

**ON NON-ANTICIPATIVE LINEAR TRANSFORMATIONS
 OF GAUSSIAN PROCESSES WITH
 EQUIVALENT DISTRIBUTIONS**

YU. A. ROZANOV

Let $\xi(t)$, $t \in T$, be a Gaussian process on a set T , and $H = H(\xi)$ be the closed linear manifold generated by all values $\xi(t)$, $t \in T$, with the inner product

$$\langle \eta_1, \eta_2 \rangle = E\eta_1\eta_2; \quad \eta_1, \eta_2 \in H.$$

We suppose that the Hilbert space H is separable.

Let \mathcal{A} be a linear operator on H ; we call a random process of the form

$$\eta(t) = \mathcal{A}\xi(t), \quad t \in T, \quad (1)$$

a *linear transformation* of the process $\xi(t)$, $t \in T$. One says that a linear transformation \mathcal{A} is *non-anticipative*, if

$$\mathcal{A}H_t(\xi) \subseteq H_t(\xi), \quad t \in T, \quad (2)$$

where $H_t(\xi)$ denotes the subspace in H , which is generated by all values $\xi(s)$, $s \leq t$.

Let P be a probability distribution of the Gaussian process $\xi = \xi(t)$, $t \in T$, in some measurable space (X, \mathfrak{B}, P) of «trajectories» $x = x(t)$, $t \in T$, where σ -algebra \mathfrak{B} is generated by all sets $\{x(t) \in B\}$ ($t \in T$), B are Borel sets on the real line, so P is determined by finite-dimensional distributions of the random process $\xi = \xi(t)$, $t \in T$. Let Q be a probability distribution of the Gaussian process $\eta = \eta(t)$, $t \in T$, represented by the formula (1). According to well known *Feldman's theorem* (see, for example, [1]), Q is equivalent to P ($Q \sim P$) if and only if the operator

$$B = \mathcal{A}^*\mathcal{A} \quad (3)$$

is invertible and $I - B \in S_2$, where S_2 denotes the class of all Hilbert-Schmidt operators in H .

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The operator \mathbf{B} connects with the correlation function $\mathbf{B}(s, t)$ of the Gaussian distribution Q as

$$\mathbf{B}(s, t) = \langle \mathbf{B}\xi(s), \xi(t) \rangle, \quad s, t \in T; \quad (4)$$

let us call \mathbf{B} the correlation operator of Q . Obviously, for any equivalent distribution Q (i.e. Q has strictly positive correlation operator \mathbf{B} , such that $I - \mathbf{B} \in S_2$.) there is a linear transformation (1), which gives us a Gaussian process $\eta(t), t \in T$, with the distribution Q : the general operator \mathcal{A} , which satisfies the condition (3), has the form

$$\mathcal{A} = V\mathbf{B}^{1/2} \quad (5)$$

where V is an arbitrary unitary operator in H .

Let us consider a linear transformation (1) with $\mathcal{A} = I - \Delta$:

$$\eta(t) = \xi(t) - \Delta\xi(t), \quad t \in T. \quad (6)$$

It is more convenient to reformulate Feldman's theorem in the following way: $Q \sim P$ if and only if $I - \mathbf{B}^{1/2} \in S_2$ and 1 does not belong to the spectrum of $I - \mathbf{B}^{1/2}$. Indeed, $I - \mathbf{B} \in S_2$ if and only if

$$(I - \mathbf{B}^{1/2}) = (I - \mathbf{B})(I + \mathbf{B}^{1/2})^{-1} \in S_2.$$

It is easy to see that for any operator $\Delta \in S_2$, which has no eigenvalue equal to 1, the random process $\eta(t), t \in T$, of the form (6) has the equivalent distribution Q with the correlation operator, because

$$I - \mathbf{B} = \Delta + \Delta^*(I - \Delta) \in S_2.$$

But the condition $\Delta \in S_2$ is not necessary for the equivalence $Q \sim P$. Namely, by the formula (5) we have

$$\Delta = I - V\mathbf{B}^{1/2}, \quad (7)$$

where V is some unitary operator and (for the equivalent distribution Q) $I - \mathbf{B}^{1/2} \in S_2$; obviously $\Delta \in S_2$ if and only if $[\Delta - (I - \mathbf{B}^{1/2})]\mathbf{B}^{-1/2} = I - V \in S_2$.

