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REMARK ON THE TRICOMI EQUATION

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§1. As an application of the Carleman-type estimation Hörmander [4], p. 221, has proved the following:

A solution (distribution) of the Tricomi equation

$$\frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x^2} = 0$$

in an open set Ω in $R^2_{x,t}$ belongs to $C^\infty(\Omega)$ if it is in $C^\infty(\Omega_-)$ where $\Omega_- = \{(x,t)\,; (x,t) \in \Omega, t < 0\}$.

In this note we shall consider the same problem for the inhomogeneous Tricomi equation

$$\frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x^2} = f(x,t)$$

in a different manner. The existence of the solution in the generalized sense is well known. Furthermore we shall consider the propagation of analyticity. More precisely, the solution u is analytic in Ω if it is analytic in Ω_- and if $f(x,t)$ is analytic in Ω (Theorem 3.1). We shall use the results of [2] and [5] in the proof.

§2. The following theorem is obtained from the results of Berezin [2].

THEOREM 2.1. *Consider the following (backward) Cauchy problem:*

$$u_{tt} + tu_{xx} = f(x,t) \quad \text{in D}\,, \tag{2.1}$$

$$u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x) \quad \text{in} \quad a \leqq x \leqq b \tag{2.2}$$

where D denotes a domain in the region $t < 0$ bounded by characteristics passing through $(a,0)$ and $(b,0), (a < b)$. Assume $f(x,t)$ and $f_x(x,t)$ are continuous in $\bar{D}$ and the initial data $\varphi(x), \psi(x)$ are thrice continuously differentiable in $[a,b]$. Then there exists one and only one solution

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$u(x,t)$ of the problem (2.1), (2.2) having continuous second derivatives in $\bar{D}$. Furthermore, if $f(x,t)$ and $\varphi(x), \psi(x)$ are infinitely differentiable in $\bar{D}$ and in $[a,b]$ respectively, then the solution $u(x,t)$ is an infinitely differentiable function in $\bar{D}$.

By virtue of Theorem 2.1 it is shown that there exists a fundamental solution $E(x,t)$ for the backward Cauchy problem for the equation $Lu = u_{tt} + tu_{xx} = 0$. That is, there exists a distribution $E(x,t)$ in the region $t \leqq 0$ such that

$$LE = E_{tt} + tE_{xx} = 0 \qquad \text{for } t < 0\,, \tag{2.3}$$

$$E(x,0) = 0\,, \qquad E_t(x,0) = \delta_x\,. \tag{2.4}$$

In fact, take $f(x,t) = 0, \varphi(x) = 0$ and

$$\psi(x) = \begin{cases} 0 & x < 0 \\ x^4/4! & x \geqq 0 \end{cases}$$

in Theorem 2.1. Then there exists a solution $v(x,t)$ for the problem (2.1), (2.2) with these data having second continuous derivatives in the region $t \leqq 0$. The desired fundamental solution is given by

$$E(x,t) = \frac{\partial^5}{\partial x^5} v(x,t) \qquad t \leqq 0\,, \tag{2.5}$$

where differentiation in x is interpreted in the sense of distributions. By Theorem 2.1 and (2.5) we have

$$\text{supp.}\, E(x,t) \subset \{(x,t)\,;\, -\tfrac{2}{3}(-t)^{3/2} \leqq x \leqq \tfrac{2}{3}(-t)^{3/2}, t \leqq 0\}\,, \tag{2.6}$$

$$E(\cdot\,,t) \in C([-T,0]\,;\mathscr{D}'(R_x))\,, \tag{2.7}$$

$$E_t(\cdot\,,t) \in C([-T,0]\,;\mathscr{D}'(R_x)) \tag{2.8}$$

for any $T > 0$, where $\mathscr{D}'(R_x)$ denotes the space of distributions in R_x.

Furthermore, by using the partial hypoellipticity of the Tricomi operator L in t (cf. [4], §§ 2.2, 4.3), we have the following.

