved in Th. 1.4 and Prop. 1.6. Canonically $\mathfrak{G}(B/T)$ is a $G$-group, and the homomorphism $G \to \operatorname{Aut}(B/T)$ induces a homomorphism $G \to \operatorname{Aut}(K)$, by restriction. By Th. 1.5, there is a homomorphism $\operatorname{Aut}(B/T) \to P(B/T)$, and this defines a $G$-group $P(B/T)$, by conjugation. Then it is evident that $P(B/T) \to \operatorname{Pic}(B)$ is a $G$-homomorphism. Next we shall show that $\mathfrak{G}(B/T) \to P(B/T)$ is a $G$-homomorphism. Let $\sigma \in \operatorname{Aut}(B/T)$, and $X \in \mathfrak{G}(B/T)$. Then $\sigma(X) \in \mathfrak{G}(B/T)$, and the image of $X$ in $P(B/T)$ is $\phi_X : X \to B, x \mapsto x$. On the other hand the image of $\sigma$ in $P(B/T)$ is $\phi_\sigma : T \to Bu_\sigma, t \mapsto tu_\sigma$. Then there is a commutative diagram

$$\begin{array}{ccc} T \otimes_T X \otimes {}_T T & \longrightarrow & Bu_\sigma \otimes {}_B B \otimes {}_B Bu_{\sigma^{-1}} \\ \sigma \downarrow \approx & & \alpha \downarrow \approx \\ \sigma(X) & \longrightarrow & B\ , \end{array}$$

where $\alpha$ is the canonical one. This shows that $\mathfrak{G}(B/T) \to P(B/T)$ is a $G$-homomorphism. It is easily seen that $U(V_B(T)) \to \mathfrak{G}(B/T)$, $d \mapsto Td$ is a $G$-homomorphism.

We denote by $\mathfrak{G}(B/T)^{(G)}$ the group $\{X \in \mathfrak{G}(B/T) \mid X(\bar\sigma) = \bar\sigma \text{ for all } \sigma \in G\}$, where $\bar\sigma$ denotes the image of $\sigma$ in $\operatorname{Aut}(B/T)$ (cf. Prop. 1.1). In § 1, we have seen that $\mathfrak{G}(B/T)^{(G)} = \{X \in \mathfrak{G}(B/T) \mid u(X, \bar\sigma, 1) \in K\} = \{X \in \mathfrak{G}(B/T) \mid$ for any $\sigma \in G$, there exists $c_\sigma \in U(K)$ such that $c_\sigma x = \sigma(x)$ for all $x \in X\}$. We denote by $P^K(B/T)^{(G)}$ the subgroup of $P^K(B/T)$ (cf. § 1), which consists of all $[\phi]$ satisfying (∗∗).

(∗∗) For any $\sigma \in G$, there exists a $B$-$B$-isomorphism $f_\sigma : M \to Bu_\sigma \otimes {}_B M \otimes {}_B Bu_{\sigma^{-1}}$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \phi\ } & M \\ & {}^{\sigma}\phi \searrow \quad \swarrow f_\sigma & \\ & Bu_\sigma \otimes {}_B M \otimes {}_B Bu_{\sigma^{-1}} & \end{array}$$

is commutative, where ${}^{\sigma}\phi$ is the map $p \mapsto u_\sigma \otimes \phi(p) \otimes u_{\sigma^{-1}} (p \in P)$. The proof that $P^K(B/T)^{(G)}$ is a subgroup is the following

PROPOSITION 3.2. *$P^K(B/T)^{(G)}$ is a subgroup of $P^K(B/T)$.*

*Proof.* Let $\phi: P \to M$ and $\psi: Q \to U$ be two representations of an element of $P^K(B/T)^{(G)}$, and let the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\psi} & U \\ {\scriptstyle\alpha}\big\downarrow{\scriptstyle\approx} & & {\scriptstyle\beta}\big\downarrow{\scriptstyle\approx} \\ P & \xrightarrow[\phi]{} & M \end{array}$$

be commutative, where $\alpha$ is a $T$-$T$-isomorphism, and $\beta$ is a $B$-$B$-isomorphism. For any $\sigma$ in $G$, there is a $B$-$B$-isomorphism $f_\sigma: M \to Bu_\sigma \otimes {}_BM \otimes {}_BBu_{\sigma^{-1}}$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi} & M \\ & {\scriptstyle{}^\sigma\phi}\searrow \quad \swarrow{\scriptstyle f_\sigma} & \\ & Bu_\sigma \otimes {}_BM \otimes {}_BBu_{\sigma^{-1}} & \end{array}$$

is commutative. Then a $B$-$B$-isomorphism $g_\sigma: U \to Bu_\sigma \otimes {}_BU \otimes {}_BBu_{\sigma^{-1}}$ is determined by the commutativity of the following diagram:

$$\begin{array}{ccccc} Q & \xrightarrow{\psi} & U & \overset{g_\sigma}{\dashrightarrow} & Bu_\sigma \otimes {}_BU \otimes {}_BBu_{\sigma^{-1}}\,, \\ {\scriptstyle\alpha}\big\downarrow{\scriptstyle\approx} & & {\scriptstyle\beta}\big\downarrow{\scriptstyle\approx} & & {\scriptstyle 1\otimes\beta\otimes 1}\big\downarrow{\scriptstyle\approx} \\ P & \xrightarrow[\phi]{} & M & \xrightarrow[f_\sigma]{} & Bu_\sigma \otimes {}_BM \otimes {}_BBu_{\sigma^{-1}} \end{array}$$

