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SURFACES WITH MEAN CURVATURE VECTOR PARALLEL IN THE NORMAL BUNDLE[1)]

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§1. Introduction. Let M be a connected surface immersed in a Euclidean m-space E^m. Let $\boldsymbol{h}$ be the second fundamental form of this immersion; it is a certain symmetric bilinear mapping $T_x \times T_x \to T_x^\perp$ for $x \in M$, where T_x is the tangent space and $T_x^\perp$ the normal space of M at x. Let $\boldsymbol{H}$ be the mean curvature vector of M in E^m. If there exists a real λ such that $\langle \boldsymbol{h}(X, Y), \boldsymbol{H}\rangle = \lambda\langle X, Y\rangle$ for all tangent vectors X, Y in T_x, then M is said to be *pseudo-umbilical at x*. If M is pseudo-umbilical at each point of M, then M is called a *pseudo-umbilical* surface. Let D denote the covariant differentiation of E^m and η be a normal vector field on M. If we denote by $D^*\eta$ the normal component of $D\eta$, then D^* defines a connection in the normal bundle. A normal vector field η is said to be *parallel in the normal bundle* if $D^*\eta = 0$.

Let h^r_{ij}; $i, j = 1, 2$; $r = 3, \cdots, m$, be the coefficients of the second fundamental form $\boldsymbol{h}$. Then the Gauss curvature K and the normal curvature K_N are given respectively by

$$(1) \qquad K = \sum_{r=3}^{m} (h^r_{11}h^r_{22} - h^r_{12}h^r_{12}) ,$$

$$(2) \qquad K_N = \sum_{r,s=3}^{2} \left[\sum_{k=1}^{m} (h^r_{1k}h^s_{2k} - h^r_{2k}h^s_{1k})\right]^2 .$$

The mean curvature vector $\boldsymbol{H}$, the Gauss curvature K, and the normal curvature K_N play the most important rôles, in differential geometry, for surfaces in Euclidean space.

We consider a surface in E^5 given by

$$c\left(\frac{yz}{\sqrt{3}}, \frac{xz}{\sqrt{3}}, \frac{xy}{\sqrt{3}}, \frac{x^2 - y^2}{2\sqrt{3}}, \frac{1}{6}(x^2 + y^2 - 2z^2)\right) ,$$

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where $x^2 + y^2 + z^2 = 3$ and c is a positive constant. This surface is a real projective plane in E^5 with $D^*H = 0$, $K = 1/3c^2$ and $K_N = 16/9c^4$. It is called the *Veronese surface*.

The main purpose of this paper is to study the surfaces in E^m with the mean curvature vector parallel in the normal bundle and to prove the following theorems.

THEOREM 1. *The Veronese surface is the only compact surface in Euclidean 5-space with $D^*H = 0$ and non-zero constant normal curvature K_N.*

THEOREM 2. *The minimal surfaces of a hypersphere of E^m, the open pieces of the product of two plane circles in E^4 and the open pieces of a circular cylinder in E^3 are the only non-minimal surfaces in Euclidean space with $D^*H = 0$ and constant Gauss curvature.*

The results obtained in this paper have been announced in [3].

§2. Lemmas. Let M be a surface immersed in Euclidean m-space E^m. We choose a local field of orthonormal frames $e_1, e_2, e_3, \cdots, e_m$ in E^m such that, restricted to M, the vectors e_1, e_2 are tangent to M (and, consequently, $e_3, \cdots, e_m$ are normal to M). With respect to the frame field of E^m chosen above, let $\omega^1, \cdots, \omega^m$ be the field of dual frames. Then the structure equations of E^m are given by

$$De_A = \sum \omega_A^B \otimes e_B\ , \qquad \omega_B^A + \omega_A^B = 0\ , \tag{3}$$

$$d\omega^A = -\sum \omega_B^A \wedge \omega^B\ , \tag{4}$$

$$d\omega_B^A = -\sum \omega_C^A \wedge \omega_B^C\ , \qquad A, B, C, \cdots = 1, 2, \cdots, m\ . \tag{5}$$

We restrict these forms to M. Then

$$\omega^r = 0,\ r, s, t, \cdots = 3, \cdots, m\ .$$

Since $0 = d\omega^r = -\sum \omega_i^r \wedge \omega^i$, by Cartan's lemma we may write

$$\omega_i^r = \sum h_{ij}^r \omega^j\ , \quad h_{ij}^r = h_{ji}^r\ , \quad i, j, k, \cdots = 1, 2\ . \tag{6}$$