COROLLARY 2.1. *Let Ω be an open set in $R^2_{x,t}$ such that $\{(x,0)\,; a < x < b\} \subset \Omega$. If $u \in \mathscr{D}'(\Omega)$ satisfies*

$$Lu = u_{tt} + tu_{xx} = 0 \qquad \text{in } \Omega\,, \tag{2.9}$$

$$u = 0 \quad \text{in} \quad \Omega_+ = \{(x,t)\,; (x,t) \in \Omega, t > 0\}\,. \tag{2.10}$$

Then $u=0$ *in* $\Omega_+ \cap (\bar{D} \cup \Omega)$ *where* D *denotes a domain in the region* $t<0$ *bounded by characteristics passing through* $(a,0)$ *and* $(b,0)$.

For the proof we apply Theorem 2.1 by regulariging u with respect to x.

§3. Let Ω be an open set in $R^2_{x,t}$ which intersects x-axis.

THEOREM 3.1. *Let* $u=u(x,t)\in\mathscr{D}'(\Omega)$ *be a solution of the equation*

$$Lu = u_{tt} + tu_{xx} = f(x,t) \qquad \text{in } \Omega \tag{3.1}$$

with $f\in C^\infty(\Omega)$. *Then* $u\in C^\infty(\Omega)$ *if it is in* $C^\infty(\Omega_-)$ *where* $\Omega_- = \{(x,t);\ (x,t)\in\Omega, t<0\}$. *Furthermore,* u *is an analytic function in* Ω *if it is analytic in* Ω_- *and if* $f(x,t)$ *is analytic in* Ω.

We shall prove this theorem in several steps. First we shall show that $u(x,0)\in C^\infty\{x;(x,0)\in\Omega\}$.

Assume $\{(x,0);0\leqq x\leqq b\}\subset\Omega,(0<b)$. If we take $T>0$ sufficiently small then the closed domain $\bar{D}$ bounded by $\{(x,0);0\leqq x\leqq b\}$, characteristics passing through $(0,0)$ and $(b,0)$ and $\{(x,-T);-\infty<x<+\infty\}$ is contained in $\Omega\cap\{(x,t);t\leqq 0\}$. Let $u(x,t)$ and $f(x,t)$ be functions given in Theorem 3.1 and b,T be sufficiently small, then by the usual way (cf. [3]) we have

$$\begin{aligned} u(x,0) &= \int E_t(x-y,-T)u(y,-T)dy - \int E(x-y,-T)u_t(y,-T)dy \\ &\quad - \iint_{-T\leq\tau\leq 0} E(x-y,\tau)f(y,\tau)dyd\tau\ , \qquad 0<x<b\ , \end{aligned} \tag{3.2}$$

where the integral is taken in the sense of distributions. We note that there exists $u(x,0)=\lim_{t\to 0} u(\cdot,t)$ in $\mathscr{D}'(0<x<b)$ by the partial hypoellipticity of L in t (cf. [4], §4). The formula (3.2) is justified because of the assumptions for u,f and the properties of $E(x,t)$: (2.6), (2.7), (2.8). Thus we have proved that $u(x,0)\in C^\infty(0,b)$, and hence

$$u(x,0)\in C^\infty\{x;(x,0)\in\Omega\}\ .$$

Similarly, if u and f are analytic in Ω_- and Ω respectively, then we see that $u(x,0)$ is analytic in $\{x;(x,0)\in\Omega\}$. We omit the detail.

In the next section we shall show that

$$u\in C^\infty(\Omega\cap\{(x,t);t\geqq 0\}) \tag{3.3}$$

from which we see that $u(x, 0)$ and $u_t(x, 0)$ are in $C^\infty\{x; (x, 0) \in \Omega\}$. Then, applying Theorem 2.1 and Corollary 2.1, we have

$$u \in C^\infty(\Omega \cap \{(x, t); t \leqq 0\}) . \tag{3.4}$$

By (3.3), (3.4) and noting that the form of the equation is $u_{tt} + tu_{xx} = f$ in Ω we have $u \in C^\infty(\Omega)$ by the usual method of calculation (cf. § 4).