that is, $g_\sigma = (1 \otimes \beta \otimes 1)^{-1} f_\sigma \beta$. It is easily seen that $g_\sigma \psi(q) = u_\sigma \otimes \psi(q) \otimes u_{\sigma^{-1}} (q \in Q)$, and hence $P^K(B/T)^{(G)}$ is well defined. It is evident that $P^K(B/T)^{(G)}$ is closed under multiplication. Finally $f_\sigma: {}_BM_B \to {}_BBu_\sigma \otimes {}_BM \otimes {}_BBu_{\sigma^{-1}B}$ induces a $B$-$B$-isomorphism $\mathrm{Hom}_r({}_BM, {}_BB) \xrightarrow{\approx} \mathrm{Hom}_r({}_BBu_\sigma \otimes {}_BM \otimes {}_BBu_{\sigma^{-1}}, {}_BB)$, and there is a canonical $B$-$B$-isomorphism $Bu_\sigma \otimes {}_B\mathrm{Hom}_r({}_BM, {}_BB) \otimes {}_BBu_{\sigma^{-1}} \to \mathrm{Hom}_r({}_BBu_\sigma \otimes {}_BM \otimes {}_BBu_{\sigma^{-1}}, {}_BB), u_\sigma \otimes h \otimes u_{\sigma^{-1}} \mapsto (u_\sigma \otimes x \otimes u_{\sigma^{-1}} \to \sigma(x^h))(x \in M)$. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_r({}_TP, {}_TT) & \xrightarrow{\gamma} & \mathrm{Hom}_r({}_BM, {}_BB) \\ & {\scriptstyle{}^\sigma\gamma}\searrow \quad \swarrow{\scriptstyle\approx} & \\ & Bu_\sigma \otimes {}_B\mathrm{Hom}_r({}_BM, {}_BB) \otimes {}_BBu_{\sigma^{-1}} & \end{array}$$

where $\gamma$ is the canonical homomorphism $f \mapsto (\phi(p) \to p^f)\ (p \in P)$. This completes the proof.

THEOREM 3.3. *There is an exact sequence*

$$U(K) \longrightarrow \mathfrak{G}(B/T)^{(G)} \longrightarrow P^K(B/T)^{(G)} \longrightarrow \mathrm{Pic}_K(B)^G .$$

*Proof.* For $X$ in $\mathfrak{G}(B/T)$, the image of $X$ in $\mathrm{Pic}^K(B/T)$ is the canonical inclusion map $\phi: X \to B$. Then ${}^\sigma\phi$ is $X \to B, x \mapsto \sigma(x)$. Therefore $[\phi]$ is in $\mathrm{Pic}^K(B/T)^{(G)}$ if and only if, for any $\sigma \in G$, there is a $c_\sigma \in U(K)$ such that $c_\sigma x = \sigma(x)$ for all $x \in X$, that is, $X \in \mathfrak{G}(B/T)^{(G)}$. Then the exactness of the present sequence follows from Th. 1.4.

THEOREM 3.4. *There is a commutative diagram with exact rows:*

$$\begin{array}{ccccccc}
U(K) & \longrightarrow & \mathfrak{G}(B/T)^{(G)} & \longrightarrow & P^K(B/T)^{(G)} & \longrightarrow & \mathrm{Pic}_K(B)^G \\
\approx\big\downarrow\alpha & (1) & \big\downarrow\beta & (2) & \big\downarrow\gamma & & \big\| \\
U(K) & \longrightarrow & \mathrm{Aut}(\Delta_1/B)^{(G)} & \longrightarrow & P_K(\Delta_1/B)^{(G)} & \longrightarrow & \mathrm{Pic}_K(B)^G
\end{array}$$

*Proof.* The isomorphism $U(K) \xrightarrow{\alpha} U(K)$ is $c \mapsto c^{-1}$. Let $X \in \mathfrak{G}(B/T)^{(G)}$. Then, for any $\sigma$ in $G$, there exists uniquely $c_\sigma \in U(K)$ such that $c_\sigma x = \sigma(x)$ for all $x \in X$. If is easily seen that $c_{\sigma\tau} = c_\sigma \cdot \sigma(c_\tau)$ for all $\sigma, \tau \in G, c_1 = 1$. Then $c_\sigma(\sigma \in G)$ defines an automorphism $\rho: \sum_\sigma b_\sigma u_\sigma \mapsto \sum_\sigma b_\sigma c_\sigma u_\sigma$. We define $\mathfrak{G}(B/T)^{(G)} \xrightarrow{\beta} \mathrm{Aut}(\Delta_1/B)^{(G)}, X \mapsto \rho$. The commutativity of (1) is easily seen. Next we shall define $P^K(B/T)^{(G)} \xrightarrow{\gamma} P_K(\Delta_1/B)^{(G)}$. Let $\phi: P \to M$ be in $P^K(B/T)^{(G)}$. Then, for any $\sigma \in G$, there exists a $B$-$B$-isomorphism $f_\sigma: M \to Bu_\sigma \otimes_B M \otimes_B Bu_{\sigma^{-1}}$ such that $f_\sigma\phi = {}^\sigma\phi$. Then $f_\sigma$ induces an isomorphism $f'_\sigma: M \otimes_B Bu_\sigma \xrightarrow{f_\sigma \otimes 1} Bu_\sigma \otimes_B M \otimes_B Bu_{\sigma^{-1}} \otimes_B Bu_\sigma \xrightarrow{*} Bu_\sigma \otimes_B M$, where $*$ is induced by the canonical map $Bu_{\sigma^{-1}} \otimes_B Bu_\sigma \to B$. As is easily seen, $f'_\sigma(\phi(p) \otimes u_\sigma) = u_\sigma \otimes \phi(p)\ (p \in P)$. Taking direct sum, we have an isomorphism $\Delta_1 \otimes_B M \xrightarrow{\approx} M \otimes_B \Delta_1$, and it is easy to check that this isomorphism satisfies the condition of Lemma 1.2. Thus we have $\bar{\phi}: M \to \Delta_1 \otimes_B M, m \mapsto 1 \otimes m$, in $P_K(\Delta_1/B)^{(G)}$ (cf. § 2). Let $\psi: Q \to U$ be another element in $P^K(B/T)^{(G)}$. Then the canonical isomorphism $\Delta_1 \otimes_B M \otimes_B U \xrightarrow{\approx} (\Delta_1 \otimes_B M) \otimes_{\Delta_1} (\Delta_1 \otimes_B U)$ is a $\Delta_1$-$\Delta_1$-isomorphism such that the diagram