From these formulas, we obtain

$$d\omega^i = -\sum \omega_j^i \wedge \omega^j\ , \qquad \omega_2^1 = -\omega_1^2\ , \tag{7}$$

$$d\omega_2^1 = K\omega^1 \wedge \omega^2\ , \tag{8}$$

$$d\omega_i^r = -\sum \omega_j^r \wedge \omega_i^j - \sum \omega_s^r \wedge \omega_i^s\ , \tag{9}$$

$$(10)\qquad d\omega_s^r = -\sum \omega_t^r \wedge \omega_s^t + \sum (h_{i1}^r h_{i2}^s - h_{i2}^r h_{i1}^s)\omega^1 \wedge \omega^2 .$$

The *second fundamental form* is given by $\boldsymbol{h} = \sum h_{ij}^r \omega^i \omega^j \boldsymbol{e}_r$ and the *mean curvature vector* is given by $\boldsymbol{H} = (1/2) \sum (h_{11}^r + h_{22}^r)\boldsymbol{e}_r$.

LEMMA 1. *Let M be a non-minimal surface in E^m with $D^*\boldsymbol{H} = 0$. Then $M = M_1 \cup M_2 \cup M_3$ such that* (i) *M_1 and M_2 are open,* (ii) $M_3 = \partial M_1 = \partial M_2$, (iii) *$M_1$ and M_3 are pseudo-umbilical in E^m,* (iv) *$K_N = 0$ on $M_2 \cup M_3$, and* (v) *M_2 is nowhere pseudo-umbilical in E^m.*

Proof. Since M is non-minimal in E^m and $\boldsymbol{H}$ is parallel in the normal bundle, the length of $\boldsymbol{H}$ is a nonzero constant. Hence we may choose our frame field in such a way that

$$(11)\qquad \boldsymbol{H} = c\boldsymbol{e}_3 , \qquad c = |\boldsymbol{H}| ,$$

$$(12)\qquad h_{12}^3 = 0 .$$

Therefore, we have

$$(13)\qquad \omega_1^3 = h_{11}^3 \omega^1 , \qquad \omega_2^3 = (2c - h_{11}^3)\omega^2 ,$$

$$(14)\qquad \omega_r^3 = 0 .$$

Taking exterior differentiation of (14) and applying (7), (9) and (13), we obtain

$$(15)\qquad h_{12}^r(c - h_{11}^3) = 0 \qquad \text{for } r = 4, \cdots, m .$$

Put $M_2 = \{p \in M ; h_{11}^3 \neq h_{22}^3\}$. Then M_2 is an open subset of M and

$$(16)\qquad h_{12}^r = 0 \quad \text{on} \quad M_2 \qquad \text{for } r = 4, \cdots, m .$$

Therefore, from (2), (12) and (16) we see that $M - M_2$ is pseudo-umbilical in E^m and $K_N = 0$ on M_2. Let $M_1 = \operatorname{Int}(M - M_2)$. Then we obtain Lemma 1.

LEMMA 2. *Let M be a non-minimal surface in E^m with $D^*\boldsymbol{H} = 0$, $K_N = 0$ and $K =$ constant, then $K \geqq 0$.*

Proof. Choose our frame field in such that a way that (11) and (12) hold. Then we have (13) and (14). Taking exterior differentiation of (13) and applying (7), (9) and (13) we obtain

$$(17)\qquad 2(c - h_{11}^3)d\omega^i = dh_{11}^3 \wedge \omega^i .$$

Since $K_N = 0$ and $h^3_{12} = 0$, we obtain from (2) that

(18) $$\omega^r_1 = h^r_{11}\omega^1 , \quad \omega^r_2 = -h^r_{11}\omega^2 , \qquad \text{for } r > 3 .$$

Taking exterior differentiation of (18) we see that

(19) $$dh^r_{11} \wedge \omega^1 + 2h^r_{11}d\omega^1 = \sum_{s=4}^{m} h^s_{11}\omega^1 \wedge \omega^r_s ,$$