In the analytic case, from the assumption the $u(x, 0)$ is analytic in $\{x; (x, 0) \in \Omega\}$ we shall show, in the next section, $u = u(x, t)$ is analytic in $\Omega \cap \{(x, t); t \geqq 0\}$ from where we have $u(x, 0), u_t(x, 0)$ are analytic in $\{x; (x, 0) \in \Omega\}$. Then by Cauchy-Kowalevski theorem and Corollary 2.1, u is analytic in a neighbourhood of the x-axis contained in Ω. On the other hand, u is analytic in $\Omega_+ = \{(x, t) \in \Omega, t > 0\}$ because it is a solution of an elliptic equation in Ω_+. Thus u is analytic in Ω.

§ 4. It remains for us to prove the regularity property of the solution u in $\Omega \cap \{(x, t); t \geqq 0\}$.

THEOREM 4.1. *Let $f \in C^\infty(\Omega)$ ($\in C^\omega(\Omega)$) and $u \in \mathscr{D}'(\Omega)$ such that*

$$Lu = u_{tt} + tu_{xx} = f(x, t) \qquad \text{in } \Omega , \tag{4.1}$$

$$u(x, 0) = \psi(x) \in C^\infty\{x; (x, 0) \in \Omega\} \qquad (\in C^\omega\{x; (x, 0) \in \Omega\}) . \tag{4.2}$$

Then we have $u \in C^\infty(\Omega \cap \{(x, t); t \geqq 0\})$ ($\in C^\omega(\Omega \cap \{(x, t); t \geqq 0\})$). Here C^ω denotes the set of analytic functions.

To prove this theorem we use the method employed in [5], §§ 5, 6. We note that it is sufficient to prove the case $u(x, 0) = \psi(x) = 0$. First we prepare the following theorem which is derived by a direct computation. Take $G = (a < x < b) \times [0, T)$ such that $\overline{G} \subset \Omega$ and introduce the notation:

$$\|v\|^2_{\mathfrak{H}(G)} = \sum_{j=0}^{2} \|D_t^j v\|^2_{L^2(G)} + \|t^{1/2} v_{xt}\|^2_{L^2(G)} + \|t^{1/2} v_x\|^2_{L^2(G)} + \|t v_{xx}\|^2_{L^2(G)} . \tag{4.3}$$

($\mathfrak{H}(G)$ is a Hilbert space with the norm $\|\cdot\|_{\mathfrak{H}(G)}$.)

THEOREM 4.2 (cf. [5], Theorem 4.2). *There exists a constant $C > 0$ such that*

$$\|v\|_{\mathfrak{H}(G)} \leqq C \|Lv\|_{L^2(G)} \tag{4.4}$$

for all $v \in \mathfrak{H}(G)$ with supp. $v \subset G$ *and* $v(x, 0) = 0$.

Suppose $f(x,t) \in C^\infty(\Omega)$, then by the partial hypoellipticity of L in t (cf. [4], § 4.3) we conclude that for any $r(\geqq 2)$ there exists a number $\beta = \beta(u, r)$ such that

$$\zeta u \in H_{(r,\beta)}(G) = H_{(r,\beta)}(R^2)|_G \tag{4.5}$$

for any $\zeta = \zeta(x,t) \in C_0^\infty(G)$, For the notation $H_{(r,\beta)}(R^2)$, we refer to [4], § 2.5.

For a real number s we define an operator T_s:

$$\widehat{T_s v}(\xi, t) = (1 + |\xi|^2)^{s/2}\, \hat{v}(\xi, t)\ ,$$

where $v \in \mathscr{S}'(R^2_{x,t} \cap \{t \geqq 0\})$ and $\hat{v}(\xi, t)$ denotes the partial Fourier transformation of v with respect to x. (cf. [4], § 1.7.)