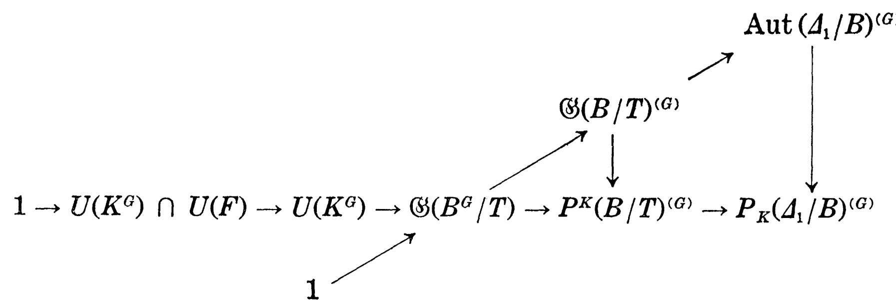
$$\begin{array}{ccc}
M \otimes_B U & \xrightarrow{\overline{\phi \otimes \psi}} & \Delta_1 \otimes_B M \otimes_B U \\
& \bar{\phi} \otimes \bar{\psi} \searrow & \big\downarrow \approx \\
& & (\Delta_1 \otimes_B M) \otimes_{\Delta_1} (\Delta_1 \otimes_B U)
\end{array}$$

is commutative. Hence the map $\phi \to \bar{\phi}$ is a homomorphism. Finally we shall show the commutativity of (2). Let $1 = \sum_i x'_i x_i\ (x'_i \in X^{-1},\ x_i \in X)$.

Then $\Delta_1 \otimes_B B \ni u_\sigma \otimes 1 = \sum_i u_\sigma x_i' \otimes x_i$, so $(u_\sigma \otimes 1)u_\tau = (\sum_i u_\sigma x_i' \otimes x_i)u_\tau = (\sum_i \sigma(x_i')u_\sigma \otimes x_i)u_\tau = \sum_i \sigma(x_i')u_\sigma u_\tau \otimes x_i = \sum_i u_\sigma x_i' u_\tau \otimes x_i = \sum_i u_\sigma x_i' u_\tau x_i \otimes 1 = \sum_i u_\sigma x_i' x_i c_\tau u_\tau \otimes 1 = u_\sigma \cdot \rho(u_\tau) \otimes 1$. Hence $\Delta_1 \otimes_B B \xrightarrow{\approx} \Delta_1 u_\rho, u_\sigma \otimes 1 \mapsto u_\sigma u_\rho$ is a $\Delta_1$-$\Delta_1$-isomorphism. Hence (2) is commutative. This completes the proof.

The next Cor. 1 is follows from Th. 3.4.

COROLLARY 1. *The following diagram is commutative, and two rows are exact:*




*where K and F are centers of B and T, respectively.*

COROLLARY 2. *If $B^G = T$ then two homomorphisms $\mathfrak{G}(B/T)^{(G)} \to$* Aut $(\Delta_1/B)^{(G)}$ *and $P^K(B/T)^{(G)} \to P_K(\Delta_1/B)^{(G)}$ are monomorphisms. Therefore, in this case, $\mathfrak{G}(B/T)^{(G)}$ is an abelian group.*

COROLLARY 3. *If $B/T$ is a finite G-Galois extension, then all vertical maps in* Th. 3.4 *are isomorphisms.*

*Proof.* It suffices to prove that $\gamma$ is surjective, by Cor. 2, Th. 1.4. and Th. 1.5, because the center of $\Delta_1$ is $F$ in this case. Let $\bar{\phi}: M \to \overline{M}$ be in $P_K(\Delta_1/B)^{(G)}$, and let $M \subseteq \overline{M}$. Then, $u_\sigma M = M u_\sigma$ $(\sigma \in G)$, and this yields a left $\Delta_1$-module $M$: $u_\sigma * m = u_\sigma m u_{\sigma^{-1}}$ $(m \in M, \sigma \in G)$. Then, by [8; Th. 1.3], $M = B \otimes_T M_0$, where $M_0 = \{m \in M \mid u_\sigma m = m u_\sigma$ for all $\sigma = G\}$. Similarly $M = M_0 \otimes_T B$, and the inclusion map $\phi: M_0 \to M$ is in $P^K(B/T)^{(G)}$, because ${}_T M_{0T} \xrightarrow{\approx} {}_T\mathrm{Hom}_r({}_{\Delta_1}B, {}_{\Delta_1}M)_T$ is a Morita module. By the proof of Th. 3.4, $\gamma(\phi) = \bar{\phi}$ is easily seen.

PROPOSITION 3.5. *If $V_B(T) = K$ then $\mathfrak{G}(B/T)^{(G)} = \mathfrak{G}(B/T)$.*

*Proof.* Let $X \in \mathfrak{G}(B/T)$, and let $1 = \sum_i a_i a_i' (a_i \in X, a_i' \in X^{-1})$, and $\sigma \in G$. Then $u = \sum_i a_i \cdot \sigma(a_i') \in V_B(T) = K$, and $u \cdot \sigma(x) = x$ for all $x \in X$ (cf. § 1).

## § 4. Morita invariance of the exact sequence in § 2.

In this section we shall cast a glance at the Morita invariance of the exact sequence in Th. 2.12. We fix two Morita modules ${}_AM_{A'} \supseteq {}_BP_{B'}$ such that $M = A \otimes_B P = P \otimes_{B'} A'$ (cf, [19]), where $B \subseteq A$ and $B' \subseteq A'$. We put $V_A(A) = L, V_{A'}(A') = L', V_B(B) = K$, and $V_{B'}(B') = K'$. There is an isomorphism $V_A(B) \to V_{A'}(B')$, $c \mapsto c'$ such that $cp = pc'$ for all $p \in P$, and this induces $L \xrightarrow{\approx} L'$ and $K \xrightarrow{\approx} K'$, by [19; Prop. 3.3]. Further, by [19; Th. 3.5], $\mathrm{Aut}\,(A/B) \xrightarrow{\approx} \mathrm{Aut}\,(A'/B')$, $\sigma \mapsto \sigma'$, where $\sum \sigma(a_i)p_i = \sum q_j \cdot \sigma'(a'_j)$ for all $\sum a_i p_i = \sum q_j a'_j (a_i \in A, p_i, q_j \in P, a'_j \in A')$ in $M$. Then it is evident the diagram

$$\begin{array}{ccc} U(V_A(B)) & \longrightarrow & \mathrm{Aut}\,(A/B) \\ \downarrow & & \downarrow \\ U(V_{A'}(B')) & \longrightarrow & \mathrm{Aut}\,(A'/B') \end{array}$$

is commutative. Let $\sigma \mapsto \sigma'$ under the isomorphism $\mathrm{Aut}\,(A/B) \to \mathrm{Aut}\,(A'/B')$. Then $Au_\sigma \otimes_A M \to M \otimes_{A'} A'u_{\sigma'}$, $u_\sigma \otimes p \mapsto p \otimes u_{\sigma'}$ $(p \in P)$ is an $A$-$A'$-isomorphism. Hence

