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Nagoya Math. J.
Vol. 47 (1972), 199-226

COSIMPLICIAL COHOMOLOGY OF COALGEBRAS

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Introduction.

The interpretations of simplicial cohomology groups for associative commutative algebras and for Lie algebras were given by Beck [2], Iwai [9] and Shimada and others [15, 16]. On the other hand, Jonah [10] gave a formulation and an interpretation of the second and third cohomology groups of an associative coalgebra after the Hochschild's treatment [5 ~ 8].

The purpose of this paper is to deal with cosimplicial cohomology groups of a coassociative (ungraded) coalgebra (over a field), with coefficient in a two sided comodule (§ 3), and to interpret their first (§ 6) and second cohomology groups (§ 5), where the dimension indices in the cosimplicial cohomology are one less than the usual.

We will describe in detail the interpretation of the second cohomology groups, while we sketch the interpretation of the first cohomology groups, since the latter is more simple and analogous to the fromer.

In the first section, generalities on coalgebras over a field and comodules are given, and, in particular, it is proved that the category $\mathscr{C}$ of coalgebras has (finite) products and difference kernels. We characterize abelian cogroup objects in the category $(A, \mathscr{C})$ in the second section. Before interpreting the second cohomology groups, we insert § 4, in which some properties of cosimplicial coalgebras are verified.

The main theorem of this paper is that $Ex^2(M, A) \approx H^2(M, A)$, where $Ex^2(M, A)$ denotes the set of all equivalence classes of two term extensions of a coalgebra A by a two sided A-comodule M and $H^2(M, A)$ the second cosimplicial cohomology group of A with coefficient in M (Theorem 5.4). It seems that furthermore complicated calculations will be needed to interpret $H^n(M, A)$ $(n \geq 3)$.

The auther is indebted to Professor Nobuo Shimada for his kind

Received December 28, 1971.
Revised February 4, 1972.

guidance, suggestion to the results in this paper and constant encouragement, to Professor Akira Iwai for his advice particularly on the treatment of abelian cogroup objects and cosimplicial coalgebras. Simultaneously, the auther would like to express thanks to Professor Hideyuki Matsumura for his critical reading and improvement of the manuscript.

1. Coalgebras and Comodules

In the sequel we assume that K is a fixed field.

DEFINITION 1.1. *A coalgebra over* K (or simply a *coalgebra*) is a K-module A together with K-module maps

$$\Delta_A : A \longrightarrow A\otimes A\ , \qquad \varepsilon_A : A \longrightarrow K$$

such that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A\otimes A \\ {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \Delta_A\otimes 1} \\ A\otimes A & \xrightarrow{1\otimes\Delta_A} & A\otimes A\otimes A\end{array}\ , \qquad \begin{array}{ccccc} K\otimes A & \xleftarrow{\varepsilon_A\otimes 1} & A\otimes A & \xrightarrow{1\otimes\varepsilon_A} & A\otimes K \\ & \nwarrow{\scriptstyle\cong} & \uparrow{\scriptstyle\Delta_A} & \nearrow{\scriptstyle\cong} & \\ & & A & & \end{array}$$

are commutative. The first diagram is called the *coassociativity* of Δ_A (p. 5 of [17]), the map Δ_A is called the *comultiplication* of A and ε_A is called the *counit* of A ([14]).

A *morphism* $f: A\rightarrow B$ of coalgebras (or simply a *coalgebra map*) is a K-module map satisfying the commutative diagrams

$$\begin{array}{ccc} A & \xleftarrow{\Delta_A} & A\otimes A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes f} \\ B & \xleftarrow{\Delta_B} & B\otimes B\end{array}\ , \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle\varepsilon_A}\searrow \quad \swarrow{\scriptstyle\varepsilon_B} & \\ & K & \end{array}$$

If we want to regard a coalgebra map f as K-module map, we shall denote this by Uf.

Suppose A is a coalgebra and V a submodule of A with $\Delta_A V \subseteq V\otimes V$. Then V is a coalgebra with its comultiplication $\Delta_A|V$ and counit $\varepsilon_A|V$, and is said to be a *subcoalgebra* of A. The following are easily proved ([17]).

PROPOSITION 1.2. (i) *The sum of a collection of subcoalgebras is a subcoalgebra.*

(ii) *The intersection of subcoalgebras is a subcoalgebra.*

(iii) *The image Uf for a coalgebra map f is a subcoalgebra.*

DEFINITION 1.3. Let A be a coalgebra. *A left A-comodule* is a K-module M together with a K-module map $\Delta^l_M : M \to A \otimes M$ such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\Delta^l_M} & A \otimes M \\ {\scriptstyle \Delta^l_M}\downarrow & & \downarrow{\scriptstyle 1 \otimes \Delta^l_M} \\ A \otimes M & \xrightarrow{\Delta_A \otimes 1} & A \otimes A \otimes M \end{array}, \qquad \begin{array}{ccc} M & \xrightarrow{\cong} & K \otimes M \\ & {\scriptstyle \Delta^l_M}\searrow \quad \nearrow{\scriptstyle \varepsilon_A \otimes 1} & \\ & A \otimes M & \end{array}$$

are commutative. Similarly, with a K-module map $\Delta^r_M : M \to M \otimes A$ we can define a *right A-comodule.* Sometimes Δ^l_M and Δ^r_M are called the *comodule structure maps of M.* Let M be both a left and a right A-comodule. If moreover $(\Delta^l_M \otimes 1)\Delta^r_M = (1 \otimes \Delta^r_M)\Delta^l_M$, then we call M a *two sided A-comodule.* In particular we can regard A as a two sided A-comodule with $\Delta^l_A = \Delta_A = \Delta^r_A$.

If M and M' are left A-comodules, then a K-module map $f : M \to M'$ is called a *left A-comodule map* if it satisfies the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Delta^l_M} & A \otimes M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1 \otimes f} \\ M' & \xrightarrow{\Delta^l_{M'}} & A \otimes M' \end{array},$$

Similarly, we can define a *right A-comodule map* and a *two sided A-comodule map.*

DEFINITION 1.4. Let A be a coalgebra, and let I be a submodule of A. We call I a *(two-sided) coideal of A* if

$$\text{(i) } \Delta_A I \subseteq A \otimes I + I \otimes A \, , \qquad \text{(ii) } \varepsilon_A I = 0 \, .$$

In this case we have the following which are easily verified (17]).

PROPOSITION 1.5. (i) *The sum of a collection of coideals is a coideal.*

(ii) *The kernel* Ker Uf *for a coalgebra map is a coideal.*

(iii) *For two coalgebra maps $f, g : A \to B$, $Im(Uf - Ug)$ is a coideal of B.*

(iv) *If A is a coalgebra and I a coideal of A, then the quotient*

A/I as K-module has a natural coalgebra structure induced by the projection $A \rightarrow A/I$.

Let $\mathscr{C}$ be the category consisting of all coalgebras over K and coalgebra maps. The following is trivial.

PROPOSITION 1.6. $\mathscr{C}$ has sums and difference cokernels. Accordingly $\mathscr{C}$ has pushouts.

In consequence $\mathscr{C}$ has direct limits (p. 38 of [19]). We shall describe the definition of cofree coalgebras and its existence following [17].

DEFINITION 1.7. Let V be a vector space over K. A pair (FV, η_V) with FV a coalgebra and a K-module map $\eta_V: FV \rightarrow V$ is called a *cofree coalgebra on* V if for any coalgebra A and a K-module map $f: A \rightarrow V$ there is a unique coalgebra map $h: A \rightarrow FV$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & FV \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \eta_V} & \\ & V & \end{array}$$

is commutative. If there exists a cofree coalgebra on V then it is unique up to isomorphism of coalgebras.

For each algebra X over K we define

$$X^0 = \{x \in X^* \mid \operatorname{Ker} x \text{ contains a cofinite ideal}\},$$

where X^* is the dual of X and a *cofinite ideal* is an ideal I in X such that X/I is finite dimensional. We can prove that X^0 is a coalgebra in X^* with $\Delta_{X^0}: X^0 \rightarrow X^0 \otimes X^0$ and $\varepsilon_{X^0}: X^0 \rightarrow K$ defined by $\Delta_{X^0} = \varphi^* \mid X^0$ and $\varepsilon_{X^0} x = x(1)$ for $x \in X^0$, where $\varphi: X \otimes X \rightarrow X$ is the multiplication of X and φ^* the dual of φ.

Given a vector space V over K, let $T(V^*)$ be the tensor algebra of V^*. Since there is the natural inclusion map $i: V^* \rightarrow T(V^*)$ we can define a K-module map $\eta: T(V^*)^0 \rightarrow V^{**}$. In this case $(T(V^*)^0, \eta)$ is the cofree coalgebra on V^{**} (see p. 126 of [17]). Let $FV = \sum W$ with the sum taken over all subcoalgebras W of $T(V^*)^0$ such that $\eta W \subset V$, and put $\eta_V = \eta \mid FV$ then (FV, η_V) is the cofree coalgebra on V. Thus we have the following.

THEOREM 1.8. *For any vector space V over K there is the cofree coalgebra on it.*

This theorem says that there is an adjoint pair (U, F) such that

$$\mathscr{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathscr{M}_K$$

where U is the underlying object functor, F the cofree coalgebra functor, and $\mathscr{M}_K$ the category of all vector spaces over K. That is, there is a natural isomorphism in C and in V

$$\lambda(C, V) : \mathrm{Hom}_{\mathscr{M}_K} \overset{\approx}{\rightarrow} \mathrm{Hom}_{\mathscr{C}}(C, FV) \tag{1.8$'$}$$

as sets for $C \in \mathscr{C}$ and $V \in \mathscr{M}_K$. In this situation U is the left adjoint of F (F the right adjoint of U), which is denoted by $U \dashv F$ in general. Define natural transformations ε and η by

$$\begin{aligned} \varepsilon(C) &= \lambda(1_{UC}) : C \to FUC \qquad \text{for } C \in \mathscr{C} , \\ \eta(V) &= \lambda^{-1}(1_{FV}) : UFV \to V \qquad \text{for } V \in \mathscr{M}_K \end{aligned} \tag{1.9}$$

with abbreviations $\lambda = \lambda(C, UC)$ and $\lambda = \lambda(FV, V)$, respectively. Then we have:

$$\begin{aligned} \lambda(f) &= F(f) \cdot \varepsilon(C) \qquad \text{for } f \in \mathrm{Hom}_{M_K}(UC, V) , \\ \lambda^{-1}(\rho) &= \eta(V) \cdot U(\rho) \qquad \text{for } \rho \in \mathrm{Hom}_{\mathscr{C}}(C, FV) , \\ \eta U \cdot U\varepsilon = 1_U &: U \to UFU \to U , \qquad F\eta \cdot \varepsilon F = 1_F : F \to FUF \to F . \end{aligned} \tag{1.10}$$

PROPOSITION 1.11. *$\mathscr{C}$ has (finite) products and difference kernels. Accordingly $\mathscr{C}$ has pullbacks.*

Proof. Let A and B be coalgebras. We have canonical projections $p_1 : F(UA \oplus UB) \to FUA$ and $p_2 : F(UA \oplus UB) \to FUB$ in $\mathscr{C}$. Define $P = p_1^{-1}(\varepsilon A) \cap p_2^{-1}(\varepsilon B)$ which is a subcoalgebra of $F(UA \oplus UB)$. The projections $p_A : p \to A$ and $p_B : P \to B$ are defined by $p_A = \varepsilon^{-1}(p_1 | P)$ and $p_B = \varepsilon^{-1}(p_2 | P)$.

Assume $f : L \to A$ and $g : L \to B$ are in $\mathscr{C}$. By the universality of $F(UA \oplus UB)$ there exists a unique coalgebra map $h : L \to F(UA \oplus UB)$ such that

$$\eta_{UA \oplus UB} \cdot h = Uf \oplus Ug , \qquad p_1 h = \varepsilon f \quad \text{and} \quad p_2 h = \varepsilon g .$$

Therefore $\mathrm{Im}\, h \subset P$, $p_A h = \varepsilon^{-1} p_1 h = f$ and $p_B h = \varepsilon^{-1} p_2 h = g$. That is, P is the product of A and B.