For any $x_0 \in (a, b)$ take $\zeta \in C_0^\infty(G)$ such that $\zeta(x_0, 0) \neq 0$ and

$$\frac{\partial \zeta}{\partial t}(x,t) = 0 \quad \text{if} \quad (x,t) \in G\ , \quad 0 \leqq t \leqq \frac{T}{2}\ .$$

Then by (4.5) we have

$$\varphi T_\beta \zeta u \in \mathfrak{H}(G) \tag{4.6}$$

for any $\varphi \in C_0^\infty(G)$. Starting with (4.6), by using the estimate (4.4) we can easily show that $\varphi T_s \zeta u \in \mathfrak{H}(G)$ for any s and $\varphi \in C_0^\infty(G)$ from where we have $\varphi D_x^j u \in \mathfrak{H}(G)$, $j = 0, 1, 2, \cdots$. And rewriting the form of the equation $u_{tt} = -t u_{xx} + f$, we have $\varphi D_t^r D_x^j \in L^2(G)$, $0 \leqq r, j < \infty$. Then we have $u \in C^\infty(G)$, from where we have $u \in C^\infty(G)$.

Next we consider the case where $f \in C^\omega(G)$ and $u(x, 0) = 0$. In this case we have $u \in C^\infty(G)$ by the above result. To obtain the analyticity of u in $\Omega \cap \{(x,t); t \geqq 0\}$, we have to estimate precisely the successive derivatives of u. We can pursuit the manner employed in [6], § 6 where the analyticity of the solutions of the equations $u_{tt} + t^{2k} u_{xx} = f$, $k = 0, 1, 2, \cdots$, was proved. In the following we shall give an outline of the reasoning.

Introduce the notations:

$$G_\varepsilon = (a + \varepsilon < x < b - \varepsilon) \times [0 \leqq t < T) \qquad 0 < \varepsilon < \mathrm{Min}\left(\frac{b-a}{2}, \frac{T}{2}\right)\ ,$$

$$G_\varepsilon^* = G_\varepsilon \backslash (a + \varepsilon < x < b - \varepsilon) \times \left[0 \leqq t < \frac{T}{2}\right)\ ,$$

$$N_\varepsilon(v) = \|v\|_{L^2(G_\varepsilon)}\ , \qquad N_\varepsilon^*(v) = \|v\|_{L^2(G_\varepsilon^*)}\ .$$

LEMMA 4.1 (cf. [4], ch. 1). *Let $\varepsilon, \varepsilon_1$ be positive numbers with $0 < \varepsilon + \varepsilon_1 < \operatorname{Min}((b-a)/2, T/2)$. Then there exists functions $\psi = \psi_{\varepsilon,\varepsilon_1} \in C_0^\infty(G_{\varepsilon_1})$ such that $\psi = \psi_{\varepsilon,\varepsilon_1} \equiv 1$ on $G_{\varepsilon+\varepsilon_1}$ and*

$$(4.7)\qquad \begin{aligned} &\operatorname{Max}|D_x^j D_t^r \psi| \leqq C_{j+r}\varepsilon^{-(j+r)} \qquad 0 \leqq j + r \leqq 2 \\ &D_t\psi \equiv 0 \qquad \text{on } (a+\varepsilon_1, b-\varepsilon_1) \times \left[0, \frac{T}{2}\right) . \end{aligned}$$

LEMMA 4.2 (cf. [6], Lemma 6.2). *There exists a constant $C > 0$ such that*

$$(4.8)\qquad \begin{aligned} &\sum_{j=0}^{2} \varepsilon^j N_{\varepsilon+\varepsilon_1}(D_t^j v) + \sum_{j=0}^{2} \varepsilon^j N_{\varepsilon+\varepsilon_1}(tD_x^j v) + N^*_{\varepsilon+\varepsilon_1}(v) \\ &\quad + \varepsilon N^*_{\varepsilon+\varepsilon_1}(D_x v) + \varepsilon^2 N^*_{\varepsilon+\varepsilon_1}(D_t D_x v) \\ &\quad \leqq C\{\varepsilon^2 N_{\varepsilon_1}(Lv) + \sum_{j=0,1} \varepsilon^j N_{\varepsilon_1}(tD_x^j v) + N^*_{\varepsilon_1}(v) + \varepsilon N^*_{\varepsilon_1}(D_t v)\} \end{aligned}$$

for all $v \in C^\infty(G)$ and $v(x, 0) = 0$. The constant C does not depend on $\varepsilon, \varepsilon_1$ under the condition mentioned previously.