$$\begin{array}{ccc} \mathrm{Aut}\,(A/B) & \longrightarrow & \mathrm{Pic}\,(A) \\ \downarrow & & \downarrow \\ \mathrm{Aut}\,(A'/B') & \longrightarrow & \mathrm{Pic}\,(A') \end{array}$$

is a commutative diagram, where $\mathrm{Pic}\,(A) \to \mathrm{Pic}\,(A'), [X] \mapsto [X']$ is the isomorphism such that $X \otimes_A M \xrightarrow{\approx} M \otimes_{A'} X'$ as $A$-$A'$-modules. There is an isomorphism $\mathfrak{S}(A/B) \to \mathfrak{S}(A'/B'), Y \mapsto Y'$ such that $YP = PY'$ (cf. [19; Prop. 3.3]). Then the following diagram is commutative:

$$\begin{array}{ccccc} U(V_A(B)) & \longrightarrow & \mathfrak{S}(A/B) & \longrightarrow & \mathrm{Pic}\,(B) \\ \approx\downarrow & & \approx\downarrow & & \approx\downarrow * \\ U(V_{A'}(B')) & \longrightarrow & \mathfrak{S}(A'/B') & \longrightarrow & \mathrm{Pic}\,(B') \end{array}$$

where $*: [W] \mapsto [W']$ is the isomorphism such that $W \otimes_B P \xrightarrow{\approx} P \otimes_{B'} W'$ as $B$-$B'$-modules. The isomorphism $P(A/B) \to P(A'/B')$, $\phi: Q \to U \mapsto \phi': Q' \to U'$ is defined by the commutativity of the diagram

$$\begin{array}{ccc} Q \otimes_B P & \underset{\alpha}{\overset{\approx}{\longleftarrow}} & P \otimes_{B'} Q' \\ \downarrow & & \downarrow \\ U \otimes_A M & \underset{\beta}{\overset{\approx}{\longleftarrow}} & M \otimes_{A'} U' \end{array}$$

for some $B$-$B'$-isomorphism $\alpha$ and some $A$-$A'$-isomorphism $\beta$. In fact, we put $Q' = \mathrm{Hom}_r({}_BP, {}_BB) \otimes_B Q \otimes_B P$ and $U' = \mathrm{Hom}_r({}_AM, {}_AA) \otimes_A U \otimes_A M$, and take the canonical isomorphisms $P \otimes_{B'} Q' \xrightarrow{\approx} Q \otimes_B P$ and $M \otimes_{A'} U' \xrightarrow{\approx} U \otimes_A M$. Then it is clear that the following diagrams are commutative:

$$\begin{array}{ccccc}
\mathrm{Aut}(A/B) & \longrightarrow & P(A/B) & \longrightarrow & \mathrm{Pic}(B) \\
\approx\downarrow & & \approx\downarrow & & \approx\downarrow \\
\mathrm{Aut}(A'/B') & \longrightarrow & P(A'/B') & \longrightarrow & \mathrm{Pic}(B')
\end{array}$$

$$\begin{array}{ccccc}
\mathfrak{G}(A/B) & \longrightarrow & P(A/B) & \longrightarrow & \mathrm{Pic}(A) \\
\approx\downarrow & & \approx\downarrow & & \approx\downarrow \\
\mathfrak{G}(A'/B') & \longrightarrow & P(A'/B') & \longrightarrow & \mathrm{Pic}(A')
\end{array}$$

We now fix a commutative diagram

$$\begin{array}{ccc}
 & \overset{J}{\nearrow} & \mathfrak{G}(A/B) \\
G & & \approx\downarrow \\
 & \underset{J'}{\searrow} & \mathfrak{G}(A'/B')
\end{array}$$

consisting of group homomorphisms. Put $\Delta = \oplus J_\sigma/B$ and $\Delta' = \oplus J'_\sigma/B'$. Then we have

THEOREM 4.1. *There exists a commutative diagram*

$$\begin{array}{ccccccccc}
U(K) & \longrightarrow & \mathrm{Aut}(\Delta/B)^{(G)} & \longrightarrow & P_K(\Delta/B)^{(G)} & \longrightarrow & \mathrm{Pic}_K(B)^G & \longrightarrow & C_0(\Delta/B) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
U(K') & \longrightarrow & \mathrm{Aut}(\Delta'/B')^{(G)} & \longrightarrow & P_{K'}(\Delta'/B')^{(G)} & \longrightarrow & \mathrm{Pic}_{K'}(B')^G & \longrightarrow & C_0(\Delta'/B')
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & B(\Delta/B) & \longrightarrow & \bar{H}^1(G, \mathrm{Pic}_0(B)) & \longrightarrow & H^3(G, U(K)) \\
 & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & B(\Delta'/B') & \longrightarrow & \bar{H}^1(G, \mathrm{Pic}_0(B')) & \longrightarrow & H^3(G, U(K'))
\end{array}$$

*where all vertical maps are isomorphisms.*

*Proof.* First we shall show that there is an isomorphism $C(\Delta/B) \xrightarrow{\approx} C(\Delta'/B'), \oplus U_\sigma/B \mapsto \oplus U'_\sigma/B'$. Put $P^* = \mathrm{Hom}_r({}_BP, {}_BB)$ and $P^* \otimes_B U_\sigma \otimes P = U'_\sigma$. Then, for any $\sigma \in G$, there is a canonical $B$-$B'$-isomorphism $f_\sigma \colon U_\sigma \otimes_B P \to P \otimes_{B'} P^* \otimes_B U_\sigma \otimes_B P = P \otimes_{B'} U'_\sigma$. The multiplicalion in $\oplus U'_\sigma/B$ is defined by the commutativity of the diagram

$$\begin{array}{ccccc} (U_\sigma \otimes_B U_\tau) \otimes_B P & \longrightarrow & U_\sigma \otimes_B P \otimes_{B'} U'_\tau & \longrightarrow & P \otimes_{B'} (U'_\sigma \otimes_{B'} U'_\tau) \\ \downarrow & & & & \downarrow \\ U_{\sigma\tau} \otimes_B P & & \longrightarrow & & P \otimes_{B'} U'_{\sigma\tau} \end{array}$$