Assume f and $g : A \to B$ are coalgebra maps. Put $D_1 = \mathrm{Ker}\,(Uf - Ug)$, and define D_2 by $D_2 = \{a \in A \mid \Delta_A a \subset D_1 \otimes D_1\}$. Since for $a \in D_2$, $a = (\varepsilon_A \otimes 1) \cdot \Delta_A a \in (\varepsilon_A \otimes 1)(D_1 \otimes D_1) \subset D_1$, we have $D_2 \subset D_1$. Inductively, we define D_n

by $D_n = \{a \in A \mid \Delta_A a \subset D_{n-1} \otimes D_{n-1}\}$, then $D_n \subset D_{n-1}$. Put $D = \bigcap_{n=1}^{\infty} D_n$, then it follows that D is a subcoalgebra of A and the natural inclusion map $i: D \to A$ is a coalgebra map.

Suppose there is a coalgebra map $h: L \to A$ with $fh = gh$. Then hL is a subcoalgebra of A contained in D. So there is a unique coalgebra map $l: L \to D$ with $il = h$. Thus D is the difference kernel for f and g.

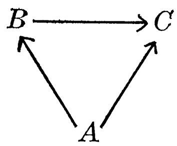
q.e.d.

(Note: In our proposition (finite) means that the first part holds even if the word "finite" is omitted.) In consequence $\mathscr{C}$ has inverse limits (p. 38 of [19]).

2. Abelian cogroup objects in $(A, \mathscr{C})$

In the sequal we assume that A is a coalgebra.

DEFINITION 2.1. We define a new category $(A, \mathscr{C})$ whose each object is $A \to B$ in $\mathscr{C}$, and each morphism is a commutative diagram



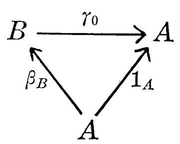
for any two objects $A \to B$ and $A \to C$ in $(A, \mathscr{C})$, where $B \to C$ is in $\mathscr{C}$. An object $\beta_B: A \to B$ in $(A, \mathscr{C})$ will be denoted by (β_B, B) (or simply by B), and β_B is called the *structure map* of B.

An *abelian cogroup object* in $(A, \mathscr{C})$ is an object B such that for any $C \in (A, \mathscr{C})$, $\mathrm{Hom}_{(A,\mathscr{C})}(B, C)$ is an abelian group and

$$\mathrm{Hom}_{(A,\mathscr{C})}(B, —): (A, \mathscr{C}) \to Ab$$

is a covariant functor, where Ab is the category of all abelian groups (for abelian group objects see [11]).

Let B be an abelian cogroup object in $(A, \mathscr{C})$. In the commutative diagram



with γ_0 the zero element of the abelian group $\mathrm{Hom}_{(A,\mathscr{C})}(B, A)$, we put M

$= \operatorname{Ker} U\gamma_0$, then M is a coideal (see Proposition 1.5), and B is isomorphic to direct sum $B \cong A \oplus M$ as K-modules.

Consider the diagram of the pushout

$$\begin{array}{ccc} A & \xrightarrow{\beta_B} & B \\ {\scriptstyle \beta_B}\downarrow & & \downarrow{\scriptstyle i_2} \\ B & \xrightarrow{i_1} & B \amalg_A B \end{array}$$

then it follows that we have a canonical direct decomposition $B \amalg_A B \cong A \oplus M \oplus M$ as K-modules. We put

$$\mu = i_1 + i_2 \in \operatorname{Hom}_{(A,\mathscr{C})}(B, B \amalg_A B) .$$

Identifying A and $\beta_B A$ we have

$$\mu a = i_1 a = i_2 a = (a, 0, 0) \qquad \text{for } a \in A$$
$$i_1 m = (0, m, 0), i_2 m = (0, 0, m), \mu m = (0, m, m) \qquad \text{for } m \in M .$$

PROPOSITION 2.2. *Under the above situation*

$$\Delta_B a = \Delta_A a \qquad \text{for } a \in A \text{ and } \Delta_B M \subseteq A \otimes M + M \otimes A .$$

Proof. Assume $\Delta_B m = \sum_i [a_i \otimes m_i + m'_i \otimes a'_i + m''_i \otimes m'''_i]$ for $m \in M$, where $a_i, a'_i \in A$ and $m_i, m'_i, m''_i, m'''_i \in M$. Since $(\mu \otimes \mu)\Delta_B = \Delta_{B \amalg_A B} \cdot \mu$ we have the following:

$$\begin{aligned} \Delta_{B\amalg_A B} \cdot \mu m &= \Delta_{B \amalg_A B}(i_1 + i_2)m = (i_1 \otimes i_1)\Delta_B m + (i_2 \otimes i_2)\Delta_B m \\ &= \textstyle\sum_i [(a_i, 0, 0) \otimes (0, m_i, m_i) + (0, m'_i, m'_i) \otimes (a'_i, 0, 0) \\ &\quad + (0, m''_i, 0) \otimes (0, m'''_i, 0) + (0, 0, m''_i) \otimes (0, 0, m'''_i)] , \end{aligned}$$

and on the other hand,

$$\begin{aligned} (\mu \otimes \mu)\Delta_B m &= \textstyle\sum_i [(a_i, 0, 0) \otimes (0, m_i, m_i) + (0, m'_i, m'_i) \otimes (a'_i, 0, 0) \\ &\quad + (0, m''_i, 0) \otimes (0, m''_i, 0) + (0, 0, m''_i) \otimes (0, 0, m''_i) \\ &\quad + (0, m''_i, 0) \otimes (0, 0, m'''_i) + (0, 0, m''_i) \otimes (0, m'''_i, 0)] . \end{aligned}$$

Therefore we have $\sum_i [(0, m''_i, 0) \otimes (0, 0, m'''_i) + (0, 0, m''_i) \otimes (0, m'''_i, 0)] = 0$, which implies that $\sum_i m''_i \otimes m'''_i = 0$. $\Delta_B a = \Delta_A a$ is clear. q.e.d.

COROLLARY 2.3. *With the above situation M is a two sided A-comodule.*

Proof. Since we can put $\Delta_B m = \sum_i [a_i \otimes m_i + m'_i \otimes a'_i]$ for $m \in M$

we define $\Delta^l_M : M \to A \otimes M$ and $\Delta^r_M : M \to M \otimes A$ by $\Delta^l_M m = \sum_i a_i \otimes m_i$ and $\Delta^r_M m = \sum_i m'_i \otimes a'_i$. By the coassociativity of Δ_B we know that M satisfies all conditions (see Definition 1.3) for a two sided A-comodule. q.e.d.

Conversely, let M be a two sided A-comodule. We define $\Delta_{A\oplus M} : A \oplus M \to (A \oplus M) \otimes (A \oplus M)$ and $\varepsilon_{A\oplus M} : A \oplus M \to K$ by

$$\Delta_{A\oplus M} = \Delta_A \text{ on } A , \qquad \Delta_{A\oplus B} = \Delta^l_M + \Delta^r_M \text{ on } M$$
$$\varepsilon_{A\oplus M}(a, m) = \varepsilon_A a \qquad \text{for } (a, m) \in A \oplus M .$$

Then $A \oplus M$ is a coalgebra, and $A \oplus M$ with the natural structure map $A \to A \oplus M$ is an object in $(A, \mathscr{C})$. The coalgebra $A \oplus B$ is called the *coidealization of* M, and we shall denote this by $A * M$.

For an object C in $(A, \mathscr{C})$ and a two sided A-comodule M a *coderivation* $f : M \to C$ is defined as a K-module map such that

$$\Delta_C f m = \sum_i [\beta_C a_i \otimes f m_i + f m'_i \otimes \beta_C a'_i] .$$

where $m \in M$ and $\Delta_{A*M} m = \sum_i [a_i \otimes m_i + m'_i \otimes a'_i]$. The set of all such coderivations $f : M \to C$ forms an abelian group denoted by $\mathrm{Coder}_M(C)$, and it gives the *coderivation functor*

$$\mathrm{Coder}_M : (A, \mathscr{C}) \to Ab . \tag{2.4}$$

As is well known, there is a canonical isomorphism

$$\mathrm{Coder}_M(C) \cong \mathrm{Hom}_{(A,\mathscr{C})}(A * M, C) . \tag{2.5}$$

We sometimes put $\mathrm{Coder}_M(C) = \mathrm{Coder}_K(M, C)$. This shows that a coidealization of a two sided A-comodule is an abelian cogroup object in $(A, \mathscr{C})$. Accordingly we have:

THEOREM 2.6. *An object B in $(A, \mathscr{C})$ is an abelian cogroup object iff there is a two sided A-comodule M such that $A * M \cong B$ as coalgebras.*

We shall denote by ${}_ACM_A$ the category consisting of all two sided A-comodules and two sided A-comodule maps. Since A is an ungraded coassociative coalgebra over a field K, we can easily check that ${}_ACM_A$ is an abelian category.

3. Cosimplicial Cohomology

We shall begin with the general theory for right derived functors. Let $\mathscr{A}$ be an arbitrary category. We define the category $\mathscr{A}^+$ whose objects are the same as those of $\mathscr{A}$ and whose morphisms are formal sum of morphisms in $\mathscr{A}$, i.e., $\mathrm{Hom}_{\mathscr{A}^+}(X, Y)$ is the free abelian group on $\mathrm{Hom}_{\mathscr{A}}(X, Y)$.

DEFINITION 3.1. Let J be a class of objects in $\mathscr{A}$. A *J-injective resolution* for $X \in \mathscr{A}$ is a cochain complex

$$X \to X_0 \to X_1 \to \cdots (*)$$

in $\mathscr{A}^+$ satisfying

(i) $X_n \in J$ for $n \geq 0$

(ii) for each $Y \in J$ the functor $\mathrm{Hom}_{\mathscr{A}^+}(\_, Y)$ transforms $(*)$ into an acyclic complex of abelian groups

$$0 \leftarrow \mathrm{Hom}_{\mathscr{A}^+}(X, Y) \leftarrow \mathrm{Hom}_{\mathscr{A}^+}(X_0, Y) \leftarrow \cdots .$$

A *class of injective models* for $\mathscr{A}$ is a class J of objects in $\mathscr{A}$ such that each $X \in \mathscr{A}$ has a J-injective resolution ([3], [18]).

If a category $\mathscr{A}$ has a class J of injective models, for each covariant functor $T : \mathscr{A} \to \mathscr{A}b$ from $\mathscr{A}$ to an abelian category $\mathscr{A}b$ right derived functors $R^nT : \mathscr{A} \to \mathscr{A}b (n \geq 0)$ of T with respect to J can be defined as follows. If X is an object in $\mathscr{A}$, then R^nTX is the n-th cohomology group of

$$0 \longrightarrow T^+X_0 \xrightarrow{T^+d_1} T^+X_1 \xrightarrow{T^+d_2} T^+X_2 \longrightarrow \cdots ,$$

where $0 \to X \to X_0 \xrightarrow{d_1} X_1 \xrightarrow{d_2} X_2 \to \cdots$ is a J-injective resolution of X and $T^+ : \mathscr{A}^+ \to \mathscr{A}b$ is a unique additive functor induced from T. Note that $T^+X = TX$ for $X \in \mathscr{A}$. The obvious comparison theorem holds for J-injective resolutions, and therefore we get a unique functor $R^nT : \mathscr{A} \to \mathscr{A}b$ up to natural equivalence, which is called the *right derived functor of T*. As is well known ([12]),

(i) $X \to X_0$ induces a natural map $TX \to R^0TX$

(ii) for $X \in J, TX \to R^0TX$ is isomorphic and $R^nTX = 0$ if $n > 0$.

It is convenient to use cosimplicial method in studying derived functors.

DEFINITION 3.2. A *cosimplicial object* X_* in a category $\mathscr{A}$ consists of

(i) an object $X_n \in \mathscr{A}$ for each $n \geq 0$.