Then we shall be interested in the linear transformation (6) with operators $\Delta \in S_2$. As we have obtained, it holds true if and only if

$$I - V \in S_2 \quad (8)$$

where V is an unitary operator connected with the operator Δ by the formula (7): $\Delta = I - V\mathbf{B}^{1/2}$. According to Feldman's theorem any trans-

formation (6) such that $\Delta \in S_2$ and 1 does not belong to the spectrum Δ gives a random process $\eta(t)$, $t \in T$, with an equivalent distribution Q .

We shall be interested also in a such property of the linear transformation (6) as *to be non-anticipative* that means

$$\Delta H_t(\xi) \subseteq H_t(\xi), \quad t \in T. \tag{9}$$

In the resent time it was paid attention for non-anticipative transformations in connection with Hitsuda's result [2] for the Wiener process $\xi(t)$, $0 \leq t \leq 1$: any Gaussian process $\eta(t)$, $0 \leq t \leq 1$, with an equivalent probability distribution can be represent in the form

$$\eta(t) = \xi(t) - \int_0^t \left[\int_0^s \Delta(u, s) d\xi(u) \right] ds \tag{10}$$

where $\Delta(t, s)$; $0 \leq t, s \leq 1$,

$$\Delta(t, s) = 0, \quad s < t, \tag{11}$$

$$\int_0^1 \int_0^1 \Delta(t, s)^2 dt ds < \infty. \tag{12}$$

Though in the paper [2] it was used some theorems on the martingales, it was clear that the representation (10) can be obtained as a result of the theory of operators in a Hilbert space: the formula (10) is given by a non-anticipative transformation (6) with $\Delta \in S_2$ in the case of Wiener process $\xi(t)$, $0 \leq t \leq 1$. The existense of such transformation in the general case follows from non-trivial Gohberg-Krein's theorems on so-called *special factorization*; namely, any positive operator B of the type

$$B = (I - F) = (I - G)^{-1} \\ (F \text{ and } G = -FB^{-1} \text{ belong } S_2)$$

can be represented in the form

$$B = (I + X)\mathcal{D}(I + X^*) \tag{13}$$

where $(I + X)$ is invertible, $X \in S_2$ and $\mathcal{D} \geq 0$; besides *the operators* X and \mathcal{D} *satisfy the condition*

$$XH_t \subseteq H_t, \quad \mathcal{D}H_t \subseteq H_t \quad (t \in T)$$

for a given monotone family of subspaces H_t , $t \in T$ ($H_s \subseteq H_t$ if $s \leq t$) (see the theorems 6.2 Ch. IV and 10.1 Ch. I in the book [3]). It is clear that for $H_t = H_t(\xi)$, $t \in T$, the operator

$$\mathcal{A} = (I + X)\mathcal{D}^{1/2} \quad (14)$$

satisfies the conditions (2) and (3), so the corresponding linear transformation (6) with $\Delta = I - \mathcal{A}$ will be *non-anticipative*. This proof of the existence of non-anticipative representations (6) for Gaussian processes $\eta(t)$, $t \in T$, with equivalent distributions was suggested recently by Kallianpur and Oodaira [4] (in the case of Wiener process $\xi(t)$, $0 \leq t \leq 1$, it was done earlier by Kailath [5]). We should like to do the following essential note: for the operator \mathcal{A} , which was mentioned above (see (14)) it holds true that

$$\Delta = I - \mathcal{A} \in S_2, \quad (15)$$

so for any Gaussian process $\xi(t)$, $t \in T$, there is a non-anticipative Gaussian process $\eta(t) = \xi(t) - \Delta\xi(t)$, $t \in T$ (where $\Delta \in S_2$ satisfies the condition (9)) with a given equivalent probability distribution.