The isomorphism $\oplus f_\sigma : (\oplus U_\sigma) \otimes_B P \to P \otimes_{B'} (\oplus U'_\sigma)$ satisfies the condition in Lemma 1.2, and $f_\sigma$ induces an isomorphism $U_\sigma \otimes_B P \to P \otimes_{B'} U'_\sigma$, that is, $\oplus U_\sigma / B$ and $\oplus U'_\sigma / B'$ defined above are equivalent as generalized crossed products. In particular, $\Delta / B$ and $\Delta' / B'$ are equivalent. The isomorphism $\mathrm{Pic}(B) \to \mathrm{Pic}(B')$ induces the isomorphism $\mathrm{Pic}_K(B)^{[G]} \to \mathrm{Pic}_{K'}(B')^{[G]}$, $[W] \mapsto [P^* \otimes_B W \otimes_B P]$, where $P^* = \mathrm{Hom}_r({}_BP, {}_BB)$. We put $W' = P^* \otimes_B W \otimes_B P$. Then $W^{*\prime} \xrightarrow{\approx} W'^*$ canonically, where $W'^* = \mathrm{Hom}_r({}_{B'}W', {}_{B'}B')$. Noting this fact, we can see that the diagram

$$\begin{array}{ccc} \mathrm{Pic}_K(B)^{[G]} & \longrightarrow & C(\Delta / B) \\ \downarrow & & \downarrow \\ \mathrm{Pic}_{K'}(B')^{[G]} & \longrightarrow & C(\Delta' / B') \end{array}$$

is commutative. The isomorphism $\mathrm{Pic}_0(B) \to \mathrm{Pic}_0(B')$ induces the isomorphism $Z^1(G, \mathrm{Pic}_0(B)) \to Z^1(G, \mathrm{Pic}_0(B'))$ (cf. Cor. to Prop. 2.9), and it is evident the diagram

$$\begin{array}{ccc} C(\Delta / B) & \longrightarrow & Z^1(G, \mathrm{Pic}_0(B)) \\ \downarrow & & \downarrow \\ C(\Delta' / B') & \longrightarrow & Z^1(G, \mathrm{Pic}_0(B')) \end{array}$$

is commutative. The facts that the isomorphism $P(\Delta / B) \to P(\Delta' / B')$ induces $P_K(\Delta / B)^{(G)} \xrightarrow{\approx} P_{K'}(\Delta' / B')^{(G)}$, and that the isomorphism $\mathrm{Aut}(\Delta / B) \to \mathrm{Aut}(\Delta' / B')$ induces $\mathrm{Aut}(\Delta / B)^{(G)} \xrightarrow{\approx} \mathrm{Aut}(\Delta' / B')^{(G)}$ are easily checked. After these remarks it is easy to complete the proof.

If we take a commutative diagram

$$\begin{array}{ccc} & \nearrow & \mathrm{Aut}(A / B) \\ G & & \downarrow \\ & \searrow & \mathrm{Aut}(A' / B') \end{array}$$

then each $g_\sigma : Au_\sigma \otimes_A M \to M \otimes_{A'} A'u'_\sigma$, $u_\sigma \otimes p \mapsto p \otimes u'_\sigma (p \in P)$ is an $A$-$A'$-isomorphism, and $\oplus g_\sigma : (\oplus Au_\sigma) \otimes_A M \to M \otimes_{A'} (\oplus A'u'_\sigma)$ satisfies the condition of Lemma 1.2, so that $\oplus Au_\sigma / B$ and $\oplus A'u'_\sigma / B'$ with trivial factor set are equivalent as generalized crossed products. Therefore Th. 4.1 is applicable to this case.

**§ 5.** In this section we fix a Morita module ${}_{A/B}M_{B^*/A^*}$ (cf. [19]) and a commutative diagram

$$\begin{array}{ccc} & \nearrow & \mathfrak{G}(A/B) \\ G & & \approx \downarrow \alpha \\ & \searrow & \mathrm{Aut}\,(B^*/A^*) \end{array}$$

of group homomorphisms, where $\alpha\colon X \mapsto \sigma$ is defined by $(xm)\cdot\sigma(b^*) = x(mb^*)(x \in X, m \in M, b^* \in B^*)$ (cf. [19; Th. 1.5]), and $A \supseteq B$ and $B^* \supseteq A^*$ are rings. For any $c$ in $V_A(B)$, there is a $c' \in V_{B^*}(A^*)$ such that $cm = mc'$ for all $m \in M$. Then the map $c \mapsto c'^{-1}$ is a group isomorphism $U(V_A(B)) \to U(V_{B^*}(A^*))$, and this induces isomorphisms $U(K) \to U(K^*)$, $U(L) \to U(L^*)$, where $K = V_B(B), K^* = V_{B^*}(B^*), L = V_A(A)$, and $L^* = V_{A^*}(A^*)$. The following diagram is commutative:

$$\begin{array}{ccc} U(V_A(B)) & \longrightarrow & \mathrm{Aut}\,(A/B) \\ \downarrow \text{(inverse)} & & \uparrow \alpha^* \\ U(V_{B^*}(A^*)) & \longrightarrow & \mathfrak{G}(B^*/A^*) \end{array}$$

where $\alpha^*: X^* \mapsto \sigma^*$ is defined by $(\sigma^*(a)m)x^* = a(mx^*)\,(x^* \in X^*, m \in M, a \in A)$, or equivalently, $\sigma^*(a)(my^*) = (am)y^*(y^* \in X^{*-1})$.

PROPOSITION 5.1. $\mathrm{Aut}\,(A/B)^{(G)} \xrightarrow{\approx} \mathfrak{G}(B^*/A^*)^{(G)}$.