(ii) morphisms $\varepsilon^i : X_{n-1} \to X_n$ $(0 \leq i \leq n)$ (*coface operators*) and $\delta^i : X_n \to X_{n-1}$ $(0 \leq i \leq n-1)$ (*codegeneracy operators*) such that

$$\varepsilon^j\varepsilon^i = \varepsilon^i\varepsilon^{j-1} \text{ if } i < j\,, \qquad \delta^j\delta^i = \delta^i\delta^{j+1} \text{ if } i \leq j\,,$$
$$\delta^j\varepsilon^i = \varepsilon^i\delta^{j-1} \text{ if } i < j\,,$$
$$\delta^i\varepsilon^i = \text{identity} = \delta^i\varepsilon^{i+1}, \delta^j\varepsilon^i = \varepsilon^{i-1}\delta^j \text{ if } i > j+1\,.$$

An *augumentation* for X_* consists of a map $\varepsilon: X_{-1} \to X_0 \in \mathscr{A}$ with $\varepsilon^1\varepsilon = \varepsilon^0\varepsilon: X_{-1} \to X_1$. If X_* is an augumented cosimplicial object of X:

$$X_*: 0 \longrightarrow X \xrightarrow{\varepsilon} X_0 \underset{\varepsilon^1}{\overset{\varepsilon^0}{\rightrightarrows}} X_1 \cdots \underset{\varepsilon^n}{\overset{\varepsilon^0}{\vdots}} X_n \cdots,$$

then there is a cochain complex

$$\text{ch}^+ X_*: 0 \longrightarrow X \xrightarrow{\varepsilon} X_0 \xrightarrow{d_1} X_1 \xrightarrow{d_2} X_2 \longrightarrow \cdots$$

in $\mathscr{A}^+$, where $d_n = \sum_{i=1}^n (-1)^i\varepsilon^j$ $(n \geq 1)$.

We next prove that the category $(A, \mathscr{C})$ has a class of injective models using a triple, and define right derived functors from $(A, \mathscr{C})$ to Ab. Recall that there is the adjoint pair $U \dashv F$ between categories $\mathscr{C}$ and $\mathscr{M}_k$ (§ 1). There is a triple (G, ε, δ) ([1]). That is, put

$$G = FU: \mathscr{C} \to \mathscr{C}\,, \quad \varepsilon: 1_{\mathscr{C}} \to G\,, \quad \delta: GG \to G\,,$$

where $\delta = F\eta U: GG = F(UF)U \to FU = G$. For

$$G_*A = \{G_nA \mid G_nA = G^{n+1}A, n \geq -1\}$$

we define the following:

$$(G_{n-1}A \xrightarrow{\varepsilon^i} G_nA) = (G^nA \xrightarrow{G^i\varepsilon G^{n-i}} G^{n+1}A) \qquad (0 \leq i \leq n)$$
$$(G_{n+1}A \xrightarrow{\delta^i} G_nA) = (G^{n+2}A \xrightarrow{G^i\delta G^{n-i}} G^{n+1}A) \qquad (0 \leq i \leq n)\,,$$

then ε^i and δ^i satisfy the identities in Definition 3.2. Therefore we obtain an augumented cosimplicial object G_*A of A with the natural augumentation $\varepsilon: G_{-1}A = A \to G_0A$, i.e.,

$$G_*A: 0 \longrightarrow A \xrightarrow{\varepsilon} GA \underset{\varepsilon^1}{\overset{\varepsilon^0}{\rightrightarrows}} G^2A \cdots \underset{\varepsilon^{n-1}}{\overset{\varepsilon^0}{\vdots}} G^nA \cdots.$$

This sequence together with codegeneracy operators δ^i is called the *augumenetd standard cosimplicial resolution of* A.

Proposition 3.3. *Let J be the class of all $C \in \mathscr{C}$ with $C \cong FV$ for some $V \in \mathscr{M}_k$, then J is a class of injective models for $\mathscr{C}$, and* $\mathrm{ch}^+ G_*A$ *is a J-injective resolution for $A \in \mathscr{C}$.*

Proof. Let us construct a contracting homotopy

$$S = \{S_n \mid S_n : \mathrm{Hom}_{\mathscr{C}^+}(G^nA, FV) \to \mathrm{Hom}_{\mathscr{C}^+}(G^{n+1}A, FV)\} \qquad (n \geq 0)\ ,$$

where $G^0A = A$. For each $f: G^nA \to FV$ in $\mathscr{C}$ we define $S_nf = F\eta_V \cdot Gf$. Setting

$$d_n^* = \mathrm{Hom}_{\mathscr{C}^+}(d_n, FV)\ (n \geq 1) \quad \text{and} \quad d_0^* = \varepsilon^* = \mathrm{Hom}_{\mathscr{C}^+}(\varepsilon, FV)$$

$$\begin{aligned} d_n^* S_n f &= \sum_{i=0}^{n} (-1)^i F\eta_V \cdot Gf \cdot \varepsilon^i = F\eta_V \cdot Gf \cdot \varepsilon^0 + \sum_{i=1}^{n} (-1)^i F\eta_V \cdot Gf \cdot \varepsilon^i \\ &= F\eta_V \cdot \varepsilon F \cdot f + \sum_{i=1}^{n} (-1)^i F\eta_V \cdot G(f \cdot G^{i-1}\varepsilon G^{n-i}) \\ &= f + \sum_{i=1}^{n} (-1)^i F\eta_V \cdot G(f \cdot \varepsilon^{i-1}) = f - S_{n-1}(d_{n-1}^* f)\ (n \geq 1)\ , \\ d_0^* S_0 f &= F\eta_V \cdot Gf \cdot \varepsilon = F\eta_V \cdot \varepsilon F \cdot f = f\ . \end{aligned}$$

(Note: By (1, 10) $F\eta_V \cdot \varepsilon F = 1$.) So we have $d_n^* \cdot S_n f + S_{n-1} \cdot d_{n-1}^* f = f$, and therefore $S = \{S_n\}$ is the required contracting homotopy. q.e.d.

Note that each G^nA is regarded canonically as an object of $(A, \mathscr{C})$ with a unique structure map $A \to G^nA$ expressed by a composition of coface operators. By the same way as in the proof of the previous proposition we can prove that the category $(A, \mathscr{C})$ has a class $J_A = \{A \to FV \in (A, \mathscr{C}) \mid V \in \mathscr{M}_k\}$ of injective models. Also, if $A \to C$ is an object in $(A, \mathscr{C})$ then $\mathrm{ch}^+ G_*C$ is a J_A-injective resolution for C.

Let $B = A * M$, which is the coidealization of a two sided A-comodule M, be an abelian cogroup object in $(A, \mathscr{C})$. Then there is the covariant functor $T = \mathrm{Hom}_{(A,\mathscr{C})}(B, \_\_): (A, \mathscr{C}) \to Ab$, and therefore we can define the right derived functor of T such that

$$R^nTA = H^n\left(\sum_{m=0}^{\infty} \mathrm{Hom}_{(A,\mathscr{C})}(B, G_mA)\right) \qquad (\text{or } H^n(\mathrm{Hom}_{(A,\mathscr{C})}(B, G_*A)) \qquad (n > 0)\ ,$$

where H is the cohomology functor. By (2.4) and (2.5) we have

$$H^n(\mathrm{Hom}_{(A,\mathscr{C})}(B, G_*A)) \cong H^n(\mathrm{Coder}_k(M, G_*A)) \left(= H^n\left(\sum_{m=0}^{\infty} \mathrm{Coder}_k(M, G_mA)\right)\right) \qquad (n \geq 0)\ ,$$

and put $H^n(\mathrm{Coder}_k(M, G_*A)) = H^n(M, A)$. $H^n(M, A)$ is called the *n-dimensional cosimplicial cohomology group with coefficient in a two sided A-comodule M.* Since the functor $\mathrm{Coder}_k(M, \_)$ is left exact we have

THEOREM 3.4. $\mathrm{Hom}_{(A,\mathscr{C})}(B, A) \cong \mathrm{Coder}_k(M, A) \cong H^0(M, A)$.

THEOREM 3.5. *If $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ is exact in ${}_ACM_A$, then there is the long exact sequence of the cohomology groups*

$$0 \longrightarrow H^0(M'', A) \longrightarrow H^0(M, A) \longrightarrow H^0(M', A) \xrightarrow{\Delta_0} H'(M'', A) \longrightarrow \cdots .$$

Proof. Let $\bar{i} : A * M' \to A * M$ and $\bar{j} : A * M \to A * M''$ be the induced maps of i and j, respectively. Assume $FV \in (A, \mathscr{C})$ for some $V \in \mathscr{M}_K$. We shall first prove that the sequence

$$0 \longleftarrow \mathrm{Hom}_{(A,\mathscr{C})}(A * M', FV) \xleftarrow{\bar{i}^*} \mathrm{Hom}_{(A,\mathscr{C})}(A * M, FV) \xleftarrow{\bar{j}^*} \mathrm{Hom}_{(A,\mathscr{C})}(A * M'', FV) \longleftarrow 0$$

is exact, where $\bar{i}^* = \mathrm{Hom}_{(A,\mathscr{C})}(\bar{i}, FV)$ and $\bar{j}^* = \mathrm{Hom}_{(A,\mathscr{C})}(\bar{j}, FV)$. We can easily check that $\bar{j}^*$ is a monomorphism. Take a map $\bar{f} \in \mathrm{Hom}_{(A,\mathscr{C})}(A * M', FV)$, then by (1.8)$'$ there is a unique K-module map $\lambda^{-1}(\bar{f}) = f : U(A * M') \to V$. Since $U(A * M) \cong U(A * M') \oplus W$ for some K-module W, we can choose a K-module map $g : U(A * M') \to V$ such that $g \cdot U\bar{i} = f$. Again, by (1.8)$'$, there exists a unique coalgebra map $\bar{g} : A * M \to FV$ in $(A, \mathscr{C})$ corresponding to g, and therefore, by (1.10) we have $\lambda^{-1}(\bar{g} \cdot \bar{i}) = \eta(V) \cdot U\bar{g} \cdot U\bar{i} = g \cdot U\bar{i} = f = \lambda^{-1}(\bar{f})$ and $\bar{g} \cdot \bar{i} = \bar{f}$. Therefore $\bar{i}^*$ is an epimorphism. (In fact, FV is an injective object in $\mathscr{C}$).

Taking G^nA for FV, the usual argument in homological algebra gives the cohomology exact sequence, as asserted. q.e.d.

We conclude this section with a proposition which will be used sometimes later on. Recall the standard cosimplicial resolution of A. We put

$$UG_*A : 0 \longrightarrow UA \xrightarrow{U\varepsilon} UGA \xrightarrow{Ud_1} UG^2A \xrightarrow{Ud_2} \cdots ,$$

where $Ud_n = U\varepsilon^0 - U\varepsilon^1 + \cdots + (-1)^n U\varepsilon^n$ $(n \geq 1)$.

PROPOSITION 3.6. *The augumented cochain complex UG_*A is acyclic.*

Proof. By (1.9) and (1.10) there are K-module maps $\eta_n = \eta_{UG_nA}$:

$UG_{n+1}A \to UG_nA$ such that $\eta_n \cdot \varepsilon^0 = 1$, $\eta_n \cdot \varepsilon^i = \varepsilon^{i-1}\eta_{n-1}$ $(0 < i \leq n+1)$ and $\eta_{-1} \cdot \varepsilon = 1$. (Note: $G_{-1}A = A$.) We can easily check $\eta_n d_{n+1} + d_n \eta_{n-1} = 1$, and therefore $\eta = \{\eta_n \mid n \geq -1\}$ is the required contracting homotopy.

q.e.d.

4. Cosimplicial Coalgebras.

A cosimplicial object $\{A_* = A_n \mid n \geq 0\}$ in the category $\mathscr{C}$ is called a *cosimplicial coalgebra*. For an augumented cosimplicial coalgebra of A:

$$A*: 0 \longrightarrow A \xrightarrow{\varepsilon} A_0 \overset{\varepsilon^0}{\underset{\varepsilon^1}{\rightrightarrows}} A_1 \cdots \overset{\varepsilon^0}{\underset{\varepsilon^n}{\vdots}} A_n \cdots ,$$

we define $t_n: UA_n \to UA_n$ and $u_n: UA_{n+1} \to UA_n$ by

$$t_n = (1 - U\varepsilon^n \cdot U\delta^{n-1}) \cdots (1 - U\varepsilon^1 \cdot U\delta^0) \ (n \geq 1) , \qquad t_0 = 1_{A_0}$$
$$u_n = U\delta^0 \cdot t_0 - U\delta^1 \cdot t_1 + \cdots + (-1)^{n-1} U\delta^{n-1} t_{n-1} \ (n \geq 1) ,$$
$$u_0 = 0 = u_{-1} .$$

For simplicity we shall omit U in $UA, U\varepsilon^i$, etc., in the sequel. Then we have the following.

PROPOSITION 4.1. (i) $t_n\varepsilon^0 = d_n \cdot t_{n-1}$, $t_n\varepsilon^i = 0$ $(0 < i \leq n)$ *and* $1 - t_n = u_n d_{n+1} + d_n u_{n-1}$.