Indeed, in the representation (13) we have $(I + X)^{-1} = I + \mathcal{F}$, $\mathcal{F} = -X(I + X)^{-1} \in S_2$, and the operator \mathcal{D} has a form

$$\mathcal{D} = (I + \mathcal{F})(I - F)(I + \mathcal{F}^*) = I + V$$

where

$$V = \mathcal{F}(I - F)(I + \mathcal{F}^*) - F(I + \mathcal{F}^*) + \mathcal{F}^* \in S_2.$$

From relations

$$\begin{aligned} \mathcal{D}^{1/2} &= (I + V)^{1/2} = I + W, \\ I + V &= (I + W)^2 = I + W(2I + W) = I + W(I + \mathcal{D}^{1/2}), \end{aligned}$$

we obtain that

$$W = V(I + \mathcal{D}^{1/2})^{-1} \in S_2,$$

so

$$\begin{aligned} \Delta &= I - \mathcal{A} = I - (I + X)\mathcal{D}^{1/2} \\ &= I - (I + X)(I + W) = -X(I + W) - W \in S_2. \end{aligned}$$

It is worth to pay attention for the following fact: the linear transformation (6) with the operator $\mathcal{A} = I - \Delta$ of the form (14) is such that

$$H_t(\xi) = H_t(\eta), \quad t \in T. \quad (16)$$

Indeed, for the invertible positive operator $\mathcal{D}^{1/2}: \mathcal{D}^{1/2}H_t(\xi) \subseteq H_t(\xi)$, we have

$$\mathcal{D}^{1/2}H_t(\xi) = H_t(\xi)$$

because in a contrary case there is an element $h \in H_t(\xi)$, such that $h \perp \mathcal{D}^{1/2}H_t(\xi)$ and $\mathcal{D}^{1/2}h = 0$. Remind that a Volterra operator X has only one point of a spectra equal to 0, so for the operator $(I + X)$ in the formula (14), $(I + X)H_t(\xi) \subseteq H_t(\xi)$, we have

$$(I + X)H_t(\xi) = H_t(\xi) .$$

Now it is obvious that the operator $\mathcal{A} = (I + X)\mathcal{D}^{1/2}$ satisfies the condition (16).

Let us consider a few examples of non-anticipative representations (6) with $\Delta \in S_2$.

EXAMPLE 1. Let $\xi(t)$, $0 \leq t \leq 1$, be a Gaussian process with stationary increments:

$$\xi(t) = \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda} \Phi(d\lambda) ,$$

which has a spectral density $f(\lambda)$ of the type:

$$0 < \lim_{\lambda \rightarrow -\infty} f(\lambda) \leq \overline{\lim}_{\lambda \rightarrow \infty} f(\lambda) < \infty$$

(if $f(\lambda) = 1/2\pi$, we deal with Wiener process $\xi(t)$, $0 \leq t \leq 1$).

The corresponding space H consists of all random variables

$$\begin{aligned} \eta &= \int_{-\infty}^{\infty} \varphi(\lambda) \Phi(d\lambda) = \int_0^1 \mathbf{c}(t) \dot{\xi}(t) dt \\ &\left(\varphi(\lambda) = \int_0^1 e^{i\lambda t} \mathbf{c}(t) dt \right) \end{aligned} \tag{18}$$

where functions $\mathbf{c}(t)$, $0 \leq t \leq 1$, belonging to $L^2[0,1]$ and $\dot{\xi}(t)$ is the generalized delivative of process $\xi(t)$; besides¹⁾

$$\|\eta\|^2 = \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\lambda \cup \int_0^1 \mathbf{c}(t)^2 dt \tag{19}$$