*Proof.* Let $X \mapsto \sigma$ under the isomorphism $\mathfrak{G}(A/B) \to \mathrm{Aut}\,(B^*/A^*)$, and let $\sigma^* \mapsto X^*$ under the isomorphism $\mathrm{Aut}\,(A/B) \to \mathfrak{G}(B^*/A^*)$. Then it suffices to prove that $X(\sigma^*) \mapsto \sigma(X^*)$ under $\mathrm{Aut}\,(A/B) \to \mathfrak{G}(B^*/A^*)$. Let $\tau \leftrightarrow \sigma(X^*)$ under $\mathrm{Aut}\,(A/B) \to \mathfrak{G}(B^*/A^*)$. There is a $u \in U(V_A(B))$ such that $X(\sigma^*)(a) = u\cdot\sigma^*(a)u^{-1}$ $(a \in A)$ (cf. § 1). Then $u\cdot\sigma^*(x) = x$ for all $x \in X$, and so $u\cdot\sigma^*(x)m = xm$ for all $m \in M$. Let $y^* \in X^{*-1}$. Then $(xm)\cdot\sigma(y^*) = x(my^*) = u\cdot\sigma^*(x)(my^*) = u((xm)y^*) = (xm)y^*u'$, so that $\sigma(y^*) = y^*u'$ for all $y^* \in X^{*-1}$, where $um = mu'$ for all $m \in M$. Then, for any $a \in A$, $\tau(a)(m \cdot \sigma(y^*)) = (am)\cdot\sigma(y^*) = (am)y^*u' = u((am)y^*) = u\cdot\sigma^*(a)(my^*) = u\cdot\sigma^*(a)u^{-1}\cdot u(my^*)$. But $u(my^*) = my^*u' = m\cdot\sigma(y^*)$. Hence $\tau(a) = X(\sigma^*)(a)$ for all $a \in A$.

PROPOSITION 5.2. *There is an isomorphism* $P(A/B) \xrightarrow{\approx} P(B^*/A^*)$.

*Proof.* Let $\phi: P \to N$ be in $P(A/B)$. Put ${}_{B^*}P'_{B^*} = \mathrm{Hom}_r\,({}_BM, {}_BB) \otimes {}_BP \otimes {}_BM$ and ${}_{A^*}N'_{A^*} = \mathrm{Hom}_r\,({}_AM, {}_AA) \otimes {}_AN \otimes {}_AM$. Then there are canonical isomorphisms ${}_BM \otimes {}_{B^*}P'_{B^*} \to {}_BP \otimes {}_BM_{B^*}$ and ${}_AM \otimes {}_{A^*}N'_{A^*} \to {}_AN \otimes {}_AM_{A^*}$. Then $\phi': N' \to P'$ in $P_{K^*}(B^*/A^*)$ is defined by the commutativity of

$$\begin{array}{ccc} M \otimes {}_{B^*}P' & \xrightarrow{\approx} & P \otimes {}_BM \\ \approx\Big\uparrow 1\otimes\phi' & & \Big\downarrow \phi\otimes 1 \\ M \otimes {}_{A^*}N' & \xrightarrow{\approx} & N \otimes {}_AM \end{array}$$

Let $\psi: Q \to U$ be another element in $P(A/B)$, and $\psi': U' \to Q'$ is the one defined by $\psi$. Then the following diagram is commutative:

$$\begin{array}{ccccc} M \otimes {}_{B^*}P' \otimes {}_{B^*}Q' & \longrightarrow & P \otimes {}_BM \otimes {}_{B^*}Q' & \longrightarrow & P \otimes {}_BQ \otimes {}_BM \\ \approx\Big\uparrow & & \approx\Big\uparrow & & \approx\Big\uparrow \\ M \otimes {}_{A^*}N' \otimes {}_{A^*}U' & \longrightarrow & N \otimes {}_AM \otimes {}_{A^*}U' & \longrightarrow & N \otimes {}_AU \otimes {}_AM \end{array}$$

On the other hand we have a diagram

$$\begin{array}{ccccc} M \otimes {}_{B^*}(P \otimes {}_BQ)' & \xrightarrow{*} & M \otimes {}_{B^*}P' \otimes {}_{B^*}Q' & \longrightarrow & P \otimes {}_BQ \otimes {}_BM \\ \Big\uparrow & (1) & \Big\uparrow & (2) & \Big\uparrow \\ M \otimes {}_{A^*}(N \otimes {}_AU)' & \longrightarrow & M \otimes {}_{A^*}N' \otimes {}_{A^*}U' & \longrightarrow & N \otimes {}_AU \otimes {}_AM \end{array}$$

where (2) and (1) + (2) are commutative, and $*$ is induced by $(P \otimes {}_BQ)' \xrightarrow{\approx} P' \otimes {}_{B^*}Q'$. Hence (1) is commutative, and this proves that the map $[\phi] \mapsto [\phi']$ is a homomorphism. Similarly we can define a homomorphism $P(B^*/A^*) \to P(A/B)$. Hence $P(A/B) \xrightarrow{\approx} P(B^*/A^*), [\phi] \mapsto [\phi']$.

THEOREM 5.3. *$\oplus J_\sigma/B$ and $\oplus B^*u_\sigma/B^*$ are equivalent by ${}_BM_{B^*}$, as generalized crossed products. Therefore* Th. 4.1 *is applicable to this case.*

*Proof.* For any $\sigma$ in $G$, the map $J_\sigma \otimes {}_BM \to M \otimes {}_{B^*}B^*u_\sigma$, $x \otimes m \mapsto xm \otimes u_\sigma$ is a $B$-$B^*$-isomorphism, and the following diagram is commutative:

$$\begin{array}{ccccc} J_\sigma \otimes {}_BJ_\tau \otimes {}_BM & \longrightarrow & J_\sigma \otimes {}_BM \otimes {}_{B^*}B^*u_\tau & \longrightarrow & M \otimes {}_{B^*}B^*u_\sigma \otimes {}_{B^*}B^*u_\tau \\ \Big\downarrow & & & & \Big\downarrow \\ J_{\sigma\tau} \otimes {}_BM & & \longrightarrow & & M \otimes {}_{B^*}B^*u_{\sigma\tau} \end{array}$$

THEOREM 5.4. *There is a commutative diagram*

$$\begin{array}{ccccccc} U(K) & \longrightarrow & \mathrm{Aut}\,(A/B)^{(G)} & \longrightarrow & P_K(A/B)^{(G)} & \longrightarrow & \mathrm{Pic}_K\,(B) \\ \approx\downarrow & (1) & \approx\downarrow & (2) & \approx\downarrow & (3) & \approx\downarrow \\ U(K^*) & \longrightarrow & \mathfrak{G}(B^*/A^*)^{(G)} & \longrightarrow & P^{K^*}(B^*/A^*)^{(G)} & \longrightarrow & \mathrm{Pic}_{K^*}\,(B^*) \end{array}$$