(ii) $\varepsilon^1 \cdot t_{n-1} = (\varepsilon^1 \cdot \delta^0 - \varepsilon^2 \delta^1)\varepsilon^0 \cdot t_n$, $\varepsilon^i t_{n-1} = (\varepsilon^i \delta^{i-1} - \varepsilon^{i-1}\delta^{n-2})\varepsilon^i t_{n-2}$ $(2 \leq i \leq n)$.

(iii) $t_n = t_{n+1} + (\varepsilon^{n+1}\delta^n - \varepsilon^n \delta^{n-1}) t_n$, *where* $d_n = \sum_{i=0}^{n} (-1)^i U\varepsilon^i$ $(n \geq 1)$.

Proof. We can prove (i) by mathematical induction (for detailes see p. 7 of [16]). (ii) and (iii) follow that $\delta^0 t_{n-1} = \delta^2 t_{n-2} (2 \leq i \leq n) = \delta^{n-1} t_n (n \geq 1) = 0$, $\delta^1 \varepsilon^0 = \varepsilon^0 \delta^0$ and $\delta^{i-2}\varepsilon^i = \varepsilon^{i-1}\delta^{i-2}$. q.e.d.

Put $\operatorname{Im} t_n = \tilde{A}_n$ $(n \geq 1)$ and $\tilde{A}_0 = A_0$, then we have the cochain complex of K-modules:

$$\tilde{A}_*: 0 \longrightarrow A \xrightarrow{\varepsilon} A_0 \xrightarrow{d_1} \tilde{A}_1 \longrightarrow \cdots ,$$

which is called the *normalized complex* of UA_* (for notation see Proposition 3.6). (Note: Since $d_n t_{n-1} = t_n \cdot \varepsilon^0$ and $t_n \cdot \varepsilon^i = 0$ $(i > 1)$, for $x \in \tilde{A}_{n-1}$ $d_n x \in \tilde{A}_n$.)

PROPOSITION 4.2. $\tilde{A}_n = \bigcap_{i=1}^{n-1} \operatorname{Ker} U\delta^i$ for $n \geq 1$.

Proof. For each $x \in \tilde{A}_n$ $t_n x = x$, and conversely $t_n x = x$ implies that

$x \in \tilde{A}_n$, because of $t_n \cdot t_n x = t_n x = x$. Take $x \in \bigcap_{i=1}^{n-1} \operatorname{Ker} U\delta^i$ then $t_n x = x$, and so $x \in \tilde{A}_n$. Conversely, take $x \in \tilde{A}_n$ then $\delta^i x = 0$, because of $\delta^i t_n x = 0$.

q.e.d.

COROLLARY 4.3. *$\tilde{A}_1$ is a coideal of A_1.*

The proof follows that $\tilde{A}_1$ is the kernel $U\delta^0$ of the coalgebra map δ^0 (Proposition 4.2).

PROPOSITION 4.4. *Two cochain complexes UA_* and $\tilde{A}_*$ are cochain equivalent.*

Proof. Since $1 - t_n = u_n t_{n-1} + d_n u_{n-1}$ and $t_n \mid \tilde{A}_n = 1_{\tilde{A}_n}$ (i.e., $t_n = 1_{\tilde{A}_n} \cdot t_n$), two cochain homomorphisms $t_n : A_n \to \tilde{A}_n$ and $1_{\tilde{A}_n} : A_n \to A_n$ are cochain homotopic with homotopy u_n. q.e.d.

DEFINITION 4.5. Let A_* be an augumented cosimplicial coalgebra of A. We define

$$\widetilde{\mathrm{Coder}}_K(M, A_n) = \{f \in \mathrm{Coder}_K(M, A_n) \mid t_n f = f\} ,$$

where M is a two sided A-comodule. In this case f is called a *normal coderivation.*

THEOREM 4.6. *$H^n(\mathrm{Coder}_K(M, A_*)) \cong H^n(\mathrm{Coder}_K(M, A_*))$ $(n \geq 0)$, where H is the cohomology functor.*

Proof. We put $(CD)_* = \mathrm{Coder}_K(M, A_*)$ and $(\tilde{C}D)_* = \mathrm{Coder}_K(M, A_*)$. Note that $(\tilde{C}D)_*$ is a cochain subcomplex of $(CD)_*$. Since t_n and u_{n-1} are linear compositions of coalgebra maps, respectively, for $f \in (CD)_n$ $t_n f \in (\tilde{C}D)_n$ and $u_{n-1} f \in (\tilde{C}D)_{n-1}$. By the same reason as in the proof of Proposition 4.4 $(CD)_*$ and $(\tilde{C}D)_*$ are cochain equivalent. q.e.d.

Put $H^n(\widetilde{\mathrm{Coder}}(M, G_* A)) = \tilde{H}^n(M, A)$. Then we know that $H^n(M, A) \cong \tilde{H}^n(M, A)$ by the above proposition. We shall conclude this section with description of two properties for cosimplicial coalgebras which are used in the next section.

PROPOSITION 4.7. *For an augumented cosimplicial coalgebra A_* of A the following hold:*

$$\text{(i) } A_1 = \varepsilon^1 A_0 \oplus \tilde{A}_1 , \qquad \text{(ii) } A_n = (\varepsilon^1)^n A_0 \oplus \tilde{A}_n \oplus I_n \ (n \geq 2) ,$$

where $\oplus$ means direct sum as K-modules and $I_n = \sum_{i=1}^{n-1} (\varepsilon^i \delta^{i-1} - \varepsilon^{i+1} \delta^i) A_n$.

Proof. (i) is clear. Put $(\varepsilon^i)^n x_0 + x_n + \sum_{i=1}^{n-1}(\varepsilon^i\delta^{i-1} - \varepsilon^{i+1}\delta^i)y_i = 0$ for $x \in A_0, x_n \in \tilde{A}_n$ and $y_i \in A_n$ $(1 \leq i \leq n-1)$. Apply $(\delta^0)^n$ from the left on both sides, then $x_0 = 0$, because of $(\delta^0)^n t_n x_n = 0 = (\delta^0)^n(\varepsilon^i\delta^{i-1} - \varepsilon^{i+1}\delta^i)y_i = 0$. Again, apply t_n from the left on both sides of the above, then $x_n = 0$, because of $t_n(\varepsilon^1)^n x_0 = 0 = t_n(\varepsilon^i\delta^{i-1} - \varepsilon^{i+1}\delta^i)y_i$.

We want to prove that A_n is spanned by $(\varepsilon^1)^n A_0, \tilde{A}_n$ and I_n. To do this it suffices to show that there exist $x_0 \in A_0$ and $x_n \in \tilde{A}_n$ for a given $x \in A_n$ such that $x = (\varepsilon^1)^n x_0 + x_n \bmod. I_n$. Since $x = \varepsilon^1\delta^0 x + t_1 x$ and $(\varepsilon^1)^{r+1} \cdot (\varepsilon^0)^{r+1} - (\varepsilon^1)^r(\varepsilon^0)^r = (\varepsilon^{r+1}\delta^r - \varepsilon^r\delta^{r-1})(\varepsilon^1)^i(\delta^0)^r \in I_{n+1} = \sum_{i=1}^r(\varepsilon^i\delta^{i-1} - \varepsilon^{i+1}\delta^i)A_n$, if we assume that $x \equiv (\varepsilon^1)^r(\delta^0)^r x + t_r x$ $(r \geq 2) \bmod. I_r$, then we have $x \equiv (\varepsilon^1)^{r+1}(\delta^0)^{r+1}x + t_{r+1}x \bmod. I_{r+1}$ by Proposition 4.1. q.e.d.

In the above proposition I_n $(n \geq 2)$ is a coideal of A_n (Proposition 1.5), so we have the quotient coalgebra $\bar{A}_n = A_n/I_n$ (Proposition 1.5). Let M be a two sided A-comodule. Then M is a two sided A_{n-1}-comodule, and hence we have the coidealization $D_{n-1} = A_{n-1} * M$ of M.

Consider two coalgebra maps $\rho_0, \rho_1 : D_{n-1} \to \bar{A}_n$ defined by

(4.8) $\rho_0 =$ the composition:

$$D_{n-1} = A_{n-1} * M \xrightarrow{\text{Projection}} A_{n-1} \xrightarrow{\varepsilon^0} A_n \xrightarrow{\text{Projection}} \bar{A}_n\ ,$$

$\rho_1 =$ the composition:

$$D_{n-1} = A_{n-1} * M \xrightarrow{(\delta^0)^{n-1} \oplus 1_M} A_0 * M \xrightarrow{((\varepsilon^1)^n \oplus f)} \bar{A}_n\ ,$$

where $f \in \widetilde{\text{Coder}}_K(M, A_n)$ with $d_{n+1}f = 0$.

PROPOSITION 4.9. $\rho_0 | A_{n-1} = d_n\tilde{A}_{n-1}$.

Proof. We have to prove that for each $x \in \tilde{A}_{n-1}$ $\varepsilon^0 x \equiv d_n x \bmod. I_n$. By Proposition 4.1 $\varepsilon^i t_{n-1}x \in I_n$ $(1 \leq i \leq n)$, and therefore $\varepsilon^0 t_{n-1}x \equiv d_n t_{n-1}x = d_n x \bmod. I_n$. q.e.d.

5. Interpretation of $H^2(M, A)$

In this section we assume that A is a coalgebra and M is a two sided A-comodule.

DEFINITION 5.1. By a (*normal*) *coderivation* 2-*cocycle* we mean a normal coderivation $f : M \to \widetilde{G^3A}$ such that $d_3 f = 0$. Two such cocycles f and f' are *cohomologous* if there exists a normal coderivation $g : M \to \widetilde{G^2A}$ such that $f - f' = d_2 g$.

DEFINITION 5.2. By a *two term extension of A by M* we mean an exact sequence (e) of K-modules:

$$(e):\ 0 \longrightarrow A \xrightarrow{\varphi_0} X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} M \longrightarrow 0$$

satisfying the conditions:

(i) X_0 is a coalgebra and φ_0 a coalgebra map,

(ii) X_1 is a two sided X_0-comodule with $(\varphi_1 \otimes 1)\Delta^l_{X_1} = (1 \otimes \varphi_1)\Delta^r_{X_1}$ and ϕ_1 a two sided X_0-comodule map,

(iii) M is a two sided X_0-comodule induced by φ_0 and φ_2 a two sided X_0-comodule map. That is, the structure of M as a two sided X_0-comodule is induced from the commutative diagrams

$$\begin{array}{ccc} X_0 \otimes M & \xleftarrow{\ \varphi_0 \otimes 1\ } & A \otimes M \\ & {\scriptstyle \Delta^l_M}\nwarrow & \uparrow {\scriptstyle \omega^l_M} \\ & & M\,, \end{array} \qquad \begin{array}{ccc} M \otimes A & \xrightarrow{\ 1 \otimes \varphi_0\ } & M \otimes X_0 \\ {\scriptstyle \omega^l_M} \uparrow & \nearrow {\scriptstyle \Delta^r_M} & \\ M & & \end{array}\ ,$$

where ω^l_M and ω^r_M are the comodule structure maps of M as a two sided A-comodule.

DEFINITION 5.3. The totality of all two term extensions of A by M forms a category whose each *morphism* (e)$\rightarrow$(e$'$) is a commutative diagram

$$\begin{array}{lccccccccc} (e): & 0 \longrightarrow & A & \xrightarrow{\varphi_0} & X_0 & \xrightarrow{\varphi_1} & X_1 & \xrightarrow{\varphi_2} & M & \longrightarrow 0 \\ & & \| & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \| & \\ (e'): & 0 \longrightarrow & A & \xrightarrow{\varphi_0'} & X_0' & \xrightarrow{\varphi_1'} & X_1' & \xrightarrow{\varphi_2'} & M & \longrightarrow 0 \end{array}$$

with ψ_0 a coalgebra map and ψ_1 a K-module map such that $(\psi_0 \otimes \psi_1)\Delta^l_{x_1}$ $\Delta^l_{x_1'}\psi_1$ and $(\psi_1 \otimes \psi_0)\Delta^r_{X_1} = \Delta^r_{X_1'}\psi_1$.

For two term extensions (e) and (e') of A by M they are said to be *equivalent*, written $(e) \sim (e')$, if they are connected by a sequence of morphisms of both direction, e.g.,

$$(e) = (e_0) \longleftarrow (e_1) \longrightarrow \cdots \longleftarrow (e_n) = (e')\ .$$

The main theorem of this paper is the following.