Multiplying (19) by h^r_{11} and summing up on r, we obtain

$$\sum_{r=4}^{m} (h^r_{11}dh^r_{11}) \wedge \omega^1 + 2\sum_{r=4}^{m} (h^r_{11}h^r_{11})\,d\omega^1 = \sum_{r,s=4}^{m} (h^r_{11}h^s_{11})\omega^1 \wedge \omega^r_s .$$

It is easy to see from $\omega^r_s = -\omega^s_r$ and above equation that

(20) $$\sum_{r=4}^{m} (h^r_{11}dh^r_{11}) \wedge \omega^1 + 2\sum_{r=4}^{m} (h^r_{11}h^r_{11})d\omega^1 = 0 .$$

On the other hand, by the assumption $K =$ constant and (18), we see that

(21) $$(c - h^3_{11})dh^3_{11} = \sum_{r=4}^{m} h^r_{11}dh^r_{11} .$$

Hence, combining (20) and (21), we obtain

(22) $$2\sum_{r=4}^{m} (h^r_{11}h^r_{11})d\omega^1 = -(c - h^3_{11})dh^3_{11} \wedge \omega^1 .$$

Substituing (17) into (22) we obtain

(23) $$\sum_{r=4}^{m} (h^r_{11}h^r_{11})d\omega^1 = -(c - h^3_{11})^2 d\omega^1 .$$

Similarly, we have

(24) $$\sum_{r=4}^{m} (h^r_{11}h^r_{11})d\omega^2 = -(c - h^3_{22})^2 d\omega^2 .$$

Put $V = \{p \in M\,;\, d\omega^1 \neq 0 \text{ or } d\omega^2 \neq 0\}$. Then V is an open subset of M. If $V = \phi$, then $d\omega^1 = d\omega^2 = 0$ identically on M. Hence, (7) and (8) imply that $K = 0$. Now, suppose that $V \neq \phi$, and let V_1 be a component of V. Then on V_1, we have

(25) $$h^r_{11} = 0 , \qquad \text{for } r = 4, \cdots, m ,$$

(26) $$c - h^3_{11} = 0 .$$

These imply that

$$\omega_1^3 = h_{11}^3\omega^1 , \qquad \omega_2^3 = h_{11}^3\omega^2 , \tag{27}$$

$$\omega_1^r = \omega_2^r = 0 , \qquad \text{for } r = 4, \cdots, m \tag{28}$$

on V_1. From (14) and (28), we can easily find that the normal subspace spanned by $e_4, \cdots, e_m$ is independent of the base point $p \in M$ and hence V_1 is contained in a 3-dimensional linear subspace E^3 of E^m. Moreover, by (27), we see that V_1 is totally umbilical in E^3. Therefore, V_1 is an open piece of a 2-sphere in E^3. From this we see that the Gauss curvature K is a positive constant on M. This completes the proof of the lemma.

LEMMA 3. *The Veronese surface is the only compact pseudo-umbilical surface in Euclidean* 5*-space with nonzero constant normal curvature, and* $\boldsymbol{H}$ *parallel.*

This lemma has been proved in [2], [5].

LEMMA 4. *If M is a non-minimal surface in E^m with K = constant $\geqq 0$, $K_N = 0$ and $D^*\boldsymbol{H} = 0$, then M is an open piece of one of the following surfaces*; (i) *a sphere in E^3*, (ii) *a circular cylinder in E^3 or* (iii) *a product of two plane circles in E^4*

This lemma has been proved in [4].

§3. Proof of Theorem 1. Suppose that M is a compact surface in Euclidean 5-space with $D^*\boldsymbol{H} = 0$, and $K_N = \text{constant} \neq 0$. Then, by Lemma 1, we see that M is pseudo-umbilical in Euclidean 5-space with nonzero constant normal curvature K_N. Hence, by Lemma 3, we see that M is a Veronese surface. This completes the proof of the theorem.

§4. Proof of Theorem 2. Suppose that M is a non-minimal surface in E^m with $D^*\boldsymbol{H} = 0$. Then, by Lemma 1, we see that $M = M_1 \cup M_2 \cup M_3$ where $M_1 \cup M_3$ is pseudo-umbilical, $K_N \equiv 0$ on $M_2 \cup M_3, M_1$ and M_2 are open, $M_3 = \partial M_1 = \partial M_2$, and M_2 is nowhere pseudo-umbilical in E^m.