*Proof.* It suffices to prove that $P(A/B) \xrightarrow{\approx} P(B^*/A^*)$ induces $P_K(A/B)^{(G)} \xrightarrow{\approx} P^{K^*}(B^*/A^*)^{(G)}$, and that (1), (2), (3) are commutative. Now, $J_\sigma \otimes {}_B M \xrightarrow{\approx} M \otimes {}_{B^*} B^* u_\sigma$, $x \otimes m \mapsto xm \otimes u_\sigma$, as $B$-$B^*$-modules. Let $\phi: P \to N$ be in $P_K(A/B)^{(G)}$. Then, for any $\sigma$ in $G$, there exists an isomorphism $f_\sigma: {}_B J_\sigma \otimes {}_B P \otimes {}_B J_{\sigma^{-1} B} \to {}_B P_B$ such that

$$\begin{array}{ccc} J_\sigma \otimes {}_B P \otimes {}_B J_{\sigma^{-1}} \otimes {}_B M & \xrightarrow{f_\sigma \otimes 1} & P \otimes {}_B M \\ & {}^{\sigma}\phi\otimes 1 \searrow & \uparrow \phi\otimes 1 \\ & & N \otimes {}_A M \end{array}$$

is commutative. Then a $B^*$-$B^*$-isomorphism $f'_\sigma: P' \to B^* u_\sigma \otimes {}_{B^*} P' \otimes {}_{B^*} B^* u_{\sigma^{-1}}$ is defined by the commutativity of

$$\begin{array}{ccc} M \otimes {}_{B^*} B^* u_\sigma \otimes {}_{B^*} P' \otimes {}_{B^*} B^* u_{\sigma^{-1}} & \xleftarrow{1 \otimes f'_\sigma} & M \otimes {}_{B^*} P' \\ & 1 \otimes {}^{\sigma}\phi' \nwarrow & \uparrow 1 \otimes \phi' \\ & & M \otimes {}_{A^*} N' \end{array}$$

Thus $[\phi']$ is in $P^{K^*}(B^*/A^*)^{(G)}$, and hence $P_K(A/B)^{(G)} \xrightarrow{\approx} P^{K^*}(B^*/A^*)^{(G)}$. The commutativity of (1) and (3) is easily seen. To prove the commutativity of (2), let $\sigma \in \mathrm{Aut}\,(A/B)^{(G)}$, and $\sigma \mapsto X$ under the isomorphism $\mathrm{Aut}\,(A/B)^{(G)} \to \mathfrak{G}(B^*/A^*)^{(G)}$. Then $MX = M \otimes {}_{A^*} X \xrightarrow{\approx} Au_\sigma \otimes {}_A M$, $m \otimes x \mapsto u_\sigma \otimes mx$ is an $A$-$A^*$-isomorphism. And it is easy to see that the diagram

$$\begin{array}{ccc} M \otimes {}_{A^*} X & \xrightarrow{\approx} & Au_\sigma \otimes {}_A M \\ \downarrow & & \uparrow \\ M \otimes {}_{B^*} B^* & \xrightarrow{\approx} & B \otimes {}_B M \end{array}$$

is commutative. Hence (2) is commutative. This completes the proof.

**§6.** PROPOSITION 6.1. *If $B/T$ is a trivial finite $G$-Galois extension then $P_K(\Delta_1/B)^{(G)} \to \mathrm{Pic}_K\,(B)^G \to 1$ is exact and splits, where $\Delta_1$ is a crossed product of $B$ and $G$ with trivial factor set* (*Cf.* [16; *Cor.* 2].)

*Proof.* $B$ is the direct sum of $(G: 1)$ copies of $T$. Put $e_\sigma = (0, \cdots, 0, 1, 0, \cdots, 0)$ (the $\sigma$-component is 1). Then $\sum_\sigma e_\sigma = 1$, $e_\sigma e_\tau = \delta_{\sigma,\tau} e_\sigma$, and $B = \sum \oplus Te_\sigma$. The operation of $G$ on $B$ is given by $\tau(e_\sigma) = e_{\tau\sigma}$. Let $[P] \in \mathrm{Pic}_K(B)^G$. Then ${}_BBu_\sigma \otimes {}_BP_B \xrightarrow{\approx} {}_BP \otimes {}_BBu_{\sigma B}$ for all $\sigma \in G$. Multiplying $e_1$ on the right, we have ${}_BBu_\sigma e_1 \otimes {}_Be_1P_B \xrightarrow{\approx} {}_BPe_\sigma \otimes {}_Be_\sigma Bu_{\sigma B}$ for all $\sigma \in G$. Hence $h_\sigma: {}_Te_1P_T \xrightarrow{\approx} {}_Te_\sigma P_T$ for all $\sigma \in G$, because ${}_Te_\sigma B_T = {}_Te_\sigma T_T \xrightarrow{\approx} {}_TT_T$, $e_\sigma t \mapsto t (t \in T)$. It is easily seen that $[e_1P] \in \mathrm{Pic}_F(T)$, where $F$ is the center of $T$. Put $e_1P = P_0$, and let $(P_0)_G$ be the module of all $G \times G$ matrices over $P_0$, and let $P'$ be its diagonal part. Then it is evident that $(P_0)_G$ is canonically a two-sided $(T)_G$-Morita module, where $(T)_G$ is the ring of all $G \times G$ matrices over $T$. Indifying $B$ with the diagonal part of $(T)_G$, ${}_BP'_B$ is isomorphic to ${}_BP_B$. And $(T)_G \otimes {}_BP' \xrightarrow{\approx} (P_0)_G$ as left $(T)_G$, right $B$-modules, canonically. Since $e_\sigma (\sigma \in G)$ is a basis for $B_T$, $\Delta_1 = \mathrm{Hom}_l(B_T, B_T) \xrightarrow{\approx} (T)_G$. Then we can easily see that the canonical map $P' \to (T)_G \otimes {}_BP'$ is in $P_K((T)_G/B)^{(G)}$.

PROPOSITION 6.2. *If $\Delta/B$ is a group ring then the sequence $P_K(\Delta/B) \to \mathrm{Pic}_K(B) \to 1$ is exact, and splits.*

*Proof.* Let $[P] \in \mathrm{Pic}_K(B)$. Then there is a $B$-$B$-isomorphism $BG \otimes {}_BP \to P \otimes {}_BBG$, $\sigma \otimes p \mapsto p \otimes \sigma (\sigma \in G)$, and this isomorphism satisfies the condition in Lemma 1.2.