THEOREM 5.4. *Let $Ex^2(M, A)$ be the set of all equivalence classes* *of all two term extensions of* A *by* M. *Then there is a bijection between* $Ex^2(M, A)$ *and* $H^2(M, A)$.

We will prove this theorem in three steps: i) to define $\alpha: \tilde{H}^2(M, A) \to Ex^2(M, A)$. ii) to define $\beta: Ex^2(M, A) \to \tilde{H}^2(M, A)$, and iii) to prove that $\beta\alpha = 1_{\tilde{H}^2(M,A)}$ and $\alpha\beta = 1_{Ex^2(M,A)}(H^2(M, A) \cong \tilde{H}^2(M, A))$.

First Step: Consider the standard cosimplicial resolution of A.

PROPOSITION 5.5. *G^3A is expressed by the direct sum* $G^3A = (\varepsilon^1)^2 G\alpha \oplus \widetilde{G^3A} \oplus \varepsilon^1\widetilde{G^2A} \oplus \varepsilon^2\widetilde{G^2A}$.

Proof of 5.5. At first we prove that $I_2 = (\varepsilon^1\delta^0 - \varepsilon^2\delta^1)G^3A = \varepsilon^1\widetilde{G^2A} \oplus \varepsilon^2\widetilde{G^2A}$. To do this we have to prove that $\varepsilon^1\widetilde{G^2A} \subset I_2 \supset \varepsilon^2\widetilde{G^2A}$. For $x \in \widetilde{G^2A}$

$$\varepsilon^1 x = (\varepsilon^1\delta^0 - \varepsilon^2\delta^1)\varepsilon^0 x \in I_2 \ni \varepsilon^2 x = (\varepsilon^2\delta^1 - \varepsilon^1\delta^0)\varepsilon^2 x \qquad (\text{Note: } \delta^0 x = 0) .$$

Conversely, for $x \in G^3A$ and $(\varepsilon^1\delta^0 - \varepsilon^2\delta^1)x \in I_2$

$$(\varepsilon^1\delta^0 - \varepsilon^2\delta^1)x = \varepsilon^1(1 - \varepsilon^1\delta^0)\delta^0 x - \varepsilon^2(1 - \varepsilon^1\delta^0)\delta^1 x \in \varepsilon^1\widetilde{G^2A} + \varepsilon^2\widetilde{G^2A} .$$

Assume that $\varepsilon^1 x = \varepsilon^2 y$ for $x, y \in G^2A$. Applying δ^1 we have $x = y$. That is, $\varepsilon^1 x = \varepsilon^2 y$ implies that $\varepsilon^1 x = \varepsilon^2 x$. Applying $(1 - \varepsilon^1\delta^0)$ we have $0 = \varepsilon^2 x$, i.e., $x = 0$. So $\varepsilon^1\widetilde{G^2A} \cap \varepsilon^2\widetilde{G^2A} = \{0\}$.

In order to complete our proof it suffices to verify that $G^3A = (\varepsilon^1)^2GA \oplus \widetilde{G^3A} \oplus I_2$. But, this is just the case $n = 2$ in Proposition 4.7. q.e.d.

Put $GA = \varepsilon A \oplus C, C = (1 - \varepsilon\eta_A)GA = \eta_{GA}d_1GA$,

$$\begin{aligned} G^2A &= \varepsilon^1 GA \oplus \widetilde{G^2A} = \varepsilon^0\varepsilon A \oplus \varepsilon^1 C \oplus N \oplus E , \\ &\qquad N = d_1 C \quad \text{and} \quad E = (1 - d_1\eta_{GA})\widetilde{G^2A} \\ &\qquad\qquad = \eta_{G^2A} d_2 \widetilde{G^2A} , \\ G^3A &= (\varepsilon^0)^2\varepsilon A \oplus (\varepsilon^1)^2 C \oplus \widetilde{G^3A} \oplus \varepsilon^1\widetilde{G^2A} \oplus \varepsilon^2\widetilde{G^2A} \\ &= (\varepsilon^0)^2\varepsilon A \oplus (\varepsilon^1)^2 C \oplus \widetilde{G^3A} \oplus \varepsilon^1 N \oplus \varepsilon^2 N \oplus \varepsilon^1 E \oplus \varepsilon^2 E^2 . \end{aligned}$$

Given a coderivation 2-cocycle f we want to calculate $\Delta_{G^2A}x$ for $x \in E$ with $d_2 x = fm$, where $m \in M$. Since G^2A is a coideal of $\widetilde{G^2A}$ (Corollary 4.3)

$$\begin{aligned} \Delta_{G^2A}x &\subset G^2A \otimes \widetilde{G^2A} + \widetilde{G^2A} \otimes G^2A \\ &= \varepsilon^0\varepsilon A \otimes \widetilde{G^2A} + \widetilde{G^2A} \otimes \varepsilon^0\varepsilon A + \varepsilon^1 C \otimes \widetilde{G^2A} + \widetilde{G^2A} \otimes \varepsilon^1 C \\ &\quad + N \otimes E + E \otimes N + E \otimes E + N \otimes N . \end{aligned}$$

So we may put

$$\begin{aligned} \Delta_{G^2A}x = \sum_i [&\varepsilon^0\varepsilon a_i \otimes x_i + x_i' \otimes \varepsilon^0\varepsilon a_i' + \varepsilon^1 c_i \otimes y_i + y_i' \otimes \varepsilon^1 c_i' \\ &+ n_i \otimes e_i + e_i' \otimes n_i' + e_i'' \otimes e_i''' + n_i'' \otimes n_i'''] , \end{aligned}$$

where $a_i, a_i' \in A, x_i, x_i', y_i, y_i' \in \widetilde{G^2A}, c_i, c_i' \in C, n_i, n_i', n_i'', n_i''' \in N$ and e_i, e_i', e_i'', $e_i''' \in E$

For convenience we put $T = \varepsilon^0 \otimes \varepsilon^0 - \varepsilon^1 \otimes \varepsilon^1 + \varepsilon^2 \otimes \varepsilon^2$. We have the following

$$\begin{aligned} T\Delta_{G^2A}x &= T\sum_i (\varepsilon^0\varepsilon a_i \otimes x_i + x_i' \otimes \varepsilon^0\varepsilon a_i') \\ &\quad + \sum_i ((\varepsilon^1)^2 c_i \otimes d_2 y_i + \varepsilon^2 d_1 c_i \otimes \varepsilon^0 y_i) \\ &\quad + \sum_i (d_2 y_i' \otimes (\varepsilon^1)^2 c_i' + \varepsilon^0 y_i' \otimes \varepsilon^2 d_1 c_i') \\ &\quad + T\sum_i (n_i \otimes e_i + e_i' \otimes n_i' + e_i'' \otimes e_i''' + n_i'' \otimes n_i''') \\ &= \Delta_{G^3A} fm \in (\varepsilon^0)^2\varepsilon A \otimes \widetilde{G^3A} + \widetilde{G^3A} \otimes (\varepsilon^0)^2\varepsilon A = W_0 , \end{aligned}$$

$$T\sum_i (\varepsilon^0\varepsilon a_i \otimes x_i + x_i' \otimes \varepsilon^0\varepsilon a_i') \in W_0 ,$$

$$\sum_i (\varepsilon^1)^2 c_i \otimes d_2 y_i \in (\varepsilon^1)^2 C \otimes \widetilde{G^3A} = W_1 ,$$

$$\sum_i \varepsilon^2 d_1 c_i \otimes \varepsilon^0 y_i \in \varepsilon^2 N \otimes \widetilde{G^3A} + \varepsilon^2 N \otimes \varepsilon^1 E + \varepsilon^2 N \otimes \varepsilon^2 E + \varepsilon^2 N \otimes \varepsilon^1 N + \varepsilon^2 N \otimes \varepsilon^2 N = W_2 ,$$

$$\sum_i d_2 y_i' \otimes (\varepsilon^1)^2 c_i' \in \widetilde{G^3A} \otimes (\varepsilon^1) c = W_3 ,$$

$$\sum_i \varepsilon^0 y_i' \otimes \varepsilon^2 d_1 c_i' \in \widetilde{G^3A} \otimes \varepsilon^2 N + \varepsilon^1 E \otimes \varepsilon^2 N + \varepsilon^2 E \otimes \varepsilon^2 N + \varepsilon^1 N \otimes \varepsilon^2 N + \varepsilon^2 N \otimes \varepsilon^2 N = W_4 ,$$

$$T\sum_i n_i \otimes e_i \in \varepsilon^1 N \otimes \widetilde{G^3A} + \varepsilon^1 N \otimes \varepsilon^2 E + \varepsilon^2 N \otimes \widetilde{G^3A} + \varepsilon^2 N \otimes \varepsilon^1 E + \varepsilon^2 N \otimes \varepsilon^2 E = W_5 ,$$

$$T\sum_i e_i' \otimes n_i' \in \widetilde{G^3A} \otimes \varepsilon^1 N + \varepsilon^2 E \otimes \varepsilon^1 N + \widetilde{G^3A} \otimes \varepsilon^2 N + \varepsilon^2 N \otimes \varepsilon^1 E + \varepsilon^2 E \otimes \varepsilon^2 N = W_6 ,$$

$$T\sum_i (e_i'' \otimes e_i''') \in \widetilde{G^3A} \otimes \widetilde{G^3A} + \widetilde{G^3A} \otimes \varepsilon^1 E + \widetilde{G^3A} \otimes \varepsilon^2 E + \varepsilon^1 E \otimes \widetilde{G^3A} + \varepsilon^2 E \otimes \widetilde{G^3A} + \varepsilon^1 E \otimes \varepsilon^2 E + \varepsilon^2 E \otimes \varepsilon^1 E + \varepsilon^2 E \otimes \varepsilon^2 E = W_7 ,$$

$$T\sum_i n_i'' \otimes n_i''' \in \varepsilon^2 N \otimes \varepsilon^2 N + \varepsilon^2 N \otimes \varepsilon^1 N + \varepsilon^1 N \otimes \varepsilon^2 N = W_8 .$$

In the above we know the following.

(i) Each term in W_0 does not appear in $W_1, \cdots, W_8$. This implies that

$$T\sum_i (\varepsilon^1 c_i \otimes y_i + y_i' \otimes \varepsilon^1 c_i' + n_i \otimes e_i + e_i' \otimes n_i' + e_i'' \otimes e_i''' + n_i'' \otimes n_i''') = 0 .$$

(ii) The term $\varepsilon^1 N \otimes \varepsilon^2 E$ in W_5 does not appear in $W_1, \cdots, W_4, W_6, \cdots, W_8$. Since ε^1 and ε^2 are monomorphisms we have $\sum_i n_i \otimes e_i = 0$.

(iii) The term $\varepsilon^2 E \otimes \varepsilon^1 N$ in W_6 does not appear in $W_1, \cdots, W_5, W_7$ and W_8. By the same reason as above we have $\sum_i e'_i \otimes n'_i = 0$.

(iv) The term $\varepsilon^2 E \otimes \varepsilon^1 E$ in W_7 does not appear in $W_1, \cdots, W_6$ and W_8. This implies that $\sum_i e''_i \otimes e'''_i = 0$.

(v) W_1 does not appear in $W_2, W_3, W_4, \cdots, W_8$. This implies that $\sum_i (\varepsilon^1)^2 c_i \otimes d_2 y_i = 0$. If we take $\{c_i\}$ as a base of C we have $d_2 y_i = 0$ for each i. So $y_i \in N$. Similarly $y'_i \in N$.