This lemma is obtained by substituting $\psi_{\varepsilon,\varepsilon_1} v$ in (4.4).

LEMMA 4.3 (cf. [4], ch. 7). *Let w be an analytic function in G. Then there exists a constant $C > 0$ such that*

$$(4.9)\qquad \varepsilon^{j+r} N_{k\varepsilon}(D_x^j D_t^r w) \leqq C^{j+r+1} \qquad \text{if } j + r < k ,$$

for all integer $k > 0$. Conversely, if $w \in C^\infty(G)$ suctisfies (4.9), then w is analytic in G.

Proof of the analyticity of u in $\Omega \cap \{(x, t)\,; t \geqq 0\}$.

First we shall show that there exists a constant $B > 0$ such that, for any $\varepsilon > 0$ and for any integer $l > 0$,

$$(4.10)\qquad \left.\begin{aligned} &\sum_{r=0}^{2} \varepsilon^{r+j} N_{l\varepsilon}(D_t^r D_x^j u) \\ &\sum_{r=0}^{2} \varepsilon^{r+j} N_{l\varepsilon}(t^{2k} D_x^{r+j} u) \\ &\sum_{r=0,1} \varepsilon^{r+j} N^*_{l\varepsilon}(D_x^{r+j} u) \\ &\quad \varepsilon^{2+j} N^*_{l\varepsilon}(D_t D_x^{j+1} u) \end{aligned}\right\} \leqq B^{l+1}$$

if $j < l$.

It we take B sufficiently large, we have (4.10) for $l = 1$ by Lemma 4.2. Next, since $f(x, t)$ is analytic in $\bar{G}$, there exists a constant $C_0 > 0$ such that

$$\varepsilon^{2+j} N_{j\varepsilon}(D_x^j f) \leqq C_0^{j+1} ,$$

for $j = 1, 2, \cdots$ and $0 < \varepsilon < (b-a)/2$.
Assuming that (4.10) have been proved for an $l > 0$, we shall prove (4.10) for $l+1$. Replacing v by $\varepsilon^l D_x^l u$ and ε_1 by $l\varepsilon$ in (4.8), we see that the terms in the left hand side of (4.10) for the case $l+1$ are smaller than $5C_0 B^{l+1}$ if $j < l+1$. Hence we have (4.10) for $l+1$ if $5C_0 B^{l+1} \leqq B^{l+2}$. This condition is satisfied for all l if $B > \max(5C_0, 1)$.

From (4.10) (cf. Lemma 4.3) we obtain

$$\sum_{r=0}^{2} \|D_t^r D_x^j u\|_{L^2(G_{\varepsilon_1})} \leqq C_1^{j+1} j^j , \qquad j = 0, 1, 2, \cdots \tag{4.11}$$

for some constant $C_1 > 0$ where $G_{\varepsilon_1} = (a + \varepsilon_1, b - \varepsilon_1) \times [0, T/2]$ with $\varepsilon_1 > 0$ sufficiently small.

To obtain the successive estimates including the derivatives in both x and t, we rewrite the equation $Lu = f$ in the form $D_t^2 u = -tD_x^2 u + f$. And using (4.11) by the usual way (cf. [6] for example) we have

$$\|D_x^j D_t^r u\|_{L^2(G_{\varepsilon_1})} \leqq C_2^{j+r+1}(j+r)^{j+r} \qquad 0 \leqq j, r < \infty$$

for some constant $C_2 > 0$, from which we have the analyticity of u in G_{ε_1} by the Sobolev lemma.

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