*Remark.* The above proposition can be generalized to the case that $\Delta = \sum \oplus Bu_\sigma$, $u_\sigma b = bu_\sigma (b \in B)$, $u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}$ with $a_{\sigma,\tau} \in U(K)$. The proof is analogous to the above one.

PROPOSITION 6.3. *Let $A, B, L$, and $K$ be rings as in § 2, and fix a group homomorphism $J: G \to \mathfrak{G}(A/B)$. Suppose that $B/K$ is separable and that $K \subseteq L$. Then*

$$P_K(A/B)^{(G)} \xrightarrow{\approx} \mathrm{Aut}(A/B)^{(G)} \times \mathrm{Pic}_K(K),$$

*and this induces*

$$P^L(A/B)^{(G)} \xrightarrow{\approx} \mathrm{Aut}(A/B \cdot L)^{(G)} \times \mathrm{Pic}_K(K).$$

*Proof.* Let $\phi: P \to M$ be in $P_K(A/B)$. Then there is an automorphism $f$ of $V_A(B)/K$ such that $f(c)\phi(p) = \phi(p)c$ for any $c \in V_A(B), p \in P$, and the map $[\phi] \mapsto f$ is a group homomorphism from $P_K(A/B)$ to $\mathrm{Aut}(V_A(B)/K)$ (cf. [19; Prop. 3.3]). Then the map $\mathrm{Aut}(A/B) \to P_K(A/B) \to \mathrm{Aut}(V_A(B)/K)$

is the restriction to $V_A(B)$. Let $U$ be a $B$-$B$-module such that $bu = ub$ for all $b \in K, u \in U$. Put $B^e = B \otimes_K B^{0P}$. Then $U$ may be considered as a left $B^e$-module. By [14; Th. 1.1], ${}_{B^e}U \xrightarrow{\approx} \mathrm{Hom}_r({}_{B^e}B^e, {}_{B^e}B) \otimes_K \mathrm{Hom}_r({}_{B^e}B, {}_{B^e}U)$, and so $U = B \otimes_K V_U(B)$. In particular, $A = B \otimes_K V_A(B)$. Hence $\mathrm{Aut}(A/B) \xrightarrow{\approx} \mathrm{Aut}(V_A(B)/K)$ by restriction. Let $\bar{f} \mid V_A(B) = f$, and assume that $\phi \in P_K(A/B)^{(G)}$. Then $J_\sigma \cdot \phi(P) = \phi(P)J_\sigma = \bar{f}(J_\sigma)\phi(P)$, because $J_\sigma = B \cdot V_{J_\sigma}(B)$. Hence $\bar{f}(J_\sigma) = J_\sigma$ for all $\sigma \in G$. Therefore the image of $\phi$ in $\mathrm{Aut}(A/B)$ belongs to $\mathrm{Aut}(A/B)^{(G)}$. Hence the map $\mathrm{Aut}(A/B)^{(G)} \to P_K(A/B)^{(G)} \to \mathrm{Aut}(A/B)^{(G)}$ is the identity map. Combining this with Prop. 2.2, we know that $P_K(A/B)^{(G)} \xrightarrow{\approx} \mathrm{Aut}(A/B)^{(G)} \times \mathrm{Im}\,\alpha$, where $\alpha: P_K(A/B)^{(G)} \to \mathrm{Pic}_K(B)^G$ is the one as in Prop. 2.2. By Remark to Lemma 2.4, $\mathrm{Pic}_K(K) \xrightarrow{\approx} \mathrm{Pic}_K(B)$, $[P_0] \mapsto [B \otimes_K P_0]$. Then the canonical map $B \otimes_K P_0 \to A \otimes_K P_0$ is in $P_K(A/B)^{(G)}$. Therefore $\mathrm{Im}\,\alpha \xrightarrow{\approx} \mathrm{Pic}_K(K)$. Thus we have the first assertion. The second assertion is obvious.

COROLLARY. *Let $L \supseteq K$ be commutative rings, and we fix a group homomorphism $G \to \mathrm{Aut}(L/K)$. Then*

$$P^L(L/K)^{(G)} = P^L(L/K) \xrightarrow{\approx} \mathrm{Pic}_K(K)\,. \quad (\text{cf. } \S\, 3)$$

*Proof.* Let $\sigma \in G$. Then, for any $[P_0] \in \mathrm{Pic}_K(K)$, $(Lu_\sigma \otimes_K P_0) \otimes_L Lu_{\sigma^{-1}} \xrightarrow{\approx} L \otimes_K P_0$, $xu_\sigma \otimes p_0 \otimes u_{\sigma^{-1}}y \mapsto xy \otimes p_0$, as $L$-$L$-modules.

*Remark.* By the above Cor, the sequence

$$\mathfrak{G}(L/K)^{(G)} \longrightarrow P^L(L/K)^{(G)} \longrightarrow \mathrm{Pic}_L(L)^G$$

is isomorphic to

$$\mathfrak{G}(L/K)^{(G)} \longrightarrow \mathrm{Pic}_K(K) \longrightarrow \mathrm{Pic}_L(L)^G\,.$$

(Cf. Th. 3.4, [8], and [16].)

PROPOSITION 6.4. *Let $A \supseteq B$ be rings, and $L$ the center of $A$. Assume that $A \otimes_L V_A(B) \mid A$ as left $A$, right $V_A(B)$-modules, and $V_A(V_A(B)) = B$. Then*

$$P^L(A/B) \xrightarrow{\approx} \mathfrak{G}(A/B) \times Im\,\alpha$$

*where $\alpha: P^L(A/B) \to \mathrm{Pic}_L(A)$ is the one as in Th.* 3.4. (*Cf.* [14], [19].)

*Proof.* By [19; Th. 1.4], $\mathrm{Aut}(V_A(B)/L) \xrightarrow{\approx} \mathfrak{G}(A/B)$, and the map

$$\mathfrak{G}(A/B) \longrightarrow P^L(A/B) \longrightarrow \mathrm{Aut}\,(V_A(B)/L) \xrightarrow{\approx} \mathfrak{G}(A/B)$$

is the identity (cf. [19; Prop. 3.3]). Then, by Th. 1.4, we can complete the proof.

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