By (i) ~ (v) above we may put

$$\Delta_{G^2A} x = \sum_i [\varepsilon^0 \varepsilon a_i \otimes x_i + x'_i \otimes \varepsilon^0 \varepsilon a'_i + \varepsilon^1 c_i \otimes n'_i \otimes \varepsilon^1 c'_i + n''_i \otimes n'''_i] ,$$

where $a_i, a'_i \in A, x_i, x'_i \in \widetilde{G^2 A}, c_i, c'_i \in C$ and $n_i, n'_i, n''_i, n'''_i N$. Since

$$T \sum_i [\varepsilon^1 c_i \otimes n_i + n'_i \otimes \varepsilon^1 c'_i + n''_i \otimes n'''_i] \in W_8 \cap W_0 = 0 ,$$

we have $T \sum_i [\varepsilon^1 c_i \otimes n_i + n'_i \otimes \varepsilon^1 c'_i + n''_i \otimes n'''_i] = 0$. It follows that

(i) $\sum_i \varepsilon^2 d_1 c_i \otimes \varepsilon^1 n_i - \sum_i \varepsilon^2 n''_i \otimes \varepsilon^1 n'''_i = 0$ (in $\varepsilon^1 N \otimes \varepsilon^1 N$) ,

(ii) $\sum_i \varepsilon^2 n'_i \otimes \varepsilon^2 d_1 c'_i - \sum_i \varepsilon^1 n''_i \otimes \varepsilon^2 n'''_i = 0$ (in $\varepsilon^1 N \otimes \varepsilon^2 N$) ,

(iii) $2 \sum_i \varepsilon^2 n''_i \otimes \varepsilon^2 n'''_i - \sum_i \varepsilon^2 d_1 c_i \otimes \varepsilon^2 n_i - \sum_i \varepsilon^2 n'_i \otimes \varepsilon^2 d_1 c'_i = 0$ (in $\varepsilon^2 N \otimes \varepsilon^2 N$) .

We take $\{c_i\}$ and $\{c'_i\}$ as bases of C. Then $\{d_1 c_i\}$ and $\{d_1 c'_i\}$ are bases of N (Proposition 3.6). Take $n''_i = d_1 c_i$, then we have $n_i = n'''_i$ from (i) above.

Put $n_i = n'''_i = d_1 c''_i, c''_i = \sum_j \alpha_{ij} c'_j$, where $\alpha_{ij} \in K$ for each i and j. Then, by (ii) above and $n''_i = d_1 c_i$ we get $n'_j = \sum_i \alpha_{ij} d_1 c_i$. These expressions of n_i, n'_i, n''_i and n'''_i satisfy (iii) above.

Using

$$n''_i = d_1 c_i, n'_i = \sum_j \alpha_{ji} d_1 c_j \quad \text{and} \quad n_i = n'''_i = d_1 c''_i = d_1 (\sum_j \alpha_{ij} c'_j) ,$$

we have

$$\begin{aligned} \sum_i \varepsilon^1 c_i \otimes n_i &= \sum_i \varepsilon^1 c_i \otimes d_1 c''_i , \\ \sum_i n'_i \otimes \varepsilon^1 c'_i &= \sum_i (\sum_j \alpha_{ji} d_1 c_j) \otimes \varepsilon^1 c'_i \\ &= \sum_i d_1 c_i \otimes \varepsilon^1 (\sum_j \alpha_{ij} c'_j) = \sum_i d_1 c_i \otimes \varepsilon^1 c''_i , \\ \sum_i n''_i \otimes n'''_i &= \sum_i d_1 c_i \otimes d_1 c''_i . \end{aligned}$$

In consequence, writing newly c'_i instead of c''_i, for $x \in E$ with $d_2 x = fm$ ($m \in M$) we have

$$\Delta_{G^2A}x = \sum [\varepsilon^0\varepsilon a_i \otimes x_i + x'_i \otimes \varepsilon^0\varepsilon a'_i + \varepsilon^1 c_i \otimes d_i c'_i + d_1 c_i \otimes \varepsilon^1 c'_i + d_1 c_i \otimes d_1 c'_i] ,$$

where $a_i, a'_i \in A$, x_i, $x'_i \in \widetilde{G^2A}$, c_i, $c'_i \in C$, $\Delta_{A*M}m = \sum_i (a_i \otimes m_i + m'_i \otimes a'_i)$, $d_2 x_i = fm_i$ and $d_2 x'_i = fm'_i$ for each i.

We put $E(f) = \{(x, m) \in \widetilde{G^2A} \times M \mid d_2 x = fm$ for some $m \in M\}$, and recall the case $n=2$ in (4.8). There exist two coalgebra maps $\rho_0, \rho_1 : D_1 = G^2A * M \to \overline{G^3A}$.

PROPOSITION 5.6. *The difference kernel E_1 for ρ_0 and ρ_1 in $\mathscr{C}$ is just* Ker $(U\rho_0 - U\rho_1)$, i.e., $E_1 = \varepsilon^1 GA \oplus E(f)$.

Proof of 5.6. Ker $(U\rho_0 - U\rho_1) = \varepsilon^1 GA \oplus E(f)$ is obvious by the definitions of ρ_0 and ρ_1. Using $\Delta_{G^2A}x$ and $\Delta_{A*M}m$, for $(x, m) \in E(f)$

$$\Delta_{D_1}(x, m) = \sum_i [(\varepsilon^0\varepsilon a_i, 0) \otimes (x_i, m_i) + (x'_i, m''_i) \otimes (\varepsilon^0\varepsilon a'_i, 0) + \varepsilon^1 c_i \otimes (d_1 c'_i, 0) + (d_1 c_i, 0) \otimes \varepsilon^1 c'_i + (d_1 c_i, 0) \otimes (d_1 c'_i, 0)] ,$$

where $d_2 x_i = fm_i, d_2 x'_i = fm'_i$ and $\Delta_{A*M}m = \sum_i (a_i \otimes m_i + m'_i \otimes a'_i)$. This implies that

$$\Delta_{D_1}(\varepsilon^1 GA \oplus E(f) \subseteq (\varepsilon^1 GA \oplus E(f)) \otimes (\varepsilon^1 GA \oplus E(f)) ,$$

and therefore $E_1 = \varepsilon^1 GA \oplus E(f)$. q.e.d.

Put $\Delta_{E_1} = \Delta_{D_1} | \varepsilon^1 GA \oplus E(f)$, then we have

$$\Delta_{E_1}(d_1 c, 0) = \sum_i [\varepsilon^1 y_i \otimes (d_1 y'_i, 0) + (d_1 y_i, 0) \otimes \varepsilon^1 y'_i + (d_1 y_i, 0) \otimes (d_1 y'_i, 0)] ,$$

$$\Delta_{E_1}(x, m) = \sum_i [(\varepsilon^0\varepsilon a_i, 0) \otimes (x_i, m_i) + (x'_i, m'_i) \otimes (\varepsilon^0\varepsilon a_i, 0) + \varepsilon^1 c_i \otimes (d_1 c'_i, 0) + (d_1 c_i, 0) \otimes \varepsilon^1 c'_i + (d_1 c_i, 0) \otimes (d_1 c'_i, 0)] ,$$

where $\Delta_{GA}C = \sum_i y_i \otimes y'_i$. Define $\Delta^l_{E(f)} : E(f) \to GA \otimes E(f)$ and $\Delta^r_{E(f)} : E(f) \to E(f) \otimes GA$ by

$$\Delta^l_{E(f)}(d_1 c, 0) = \sum_i y_i \otimes (d_1 y'_i, 0) ,$$

$$\Delta^l_{E(f)}(x, m) = \sum_i [\varepsilon a_i \otimes (x_i, m_i) + c_i \otimes (d_1 c'_i, 0)]$$

$$\Delta^r_{E(f)}(d_1 c, 0) = \sum_i (d_1 y_i, 0) \otimes y'_i ,$$

$$\Delta^r_{E(f)}(x, m) = \sum_i [(x'_i, m'_i) \otimes \varepsilon a'_i + (d_1 c_i, 0) \otimes c'_i]$$

then $E(f)$ becomes a two sided GA-comodule by the coassociativity of Δ_{E_1}. In consequence we have an exact sequence of K-modules:

$$(e_f): 0 \longrightarrow A \xrightarrow{\varepsilon} GA \xrightarrow{d_1'} E(f) \xrightarrow{\varphi} M \longrightarrow 0 ,$$

where $d_i'x = (d_1x, 0)$ and $\varphi(x, m) = m$ for $x \in GA$ and $(x, m) \in E(f)$. This sequence (e_f) satisfies the conditions for a two term extension of A by M, which is called a *standard two term extension of A by M*.

Next, consider the commutative diagram

$$\begin{array}{lccccccccccc} (e_f): & 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & GA & \xrightarrow{d_1'} & E(f) & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\ & & & \| & & \| & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle f} & & \\ & 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & GA & \xrightarrow{d_1} & G^2A & \xrightarrow{d_2} & G^3A & \xrightarrow{d_3} & G^4A \end{array}$$

where $\psi_1(x, m) = x$. By the property of $E(f)$ we know that

$$\begin{aligned} \Delta_{G^2A}\psi_1 &= (\varepsilon^1 \otimes \psi_1)\Delta^l_{E(f)} + (\psi_1 \otimes \varepsilon^1)\Delta^r_{E(f)} + (d_1 \otimes \psi_1)\Delta^l_{E(f)} \\ &(= (\varepsilon^0 \otimes \psi_1)\Delta^l_{E(f)} + (\psi_1 \otimes \varepsilon^1)\Delta^r_{E(f)}) . \end{aligned} \tag{5.7}$$

PROPOSITION 5.8. *If two coderivations of 2-cocycles f and f' are cohomologous, then $(e_f) \sim (e_f')$.*

Proof of 5.8. By our assumption there is a normal coderivation $g: M \to \widetilde{G^2A}$ such that $f - f' = d_2g$. Define $\psi: E(f) \to E(f')$ by $\psi(x, m) = (x - gm, m)$ for $(x, m) \in E(f)$. Since $d_2(x - gm) = f'm$ $(x - gm, m) \in E(f')$. So we have the commutative diagram

$$\begin{array}{lccccccccccc} (e_f): & 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & GA & \xrightarrow{d_1'} & E(f) & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\ & & & \| & & \| & & \downarrow{\scriptstyle\psi} & & \| & & \\ (e_f'): & 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & GA & \xrightarrow{d_1'} & E(f') & \xrightarrow{\varphi'} & M & \longrightarrow & 0 . \end{array}$$

We want to prove that the diagram is a morphism $(e_f) \to (e_{f'})$. Note that $\Delta^l_{E(f')}\psi = \Delta^l_{E(f')}(\psi_1 - g\varphi, \varphi)$, which is defined by $\Delta_{G^2A}\psi_1 - \Delta_{G^2A}g\varphi$ and $\Delta_{GA*M}\varphi$, where $\psi_1: E(f) \to G^2A$ such that $\psi_1(x, m) = x$ for $(x, m) \in E(f)$. By (5.7), $\Delta_{G^2A}g\varphi = (\varepsilon^1 \otimes g\varphi)\Delta^l_{E(f)} + (g\varphi \otimes \varepsilon^1)\Delta^r_{E(f)}$ and $\Delta_{GA*M}\varphi = (1 \otimes \varphi)\Delta^l_{E(f)} + (\varphi \otimes 1)\Delta^r_{E(f)}$ we have

$$\Delta^l_{E(f')}\psi = \Delta^l_{E(f')}(\psi_1 - g\varphi, \varphi) = (1 \otimes (\psi_1 - g\varphi, \varphi))\Delta^r_{E(f)} = (1 \otimes \psi)\Delta^l_{E(f)} .$$

Similarly we have $\Delta^r_{E(f')}\psi = (\psi \otimes 1)\Delta^r_{E(f)}$. q.e.d.

Summarizing the above we get a map $\alpha: \tilde{H}^2(M, A) \rightarrow Ex^2(M, A)$ such that $\alpha[f] = [(e_f)]$, where $[f]$ means the cohomology class containing f, and $[(e)]$ the equivalence class of (e).

Second step: For a two term extension (e) of A by M we consider the diagram:

$$
\begin{array}{ccccccccccccc}
(e): & 0 & \longrightarrow & A & \underset{\xi_0}{\overset{\varphi_0}{\rightleftarrows}} & X_0 & \underset{\xi_1}{\overset{\varphi_1}{\rightleftarrows}} & X_1 & \underset{\xi_2}{\overset{\varphi_2}{\rightleftarrows}} & M & \longrightarrow & 0 & \\
 & & & \| & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle f} & & & \\
 & 0 & \longrightarrow & A & \underset{\eta_A}{\overset{\varepsilon}{\rightleftarrows}} & GA & \underset{\eta_{GA}}{\overset{d_1}{\rightleftarrows}} & G^2A & \xrightarrow{d_2} & G^3A & \xrightarrow{d_3} & G^4A &
\end{array}
$$

where ξ_0, ξ_1 and ξ_2 are K-module maps such that $\xi_0\varphi_0 = 1_A, \varphi_0\xi_0 + \xi_1\varphi_1 = 1_{x_0}$, $\varphi_1\xi_1 + \xi_2\varphi_2 = 1_{x_1}$ and $\varphi_2\xi_2 = 1_M$. Since $\xi_0 \in \operatorname{Hom}(UX_0, UA)$ there is a unique coalgebra map $\psi_0: X_0 \rightarrow GA$ such that $\eta_A\psi_0$ (see (1.7)). Since $\eta_A\varepsilon = \eta_A\psi_0\varphi_0 = 1_A$ we have $\varepsilon = \psi_0\varphi_0$ (see (1.7)).