¹⁾ The relation $\alpha \cup \beta$ between variables α, β means that

$$0 < \mathbf{c}_1 \leq \frac{\alpha}{\beta} \leq \mathbf{c}_2 < \infty .$$

(see, for example, [1] or [6]), and the formula (18) gives us the isomorphism between H and $L^2[0, 1]$ such that

$$H_t(\xi) \leftrightarrow L^2[0, t], \quad 0 \leq t \leq 1, \quad (20)$$

where $L^2[0, t]$ denotes the subspace of all functions $\mathbf{c}(s)$, $0 \leq s \leq 1$: $\mathbf{c}(s) = 0$ for $s > t$. As it follows from the conditions (19) and (20), the formula

$$\Delta\eta = \int_0^1 [\tilde{\Delta}\mathbf{c}(t)]\dot{\xi}(t)dt$$

gives us the isomorphism $\Delta \leftrightarrow \tilde{\Delta}$ between Hilbert-Schmidt operators in H and $L^2[0, 1]$, and an operator Δ satisfies the condition (9) if and only if

$$\tilde{\Delta}L^2[0, t] \subseteq L^2[0, t], \quad 0 \leq t \leq 1,$$

that is equivalent to the condition (11) for a corresponding kernel $\Delta(t, s)$:

$$\tilde{\Delta}\mathbf{c}(t) = \int_0^1 \Delta(t, s)\mathbf{c}(s)ds, \quad 0 \leq t \leq 1,$$

(remind $\tilde{\Delta} \in S_2$ if and only if $\Delta(t, s)$; $0 \leq t, s \leq 1$, satisfies the condition (12)). Thus any *non-anticipative* operator $\Delta \in S_2$ can be described by the formula

$$\Delta\eta = \int_0^1 \left[\int_0^1 \Delta(t, s)\mathbf{c}(s)ds \right] \dot{\xi}(t)dt \quad (21)$$

with a Volterra, Hilbert-Schmidt kernel $\Delta(t, s)$; $0 \leq t \leq 1$. For variables $\xi(t)$, $0 \leq t \leq 1$, which correspond to the functions

$$\mathbf{c}(s) = \begin{cases} 1, & 0 \leq s \leq t, \\ 0, & s > t, \end{cases}$$

we obtained from the formula (21) a general *non-anticipative* transformation (6) with $\Delta \in S_2$ as

$$\eta(t) = \xi(t) + \int_0^t \left[\int_0^s \Delta(u, s)\dot{\xi}(u)du \right] ds, \quad 0 \leq t \leq 1, \quad (22)$$

that gives us Hitsuda's representation (10) in the case of Wiener process $\xi(t)$, $0 \leq t \leq 1$.

EXAMPLE 2. Let $\xi(t)$, $0 \leq t \leq 1$, be a Gaussian stationary process:

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \Phi(d\lambda)$$

with a spectral density $f(\lambda)$ of the type

$$0 < \liminf_{\lambda \rightarrow \infty} \lambda^{2n} f(\lambda) \leq \overline{\lim}_{\lambda \rightarrow \infty} \lambda^{2n} f(\lambda) < \infty . \tag{23}$$

It will be convenient to introduce the process

$$\begin{aligned} \zeta(t) &= \sum_{k=0}^{n-1} {}_n C_{k+1} [\xi^{(k)}(t) - \xi^{(k)}(0)] + \int_0^t \xi(s) ds \\ &= \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda} (1 + i\lambda)^n \Phi(d\lambda) , \quad 0 \leq t \leq 1 . \end{aligned} \tag{24}$$

Obviously, the spectral density of this process $\zeta(t)$ with stationary increments satisfies the condition (17) and we can use results of our example 1 for the process $\zeta(t)$, $0 \leq t \leq 1$.

