H. Morikawa
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A NOTE ON HOLOMORPHIC VECTOR BUNDLES OVER QUOTIENT MANIFOLDS WITH RESPECT TO NILPOTENT GROUPS

HISASI MORIKAWA

1. A holomorphic vector bundle E over a complex analytic manifold $\mathscr{D}$ is said to be simple, if its global endomorphism ring $\mathrm{End}_C(E)$ is isomorphic to C. Projectifying the fibers of E, we get the associated projective bundle $P(E)$ of E. If we can choose a system of constant transition functions of $P(E)$, the projective bundle $P(E)$ is said to be locally flat.

In the present note we shall prove the following the theorem:

THEOREM 1. *Let Γ be a finitely generated nilpotent subgroup in the group of automorphisms of a complex analytic manifold $\mathscr{D}$. Assume that Γ acts properly discontinuously on $\mathscr{D}$ without fixed points. Let E be a holomorphic vector bundle over the quotient manifold $\mathscr{D}/\Gamma$ such that* i) *the inverse image of E with respect to the natural map $\mathscr{D} \to \mathscr{D}/\Gamma$ is trivial,* ii) *the associated projective bundle $P(E)$ is locally flat and* iii) *E is simple. Then there exists a subgroup Δ of finite index in Γ and a line bundle L over the quotient $\mathscr{D}/\Delta$ such that E is isomorphic to the direct image of L with respect to the natural map $\mathscr{D}/\Delta \to \mathscr{D}/\Gamma$.*

A complex nilmanifold is defined as the quotient of simply connected nilpotent complex Lie group G with respect to a discrete subgroup Γ of G. The finiteness of $\dim G$ implies the finite generation of Γ, and G is biholomorphic to a complex vector space. Hence, applying Theorem 1 to $\mathscr{D} = G$, we conclude that

THEOREM 2. *Let Γ be a discrete subgroup in a simply connected nilpotent complex Lie group G. Let E be a holomorphic vector bundle*

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over the nilmanifold G/Γ such that i) *the associated projective bundle $P(E)$ is locally flat and* ii) *E is simple. Then there exists a subgroup Δ of finite index in Γ and a line bundle L over G/Δ such that E is isomorphic to the direct image of L with respect to the natural map $G/\Delta \to G/\Gamma$.*

2. We need two algebraic lemmas.

LEMMA 1. *Let Γ be a finitely generated nilpotent group and let Z be its center. If the exponent of Z is finite, then Γ is a finite group.*

Proof. First we show that the exponent of Γ is finite. Denote by

$$Z^{(r)} = \Gamma \supset Z^{(r-1)} \supset \cdots \supset Z^{(1)} \supset Z^{(0)} = \{1\}$$

the upper central series of Γ. By the assumption the exponent of $Z^{(1)}/Z^{(0)}$ is finite. Assume that the exponent of $Z^{(s)}/Z^{(s-1)}$ is finite, say n. Since $(\Gamma, Z^{(s+1)}) \subset Z^{(s)}$ and $(\Gamma, Z^{(s)}) \subset Z^{(s-1)}$, it follows that for $a \in Z^{(s+1)}$ and $b \in \Gamma$

$$a^{-1}b^{-1}a = (a, b)b^{-1}, \qquad (a, b) \in Z^{(s)},$$
$$a^{-1}(a, b)a \equiv (a, b) \qquad \bmod Z^{(s-1)}.$$

Hence

$$a^{-n}b^{-1}a^n \equiv (a, b)^n b^{-1} \equiv b^{-1} \qquad \bmod Z^{(s-1)},$$

and thus

$$a^n b \equiv b a^n \qquad \bmod Z^{(s-1)}.$$

This means that $a^n \in Z^{(s)}$ for $a \in Z^{(s+1)}$ and the exponent of $Z^{(s+1)}/Z^{(s)}$ is finite. Therefore the exponents of $Z^{(s)}/Z^{(s-1)}$ $(1 \leq s \leq r)$ are finite and consequently the exponent of Γ is finite. To prove the finiteness of the order of Γ, we need the lower central series

$$\Gamma = \Gamma_{(0)} \supset \Gamma_{(1)} \supset \cdots \supset \Gamma_n = \{1\}.$$