Let $X_0 + X_1$ be the direct sum of X_0 and X_1 as K-modules. We give $X_0 + X_1$ the structure of a coalgebra as follows. Define $\Delta_{X_0+X_1}$ by

$$\Delta_{X_0+X_1} = \Delta_{X_0} \text{ on } X_0 , \qquad \Delta_{X_0+X_1} = \Delta^l_{X_1} + \Delta^r_{X_1} + (\varphi_1 \otimes 1)\Delta^l_{X_1} \text{ on } X_1$$

Then we can easily check the coassociativity of $\Delta_{X_0+X_1}$ by the coassociativity of Δ_{X_0}, the properties of X_1 and φ_1. If we define $\varepsilon_{X_0+X_1}(x + y)$ $\varepsilon_{X_0}x$ for $x + y \in X_0 + X_1$ then $X_0 + X_1$ is a coalgebra.

Take $\rho' \in \operatorname{Hom}(U(X_0 + X_1), UGA)$ such that $\rho'(x + y) = \psi_0\varphi_0\xi_1 x + \psi_0\xi_1 y = \varepsilon\eta_A\psi_0 x + \psi_0\xi_1 y$. (Note: $\psi_0\varphi_0\xi_0 = \varepsilon\xi_0 = \varepsilon\eta_A\psi_0$.) Then there is a unique coalgebra map $\rho: X_0 + X_1 \rightarrow G^2A$ such that $\rho' = \eta_{GA}\cdot\rho$ (see (1.7)).

Proposition 5.9. *The coalgebra map ρ is denoted by $\rho(x + y) = \varepsilon^1\psi_0 x + \bar{\psi}_1 y$, where $x + y \in X_0 + X_1$ and $\bar{\psi}_1: X_1 \rightarrow G^2A$ is a K-module map satisfying $d_1\psi_0 = \bar{\psi}_1\varphi_1$ and $\Delta_{G^2A}\bar{\psi}_1 = (\varepsilon^1\psi_0 \otimes \bar{\psi}_1)\Delta^l_{X_1} + (\bar{\psi}_1 \otimes \varepsilon^1\psi_0)\Delta^r_{X_1} + (d_1\psi_0 \otimes \bar{\psi}_1)\Delta^l_{X_1}$.*

Proof of 5.9. Define θ^0 and $\theta^1: X_0 \rightarrow X_0 + X_1$ by $\theta^0 x = x + \varphi_1 x$ and $\theta^1 x = x$ for $x \in X_0$. From the definition of $X_0 + X_1$ and the property of φ_1 we can prove that θ^0 and θ^1 are coalgebra maps. Since G^2A is the cofree coalgebra on UGA

$$
\begin{aligned}
\eta_{GA}\cdot\rho\theta^0 x &= \eta_{GA}\rho(x + \varphi_1 x) = \rho'(x + \varphi_1 x) = \varepsilon\eta_A\psi_0 x + \psi_0\xi_1\varphi_1 x \\
&= \varepsilon\eta_A\psi_0 x + \psi_0(1 - \varphi_0\xi_0)x = \psi_0 x = \eta_{GA}\cdot\varepsilon^0\psi_0 x .
\end{aligned}
$$

By the universal property of G^2A $\rho\theta^0 = \varepsilon^0\psi_0$. Similarly we get $\rho\theta' = \varepsilon^1\psi_0$. Thus

$$\rho(x + \varphi_1 x) = \varepsilon^1\psi_0 x + (\varepsilon^0 - \varepsilon^1)\psi_0 x = \varepsilon^1\psi_0 x + \bar{\psi}_1\varphi_1 x \ ,$$

which implies that $\bar{\psi}_1\varphi_1 = d_1\psi_0$. Since ρ is coalgebra map

$$\begin{aligned} \Delta_{G^2A}\bar{\psi}_1 = (\rho \otimes \rho)(\Delta_{X_0+X_1}|X_1) &= (\varepsilon^1\psi_0 \otimes \bar{\psi}_1)\Delta^l_{X_1} \\ &+ (\bar{\psi}_1 \otimes \varepsilon^1\psi_0)\Delta^r_{X_1} + (d_1\psi_0 \otimes \bar{\psi}_1)\Delta^l_{X_1} \ . \end{aligned}$$

q.e.d.

By (ii) of Definition 5.2 we know that for $y \in X_1$

$$\begin{aligned} \Delta^l_{X_1} y &= \sum_i (\varphi_0 a_i \otimes y_i + z_i \otimes \varphi_1 z'_i) \ , \\ \Delta^r_{X_1} y &= \sum_i (y'_i \otimes \varphi_0 a'_i + \varphi_1 z_i \otimes z'_i) \ , \end{aligned} \tag{5.10}$$

where $a_i, a'_i \in A, y_i, y'_i \in X_1$ and $z_i, z'_i \in Z = (1 - \phi_0\xi_0)X_0$. This implies that

$$\begin{aligned} \Delta_{G^2A}t_1\bar{\psi}_1 &= \Delta_{G^2A}\bar{\psi}_1 - (\varepsilon^1\delta^0 \otimes \varepsilon^1\delta^0)\Delta_{G^2A}\bar{\psi}_1 \\ &= (\varepsilon^1\psi_0 \otimes t_1\bar{\psi}_1)\Delta^l_{X_1} + (t_1\bar{\psi}_1 \otimes \varepsilon^1\psi_0)\Delta^r_{X_1} + (d_1\psi_0 \otimes t_1\bar{\psi}_1)\Delta^l_{X_1} \ , \end{aligned}$$

where $t_1 = 1 - \varepsilon^1\delta^0 : G^2A \to G^2A$. Put $\psi_1 = t_1\bar{\psi}_1$, then it follows that $\psi_1\varphi_2 = d_1\psi_0$ and

$$\begin{aligned} \Delta_{G^2A}\psi_1 &= (\varepsilon^1\psi_0 \otimes \psi_1)\Delta^l_{X_1} + (\psi_1 \otimes \varepsilon^1\psi_0)\Delta^r_{X_1} + (d_1\psi_0 \otimes \psi_1)\Delta^l_{X_1} \\ &= (\varepsilon^0\psi_0 \otimes \psi_1)\Delta^l_{X_1} + (\psi_1 \otimes \varepsilon^1\psi_0)\Delta^r_{X_1} \ . \end{aligned} \tag{5.11}$$

PROPOSITION 5.12. *$(1 \otimes d_2\psi_1)\Delta^l_{X_1}\xi_2 = (\varphi_0 \otimes d_2\psi_1\xi_2)\omega^l_M$ and $(d_2\psi_1 \otimes 1)\Delta^r_{X_1}\xi_2 = (d_2\psi_1\xi_2 \otimes \varphi_0)\omega^r_M$, where $\omega^l_M : M \to A \otimes M$ and $\omega^r_M : M \to M \otimes A$ are the comodule structure maps of M as a two sided A-comodule.*

Proof of 5.12. Recall that $1_{X_1} = \varphi_1\xi_1 + \xi_2\varphi_2$ and $\varphi_2\xi_2 = 1_M$ (see the above diagram). Assume $m \in M$ and $\Delta^l_M m = \sum_i \varphi_0 a_i \otimes m_i$. We put $y = \xi_2 m$ (see (5.10)), then

$$\begin{aligned} \Delta^l_{X_1}\xi_2 m = \Delta^l_{X_1} y &= \sum_i (\varphi_0 a_i \otimes y_i + z_i \otimes \varphi_1 z'_i) \\ &= \sum_i \varphi_0 a_i \otimes \xi_2 m_i + \sum_i (\varphi_0 a_i \otimes \varphi_1 x_i + z_i \otimes \varphi_1 z'_i) \ , \end{aligned}$$

where $y_i = \xi_2 m_i + \varphi_1 x_i$ for some $x_i \in X_0$. Since $\psi_1\varphi_1 = d_1\psi_0$ we have

$$(1 \otimes d_2\psi_1)\Delta^l_{X_1}\xi_2 = (1 \otimes d_2\psi_1)(\varphi_0 \otimes \xi_2)\omega^l_M = (\varphi_0 \otimes d_2\psi_1\xi_2)\omega^l_M \ .$$

Similarly, we get $(d_2\psi_1 \otimes 1)\Delta^r_{X_1}\xi_2 = (d_2\psi_1\xi_2 \otimes \varphi_0)\omega^r_M$. q.e.d.

PROPOSITION 5.13. *Let us put $f = d_2\psi_1\xi_2$, then f is a coderivation 2-cocycle.*

Proof of 5.13. It is obvious that $f: M \to \widetilde{G^3A}$. Put $T = \varepsilon^0 \otimes \varepsilon^0 - \varepsilon^1 \otimes \varepsilon^1 + \varepsilon^2 \otimes \varepsilon^2$. Using (5.11)

$$\begin{aligned} T\Delta_{G^2A}\psi_1 = (\varepsilon^0\varepsilon^0\psi_0 \otimes d_2\psi_1)\Delta^l_{X_1} + (d_2\psi_1 \otimes \varepsilon^1\varepsilon^1\psi_0)\Delta^r_{X_1} \\ + (\varepsilon^0\psi_1 \otimes \varepsilon^2\psi_1)((1 \otimes \varphi_1)\Delta^r_{X_1} - (\varphi_1 \otimes 1)\Delta^l_{X_1}) \,. \end{aligned}$$

Since $(1 \otimes \varphi_1)\Delta^r_{X_1} = (\varphi_1 \otimes 1)\Delta^l_{X_1}$ we have

$$T\Delta_{G^2A}\psi_1 = (\varepsilon^0\varepsilon^0\psi_0 \otimes d_2\psi_1)\Delta^l_{X_1} + (d_2\psi_1 \otimes \varepsilon^1\varepsilon^1\psi_0)\Delta^r_{X_1} \,.$$

By Proposition 5.12 we have

$$\begin{aligned} \Delta_{G^3A}f &= T\Delta_{G^2A}\psi_1\xi_2 = (\varepsilon^0\varepsilon^0\psi_0 \otimes d_2\psi_1)\Delta^l_{X_1}\xi_2 + (d_2\psi_1 \otimes \varepsilon^1\varepsilon^1\psi_0)\Delta^r_{X_1}\xi_2 \\ &= (\varepsilon^0\varepsilon^0\varepsilon \otimes d_2\psi_1\xi_2)\omega^l_M + (d_2\psi_1\xi_2 \otimes \varepsilon^0\varepsilon^0\varepsilon)\omega^r_M \,, \end{aligned}$$

i.e., f is a coderivation with $d_3 f = 0$. q.e.d.

In consequence we get a map β' from the set of all two term extensions of A by M to the set of all coderivations of 2-cocycles. By the definitions of β' and (e_f) we have $\beta': (e_f) \mapsto f$.