Case (i). If $M_2 = \phi$, then $M_3 = \phi$, and M is pseudo-umbilical in E^m. Therefore, by the assumption $D^*\boldsymbol{H} = 0$, we see from Proposition 1 of [1] that M is a minimal surface in a hypersphere of E^m, with radius $1/|\boldsymbol{H}|$.

Case (ii). If $M_1 = \phi$, then $M_3 = \phi$ and $K_N \equiv 0$ on M. Therefore, by the assumption K = constant and Lemma 2, we see that $K \geqq 0$. Applying Lemma 4, we see that M is an open piece of one of the surfaces given in Lemma 4. Hence the theorem is true in this case.

Case (iii). If $M_1 \neq \phi$ and $M_2 \neq \phi$, then, by Lemma 2, we see that $K \geqq 0$. If $K > 0$, then by Lemma 4, we see that every component of M_2 is an open piece of a two sphere with radius $1/|\boldsymbol{H}|$ in a 3-space. This implies that M_2 is pseudo-umbilical in E^m. This is a contradiction. Therefore, we have $K = 0$ identically on M. Since $M_1 \neq \phi$ and $M_2 \neq \phi$ and both of M_1 and M_2 are open, we see that $M_3 \neq \phi$. Let $p \in M_3$. Then there exists a component U_1 of M_1 and a component U_2 of M_2 such that $p \in$ closure (U_1) and $p \in$ closure (U_2). By Case (i) we see that U_1 is a minimal surface of a hypersphere of radius $1/|\boldsymbol{H}|$ in E^m. Therefore, by a simple, direct computation, we know that the second fundamental form in the direction of $\boldsymbol{H} = |\boldsymbol{H}|\boldsymbol{e}_3$ is given by

$$(29) \qquad (h^3_{ij}) = \begin{bmatrix} |\boldsymbol{H}| & 0 \\ 0 & |\boldsymbol{H}| \end{bmatrix} .$$

Therefore, by the continuity of the second fundamental form $\boldsymbol{h}$, we see that the second fundamental form at p in the direction of $\boldsymbol{H} = |\boldsymbol{H}|\boldsymbol{e}_3$ is also given by (29). On the other hand, by Case (ii), we see that U_2 is either an open piece of a circular cylinder or an open piece of a product surface of two plane circles with different radius (this follows from "U_2 is nowhere pseudo-umbilical"). By a direct computation, if we choose $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the principal directions of $\boldsymbol{H}$, then we see that the second fundamental form in the direction of $\boldsymbol{H} = |\boldsymbol{H}|\boldsymbol{e}_3$, for every point in U_2 and hence for p, are given by one of the following forms:

$$(30) \qquad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad a \neq b , \quad a, b \text{ are constants.}$$

$$(31) \qquad \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix}, \quad d \text{ is constant.}$$

This is a contradiction. Therefore, we prove Theorem 2 completely.

§5. Corollaries. In this section, we give the following

COROLLARY 1. *Let M be a compact surface in Euclidean 5-space with nonzero constant normal curvature. If there exists a unit normal vector field η over M which is parallel in the normal bundle and parallel to the mean curvature vector $\boldsymbol{H}$, then M is a Veronese surface.*

Proof. Set $V = \{p \in M; \boldsymbol{H} \neq 0 \text{ at } p\}$. Then V is open. We choose our frame field in such a way that $e_3 = \eta$ and $h^3_{12} = 0$. Then we can prove, by a similar argument of Lemma 1, that $h^3_{11} = h^3_{22}$ and $\omega^3_r = 0$ on V. From this we can easily prove that $dh^3_{11} = 0$. This implies that $V = M$ and $D^*\boldsymbol{H} = 0$. Therefore, by Theorem 1, we obtain the corollary.

COROLLARY 2. *Let M be a non-minimal surface in E^4 with $D^*\boldsymbol{H} = 0$ and constant Gauss curvature. Then M is an open piece of one of the following surfaces*; (i) *a 2-sphere in E^3*, (ii) *a circular cylinder in E^3 or* (iii) *a product surface of two plane circles.*

This corollary follows immediately from Theorem 2 and the fact that the open pieces of a 2-sphere or a Clifford torus are the only minimal surfaces of a 3-sphere with constant Gauss curvature.

COROLLARY 3. *Let M be a non-minimal surface in E^4. If M has constant negative Gauss curvature, then there exists no open subset U of M such that $D^*\boldsymbol{H} = 0$ on U.*

This corollary follows immediately from Corollary 2.

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