Since $\Gamma/\Gamma_{(1)}$ is a finitely generated abelian group and its exponent is finite, the group $\Gamma/\Gamma_{(1)}$ is a finite group. Assume that $\Gamma/\Gamma_{(s)}$ is a finite group. It is enough to show that $\Gamma/\Gamma_{(s+1)}$ is also a finite group. Let $\{\bar{a}_1, \cdots, \bar{a}_m\} = \Gamma/\Gamma_{(s)}$ and $\{\bar{b}_1, \cdots, \bar{b}_l\} = \Gamma_{(s-1)}/\Gamma_{(s)}$. Let $\{a_1, \cdots, a_m\}$ and $\{b_1, \cdots, b_l\}$ be representatives of $\{\bar{a}_1, \cdots, \bar{a}_m\}$ and $\{\bar{b}_1, \cdots, \bar{b}_l\}$ in $\Gamma/\Gamma_{(s+1)}$. Since $\Gamma_{(s)}/\Gamma_{(s+1)}$ is contained in the center of $\Gamma/\Gamma_{(s+1)}$, the commutators (a_i, b_j) $(1 \leq i \leq m, 1 \leq j \leq l)$ do not depends on the choice of the representatives. This shows that $\Gamma_{(s)}/\Gamma_{(s+1)}$ is an abelian group generated by (a_i, b_j) $(1 \leq i \leq m, 1 \leq j \leq l)$ and its exponent is finite. Hence $\Gamma_{(s)}/\Gamma_{(s+1)}$ is a finite group, and thus $\Gamma/\Gamma_{(s+1)}$ is a finite group. This completes the proof of Lemma 1.

LEMMA 2. *Let $\tilde{\Gamma}$ be a nilpotent subgroup in $GL(n, C)$ and let $\tilde{Z}$ be its center. Assume that $\tilde{\Gamma}/\tilde{Z}$ is finitely generated and the commutor of $\tilde{\Gamma}$ in $(C)_{n\times n}$ consists of scalar matrices. Then* i) *$\tilde{\Gamma}/\tilde{Z}$ is a finite group,* ii) *$\tilde{\Gamma}$ is an irreducible matric group and* iii) *$\tilde{\Gamma}$ is equivalent to a matric group whose elements are monomial matrices.*

Proof. Denote by

$$\tilde{Z}^{(r)} = \tilde{\Gamma} \supset \tilde{Z}^{(r-1)} \supset \cdots \supset \tilde{Z}^{(2)} \supset \tilde{Z}^{(1)} \supset \tilde{Z}^{(0)} = \{I\}$$

the upper central series of $\tilde{\Gamma}$. We mean by $\chi(\tilde{\alpha}, \tilde{a})$ $(\tilde{\alpha} \in \tilde{\Gamma}, \tilde{a} \in \tilde{Z}^{(2)})$ the scalars such that

$$(\tilde{\alpha}, \tilde{a}) = \chi(\tilde{\alpha}, \tilde{a})I \qquad (\tilde{\alpha} \in \tilde{\Gamma}, \tilde{a} \in \tilde{Z}^{(2)}) .$$

Since

$$\begin{aligned}(\tilde{\alpha}\tilde{\beta}, \tilde{a}) &= \tilde{\beta}^{-1}(\tilde{\alpha}, \tilde{a})\tilde{\beta}(\tilde{\beta}, \tilde{a}) , \\ (\tilde{\alpha}, \tilde{a}\tilde{b}) &= (\tilde{\alpha}, \tilde{b})\tilde{b}^{-1}(\tilde{\alpha}, \tilde{a})\tilde{b}\end{aligned}$$

and

$$\det(\tilde{\alpha}, \tilde{a}) = 1 \qquad (\tilde{\alpha}, \tilde{\beta} \in \tilde{\Gamma}\,;\, \tilde{a}, \tilde{b} \in \tilde{Z}^{(2)}) ,$$