PROPOSITION 5.14. *Assume $\beta': (e) \mapsto f$, then $(e) \sim (e_f)$.*

Proof of 5.14. By our assumption there is the commutative diagram

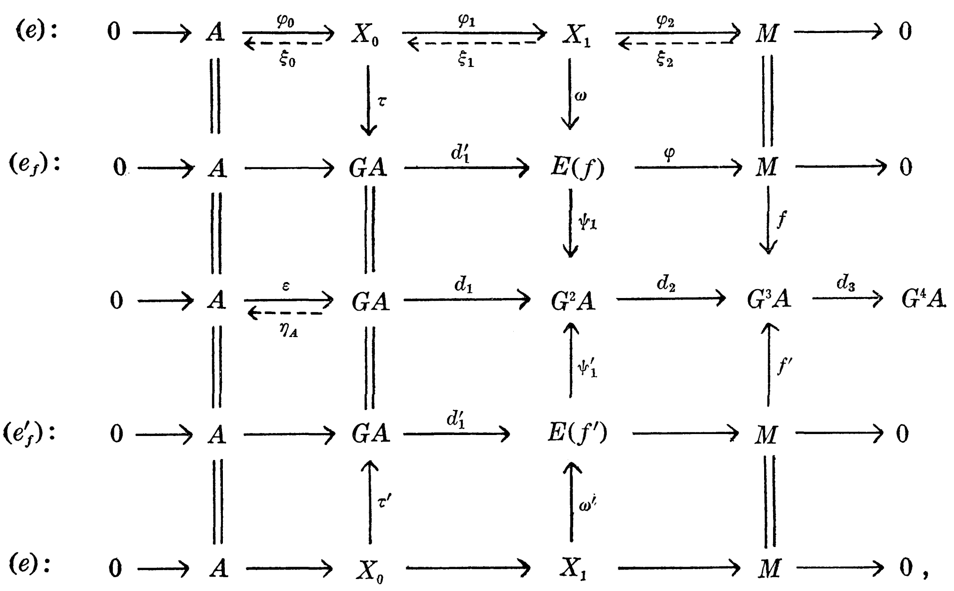
$$\begin{array}{lccccccccc} (e): & 0 \longrightarrow & A & \xrightarrow{\varphi_0} & X_0 & \xrightarrow{\varphi_1} & X_1 & \xrightarrow{\varphi_2} & M & \longrightarrow 0 \\ & & \| & & \downarrow{\scriptstyle\psi_0} & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle f} & \\ & 0 \longrightarrow & A & \xrightarrow{\varepsilon} & GA & \xrightarrow{d_1} & G^2A & \xrightarrow{d_2} & G^3A & \xrightarrow{d_3} G^4A \,. \end{array}$$

Define $\omega: X_1 \to E(f)$ by $\omega x = (\psi_1 x, \varphi_2 x)$ for $x \in X_1$. Since $d_2\psi_1 x = f\varphi_2 x$ $\omega x \in E(f)$. So we have the commutative diagram

$$\begin{array}{lccccccccc} (e): & 0 \longrightarrow & A & \xrightarrow{\varphi_0} & X_0 & \xrightarrow{\varphi_1} & X_1 & \xrightarrow{\varphi_2} & M & \longrightarrow 0 \\ & & \| & & \downarrow{\scriptstyle\psi_0} & & \downarrow{\scriptstyle\omega} & & \| & \\ (e_f): & 0 \longrightarrow & A & \xrightarrow{\varepsilon} & GA & \xrightarrow{d'_1} & E(f) & \xrightarrow{\phi} & M & \longrightarrow 0 \end{array}$$

By the definition of $\Delta^l_{E(f)}$, (5.11) and $\Delta^l_M\varphi_2 = (1 \otimes \varphi_2)\Delta^l_{X_1}$ we have $\Delta^l_{E(f)}\omega = (\psi_0 \otimes \omega)\Delta^l_{X_1}$. Similarly we get $\Delta^r_{E(f)}\omega = (\omega \otimes \psi_0)\Delta^r_{X_1}$. So $(e) \sim (e_f)$. q.e.d.

PROPOSITION 5.15. *If $(e_f) \sim (e_{f'})$ then f and f' are cohomologous.*



Proof of 5.15. Let us prove that if $(e_f) \leftarrow (e) \rightarrow (e_{f'})$ then f and f' are cohomologous. With this fact and (5.14) all cases in our proposition can be proved. Under our hypothesis is the commutative diagram where ξ_0, ξ_1, ξ_2 are K-module maps such that $\varphi_0\xi_0 + \xi_1\varphi_1 = 1_{X_0}, \xi_0\varphi_0 = 1_A = \eta_A\varepsilon$ and $\varphi_2\xi_2 = 1_M$. Define the coalgebra $X_0 + X_1$ by the same way as in the upper part of Proposition 5.9 and $\rho' : U(X_0 + X_1) \rightarrow UA$ by $\rho'(x + y) = \eta_A\tau x + \eta_A\tau'\xi_1 y$ for $x + y \in X_0 + X_1$, then there is a unique coalgebra map $\rho : X_0 + X_1 \rightarrow GA$ such that $\eta_A \cdot \rho = \rho'$. By the same way as in the proof of Proposition 5.9 we can prove that $\rho(x + y) = \tau x + gy$ for $x + y \in X_0 + X$, where $g : X_1 \rightarrow GA$ is a K-module map satisfying $g\phi_1 = \tau' - \tau$ and $\Delta_{GA}g = (\tau \otimes g)\Delta^l_{X_1} + (g \otimes \tau)\Delta^r_{X_1} + ((\tau' - \tau) \otimes g)\Delta^l_{X_1} = (\tau' \otimes g)\Delta^l_{X_1} + (g \otimes \tau)\Delta^r_{X_1}$. Put $\chi = \psi'_1\omega' - \psi_1\omega - d_1 g : X_1 \rightarrow \widetilde{G^2A}$, then $\chi\varphi_1 = 0$. In this case, by a straightforward calculation we have

$$\begin{aligned}\Delta_{G^2A}\chi &= \Delta_{G^2A}\psi'_1\omega' - \Delta_{G^2A}\psi_1\omega - (\varepsilon^0 \otimes \varepsilon^0 - \varepsilon^1 \otimes \varepsilon^1)\Delta_{GA}g \\ &= [\varepsilon^0\tau' \otimes (\psi'_1\omega' - \psi_1\omega - d_1 g)]\Delta^l_{X_1} + [(\psi'_1\omega - \psi_1\omega - d_1 g) \otimes \varepsilon^1\tau]\Delta^r_{X_1} \\ &\quad + [\varepsilon^0(\tau' - \tau) \otimes \psi_1\omega - d_1\tau' \otimes \varepsilon^1 g]\Delta^l_{X_1} \\ &\quad + [\psi'_1\omega' \otimes \varepsilon^1(\tau' - \tau) - \varepsilon^0 g \otimes d_1\tau]\Delta^r_{X_1} .\end{aligned}$$

Since $(\varepsilon^0(\tau' - \tau) \otimes \psi_1\omega)\Delta^l_{X_1} = (\varepsilon^0 g \otimes d_1\tau)\Delta^r_{X_1}$ and $(\psi'_1\omega' \otimes \varepsilon'(\tau' - \tau))\Delta^r_{X_1} = (d_1\tau' \otimes \varepsilon^1 g)\Delta^l_{X_1}$ (refer 5.10) we have $\Delta_{G^2A}\chi = (\varepsilon^0\tau' \otimes \chi)\Delta^l_{X_1} + (\chi \otimes \varepsilon^1\tau)\Delta^r_{X_1}$. Using

$\chi\varphi_1 = 0$ we can prove that $(1 \otimes \chi)\Delta^l_{X_1}\xi_2 = (\varphi_0 \otimes \chi\xi_2)\omega^l_M$ and $(\chi \otimes 1)\Delta^r_{X_1}\xi_2 = (\chi\xi_2 \otimes \varphi_0)\omega^r_M$ (refer Proposition 5.12). $\Delta_{G^2A}\chi\xi = (\varepsilon^0\varepsilon \otimes \chi\xi_2)\omega^l_M + (\chi\xi_2 \otimes \varepsilon^0\varepsilon)\omega^r_M$, which implies that $\chi\xi_2$ is a normal coderivation. Since $d_2\chi\xi_2 = f' - f, f$ and f' are cohomologous. q.e.d.

With Propositions 5.14 and 5.15 we can define $\beta: Ex^2(M, A) \to \tilde{H}^2(m, A)$ by $\beta[(e)][= [\beta'(e)]$. In this case $\beta[(e_f)] = [f]$.

Third step: By the definitions of α and β we have

$$\beta\alpha[f] = \beta[(e_f)] = [f]\,, \qquad \text{i.e., } \beta\alpha = 1_{\tilde{H}^2(M,A)}\,,$$
$$\alpha\beta[(e)] = \alpha[\beta'(e) = f] = \alpha[f] = [(e_f)] = [(e)]$$
$$\text{(see Proposition 5.14),}$$

i.e., $\alpha\beta = 1_{Ex^2(M,A)}$, and therefore we complete the proof of Theorem 5.4. (Note that $H^2(M, A) \cong \tilde{H}^2(M, A)$ (see 4.6).)

6. Interpretation of $H^1(M, A)$.

The arguments in this section are analogous to those in the preceding section. The detailed description will be omitted. In this section we assume that A is a coalgebra and M a two sided A-comodule.

DEFINITION 6.1. By a *(normal) coderivation* 1-*cocycle* we mean a normal coderivation $f: M \to \widetilde{G^2A}$ such that $d_2 f = 0$. Two such cocycles f and f' are *cohomologous* if there exists a coderivation $g: M \to GA$ such that $f - f' = d_1 g$.

Assume f is a coderivation 1-cocycle. Since $\overline{G^2A} = G^2A = \varepsilon^1 GA \oplus \widetilde{G^2A}$ coalgebra maps ρ_0 and $\rho_1: GA * M \to \overline{G^2A}$ are defined by $\rho_0 =$ the composition $GA * M \xrightarrow{\text{Projection}} GA \xrightarrow{\varepsilon^0} G^2A$ and $\rho_1 = (\varepsilon^1, f)$ (see (4.8)). Put

$$E(f) = \operatorname{Ker}(U\rho_0 - U\rho_1)$$
$$= \{(x, m) \in GA * M \mid d_1 x = fm \text{ for some } m \in M\}\,.$$

Using the direct sum decompositions $GA = \varepsilon A \oplus C(C = (1 - \varepsilon\eta_A)GA)$, $G^2A = \varepsilon^0\varepsilon A \oplus \varepsilon^1 C \oplus \widetilde{G^2A}$ and the monomorphism $d_1 | C$ (Proposition 3.5) we have

$$\Delta_{GA*M}(x, m) = \sum_i [(\varepsilon a_i, 0) \otimes (x_i, m_i) + (x'_i, m'_i) \otimes (\varepsilon a'_i, 0)]$$

for $(x, m) \in E(f)$, where $x_i, x'_i \in GA, \Delta_{A*M} m = \sum_i (a_i \otimes m_i + m'_i \otimes a'_i), d_1 x_i = m_i$ and $d_1 x'_i = m'_i$. This just indicates that $E(f)$ is a subcoalgebra of $GA * M$. So we obtain an exact sequence of K-modules

$$(e_f):\quad 0 \longrightarrow A \xrightarrow{\varepsilon'} E(f) \xrightarrow{\varphi} M \longrightarrow 0\ ,$$

where $\varepsilon' a = (\varepsilon a, 0)$ and $\varphi(x, M) = m$ for $a \in A, (x, m) \in E(f)$.

DEFINITION 6.2. By an *extension of A by M* we mean an exact sequence of K-modules:

$$(e):\quad 0 \longrightarrow A \xrightarrow{\varphi_0} X \xrightarrow{\varphi_1} M \longrightarrow 0$$

satisfying the conditions:

(i) X is a coalgebra and φ_0 a coalgebra map,

(ii) M is a two sided X-comodule induced by φ_0, and φ_1 a two sided X-comodule map. That is,

$$\Delta^l_M = (\varphi_0 \otimes 1)\omega^l_M\ , \qquad \Delta^r_M = (1 \otimes \varphi_0)\omega^r_M\ ,$$

where $\omega^l_M : N \to A \otimes M$ and $\omega^r_M : M \to M \otimes A$ are the comodule structure maps of M as a two sided A-comodule.

The above sequence (e_f) is an extension of A by M, which is called a *standard extension of A by M*.

DEFINITION 6.3. For two extensions (e) and (e') of A by M, if there is an isomorphism $\psi : X \to X'$ of coalgebras satisfying the commutative diagram

$$\begin{array}{lccccccccc}
(e): & 0 & \longrightarrow & A & \xrightarrow{\varphi_0} & X & \xrightarrow{\varphi_1} & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle\psi} & & \| & & \\
(e'): & 0 & \longrightarrow & A & \xrightarrow{\varphi'_0} & X' & \xrightarrow{\varphi'_1} & M & \longrightarrow & 0\ ,
\end{array}$$

then we say that (e) is *isomorphic* to (e'), written $(e) \approx (e')$.

Given an extension (e) of A by M there is a coderivational 1-cocycle f such that $(e) \approx (e_f)$, where the standard extension (e_f) corresponds to f. In particular, for two standard extensions (e_f) and (e'_f) if f and f' are cohomologous then $(e_f) \approx (e'_f)$, and conversely if $(e_f) \approx (e'_f)$ then f and f' are cohomologous.

Let us denote the set of all isomorphism classes of extensions of A by M by $Ex^1(M, A)$. Summarizing the above we have:

THEOREM 6.4. *There is an one-to-one correspondence between* $H^1(M,A)$ *and* $Ex^1(M, A)$.

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