As is known (see, for example, [1] or [6]) the Hilbert space $H = H(\xi)$ consists of all variables

$$\begin{aligned} \eta &= \int_{-\infty}^{\infty} \varphi(\lambda) \Phi(d\lambda) \\ \left(\varphi(\lambda) &= \sum_{k=0}^{n-1} c_k (i\lambda)^k + (1 + i\lambda)^n \int_0^1 e^{i\lambda t} \mathbf{c}(t) dt \right) \end{aligned}$$

where c_1, \dots, c_{n-1} are arbitrary constants and $\mathbf{c}(t) \in L^2[0, 1]$ or

$$\eta = \sum_{k=0}^{n-1} c_k \xi^{(k)}(0) + \int_0^1 \mathbf{c}(t) \dot{\zeta}(t) dt \tag{25}$$

where $\dot{\zeta}(t)$ denotes the generalized derivative of the process $\zeta(t)$ determined by the transformation (24).

If we consider in the general formula (25) only functions $\mathbf{c}(s) \in L^2[0, t]$, we obtain the corresponding subspace $H_t(\xi)$, $0 \leq t \leq 1$, and it shows that $H_t(\xi)$ is a *direct sum* of the subspace

$$H_{0+}(\xi) = \bigcap_{t>0} H_t(\xi) ,$$

which consists of all variables $\eta = \sum_{k=0}^{n-1} c_k \xi^{(k)}(0)$, and the subspace $H_t(\zeta)$ of all variables $\eta = \int_0^t \mathbf{c}(s) \dot{\zeta}(s) ds :$

$$H_t(\xi) = H_{0+}(\xi) + H_t(\zeta) , \quad 0 \leq t \leq 1 ;$$

in particular

$$H(\xi) = H_{0+}(\xi) + H(\zeta) .$$

Let P be a projector on the subspace $H(\zeta)$ parallel to the subspace $H_{0+}(\xi)$. If $\Delta \in S_2$ then $P\Delta P \in S_2$; obviously, if Δ satisfies the condition (9) then $P\Delta P$ satisfies to the similar condition with respect to $H_t(\zeta)$, $0 \leq t \leq 1$. As it has been shown (see (21)), the non-anticipative operator $P\Delta P$ in $H(\zeta)$ can be described by a Volterra, Hilbert-Schmidt kernel $\Delta(t, s)$; $0 \leq t, s \leq 1$:

$$P\Delta P\eta = \int_0^1 \left[\int_t^1 \Delta(t, s) \mathbf{c}(s) ds \right] \dot{\zeta}(t) dt \quad (26)$$

where $\eta \in H$ is given by the formula (25) and

$$P\eta = \int_0^1 \mathbf{c}(t) \dot{\zeta}(t) dt .$$

For any non-anticipative operator Δ in $H(\xi)$ we have

$$\Delta H_{0+} = \Delta \left(\bigcap_{t>0} H_t \right) \subseteq \bigcap_t (\Delta H_t) \subseteq \bigcap_t H_t = H_{0+}$$

that is equivalent to the condition

$$(I - P)\Delta(I - P) = \Delta(I - P) .$$

Then

$$\Delta = (I - P)\Delta P + \Delta(I - P) + P\Delta P = (I - P)\Delta + P\Delta P$$

where the finite-dimensional operator $(I - P)\Delta$, mapping $H(\xi)$ on the subspace $H_{0+}(\xi)$, has the form

$$(I - P)\Delta\eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) \quad (27)$$

($\eta_0, \eta_1, \dots, \eta_{n-1}$ are some fixed elements in H_{0+}). Combining formulas (26) and (27), we obtain a general *non-anticipative* operator $\Delta \in S$ as

$$\Delta\eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) + \int_0^1 \left[\int_t^1 \Delta(t, s) \mathbf{c}(s) ds \right] \dot{\zeta}(t) dt ; \quad (28)$$

in particular, for $\eta \in H_t(\xi)$

$$\Delta\eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) + \int_0^t \left[\int_0^s \Delta(u, s) \dot{\zeta}(u) du \right] \mathbf{c}(s) ds . \quad (29)$$

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Steklov Mathematical Institute, Moscow