it follows that

$$\begin{aligned}\chi(\tilde{\alpha}\tilde{\beta}, \tilde{a}) &= \chi(\tilde{\alpha}, \tilde{a})\chi(\tilde{\beta}, \tilde{a}) , \\ \chi(\tilde{\alpha}, \tilde{a}\tilde{b}) &= \chi(\tilde{\alpha}, \tilde{a})\chi(\tilde{\alpha}, \tilde{b}) , \\ \chi(\tilde{\alpha}, \tilde{a}^n) &= \chi(\tilde{\alpha}, \tilde{a})^n = \det(\tilde{\alpha}, \tilde{a}) = 1 \\ &\qquad (\tilde{\alpha}, \tilde{\beta} \in \Gamma\,;\, \tilde{a}, \tilde{b} \in \tilde{Z}^{(2)}) .\end{aligned}$$

This shows that $\tilde{\alpha}\tilde{a}^n = \tilde{a}^n\tilde{\alpha}$ $(\tilde{\alpha} \in \Gamma, \tilde{a} \in \tilde{Z}^{(2)})$, namely $\tilde{a}^n \in \tilde{Z}^{(1)}$ for $\tilde{a} \in \tilde{Z}^{(2)}$. Applying Lemma 1 to the quotient group $\tilde{\Gamma}/\tilde{Z}^{(1)}$. We conclude that the order of $\tilde{\Gamma}/\tilde{Z}^{(1)}$ is finite. Denote by Γ the quotient group $\tilde{\Gamma}/\tilde{Z}^{(1)}$ and choose a system of representatives $\{\tilde{\alpha} \mid \alpha \in \Gamma\}$ in $\tilde{\Gamma}$, where $\tilde{\alpha}$ corresponds to $\tilde{\alpha}$. Then we get a 2-cocycle η of Γ with coefficients in the multiplicative group $C^\times$ such that

$$\tilde{\alpha}\tilde{\beta} = \eta(\alpha, \beta)\widetilde{\alpha\beta} \qquad (\alpha, \beta \in \Gamma) .$$

Since Γ is a finite group, multiplying non-zero scalars λ_α to $\tilde{\alpha}$, we have a system of matrices $\{\mu_\alpha = \lambda_\alpha \tilde{\alpha} \mid \alpha \in \Gamma\}$ such that $\mu_{\alpha\beta}\mu_\beta^{-1}\mu_\alpha^{-1}$ $(\alpha, \beta \in \Gamma)$ are roots of unity, Denote by Γ^* the matric group generated by the matrices $\mu_\alpha (\alpha \in \Gamma)$. Then Γ^* is a finite group of matrices such that the commutor of Γ^* in $(C)_{n\times n}$ consists of scalar matrices. This means that Γ^* is an irreducible matric group. Since Γ^* is a finite nilpotent group, the irreducibility of Γ^* implies that Γ^* is equivalent to a matric group whose elements are monomial matrices[1].

3. We now prove Theorem 1. Let $\mathscr{D}$ be a complex analytic manifold and let Γ be a finitely generated nilpotent subgroup in the group of automorphisms of $\mathscr{D}$ such that Γ acts properly discontinuously on $\mathscr{D}$ without fixed points. Let φ be the natural map $\mathscr{D} \to \mathscr{D}/\Gamma$ and let E be a holomorphic vector bundle over $\mathscr{D}/\Gamma$ such that i) the inverse image $\varphi^*(e)$ of E is trivial, ii) the associated projective bundle $P(E)$ is locally flat, and iii) E is simple. The inverse image $\varphi^*(E)$ can be identified with $\mathscr{D} \times C^n$ and the automorphisms $\alpha \in \Gamma$ of $\mathscr{D}$ induce bundle automorphisms

$$(z, v) \to (z\alpha, v\mu_\alpha(z)) \qquad (\alpha \in \Gamma) ,$$

where $\mu_\alpha(z)$ $(\alpha \in \Gamma)$ are holomorphic $n \times n$-matric functions such that

1) $\det \mu_\alpha(z) \neq 0$ everywhere on $\mathscr{D}$,
2) $\mu_\alpha(z)\mu_\beta(z\alpha) = \mu_{\alpha\beta}(z)$, $(\alpha, \beta \in \Gamma)$

The local flatness of $P(E)$ is equivalent to

3) $\mu_\alpha(z) = \mu_\alpha \xi_\alpha(z)$ $(\alpha \in \Gamma)$ with scalar functions $\xi_\alpha(z)$ and constant $n \times n$-matrices μ_α.

The simplicity of E is equivalent to

4) the commutor of $\{\mu_\alpha \mid \alpha \in \Gamma\}$ in $(C)_{n\times n}$ consists of scalar matrices.

Let $\tilde{\Gamma}$ be the matric group generated by $\{\mu_\alpha \mid \alpha \in \Gamma\}$ and let $\tilde{Z}$ be its center. Then from 2) and 3) the quotient group $\tilde{\Gamma}/\tilde{Z}$ is isomorphic to a quotient group of Γ, and thus $\tilde{\Gamma}/\tilde{Z}$ is finitely generated. Therefore by virtue of Lemma 2, $\tilde{\Gamma}$ is a matric group such that i) $\tilde{\Gamma}/\tilde{Z}$ is a finite group, ii) $\tilde{\Gamma}$ is an irreducible matric group and iii) $\tilde{\Gamma}$ is equivalent to a group of monomial matrices. After suitable change of the base of the vector space C^n, we may assume that $\mu_\alpha (\alpha \in \Gamma)$ are monomial matrices. Denote by μ_α^* the $n \times n$-matrix obtained by replacement of non-zero entries of μ_α with 1. Then $\Gamma^* = \{\mu_\alpha^* \mid \alpha \in \Gamma\}$ form a group of

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1) See [1] VII 52. 1.

permutation matrices. Since the matric group $\tilde{\Gamma}$ is irreducible the permutation group Γ^* is transitive. If we denote by Δ the subgroup of Γ consisting of α such that

$$\mu_\alpha^* = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix},$$

then from the transitivity we can conclude $[\Gamma : \Delta] = n$. If we decompose $\mu_\gamma(z)$ as

$$\mu_\gamma(z) = \begin{pmatrix} \nu_\gamma(z) & 0 \\ 0 & \mu_\gamma^{(1)}(z) \end{pmatrix} \qquad (\gamma \in \Delta) ,$$

then the group Δ acts on $\mathscr{D} \times C$ and $\mathscr{D} \times C^{n-1}$ as follows

$$(z, u) \to (z\gamma, u\nu_\gamma(z))$$

and

$$(z, v) \to (z\gamma, v\mu_\gamma^{(1)}(z)) \qquad (\gamma \in \Delta) .$$

Using these actions of Δ we get a line bundle L and a vector bundle $E^{(1)}$ of rank $n-1$ over $\mathscr{D}/\Delta$ as the quotients

$$L = \mathscr{D} \times C/\Delta$$

and

$$E^{(1)} = \mathscr{D} \times C^{n-1}/\Delta$$

such that

$$\psi^*(E) = L \oplus E^{(1)} ,$$

where ψ is the natural map $\mathscr{D}/\Delta \to \mathscr{D}/\Gamma$. Taking the direct images of of both sides, we have

$$E \overbrace{\oplus \cdots \oplus}^{n} E = \psi_*\psi^*(E) = \psi_*(L) \oplus \psi_*(E^{(1)}) .$$

Since $[\Gamma : \Delta] = n$ and the linear hull of $\{\mu_\alpha | \alpha \in \Gamma\}$ is the full matric ring $(C)_{n\times n}$, $\psi_*(L)$ is simple and $\psi_*(L) = n$. By the Krull-Remark-Schmidt theorem for vector bundles,

$$E \simeq \psi_*(L) .$$

REFERENCES

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Nagoya University