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NON-ESSENTIAL CLUSTER VALUES AND NORMAL FUNCTIONS

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1. Introduction

We consider continuous functions f which map the open unit disk D into the Riemann sphere W. For a point ζ on the unit circle C, we say that χ is a *chord at* ζ if χ is a chord of C having one endpoint at ζ and that Δ is a *Stolz angle at* ζ if Δ is a Stolz angle with vertex ζ. Suppose S denotes either a chord at ζ, a Stolz angle at ζ, or D. Then, letting σ denote the chordal metric on W and setting

$$S_r = S \cap \{z \in D : |z - \zeta| < r\} \qquad (r > 0)\ ,$$

we define the *cluster set* $C(f, \zeta, S)$ *of* f *at* ζ *relative to* S and the *essential cluster set* $C_e(f, \zeta, S)$ *of* f *at* ζ *relative to* S as follows: the point $w^* \in W$ is in $C(f, \zeta, S)$ if, for every $\varepsilon > 0$ and every $r > 0$,

$$S_r \cap f^{-1}(\{w \in W : \sigma(w, w^*) < \varepsilon\}) \neq \phi\ ;$$

whereas w^* is in $C_e(f, \zeta, S)$ if, for every $\varepsilon > 0$,

$$\limsup_{r \to 0} \frac{m[S_r \cap f^{-1}(\{w \in W : \sigma(w, w^*) < \varepsilon\})]}{mS_r} > 0\ ,$$

where m denotes linear Lebesgue measure m_1 if S is a chord at ζ and denotes 2-dimensional Lebesgue measure m_2 if S is either a Stolz angle at ζ or D. The abbreviated notations $C(f, \zeta)$ and $C_e(f, \zeta)$ are used in place of $C(f, \zeta, D)$ and $C_e(f, \zeta, D)$. We remark that both $C(f, \zeta, S)$ and $C_e(f, \zeta, S)$ are closed subsets of W with $C_e(f, \zeta, S) \subset C(f, \zeta, S)$ and that

$$\lim_{r \to 0} \frac{m[S_r \cap f^{-1}(G)]}{mS_r} = 1$$

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for each open subset G of W containing $C_e(f, \zeta, S)$. (Note that the latter fact is trivially true for open sets G containing $C(f, \zeta, S)$.)

The object of our study is the open set $C(f, \zeta, S) - C_e(f, \zeta, S)$, which we call the set of *non-essential cluster values of f at ζ relative to S.* In section 2 (3) we give a necessary condition for a point w to be a non-essential cluster value of f at ζ relative to a chord (Stolz angle) at ζ. In both of these sections we make use of the following lemma of Lappan [3, Lemma 2, p. 46] (see also Rung [7, p. 424]) concerning the non-Euclidean hyperbolic distance between points a and b in D given by

$$\rho(a, b) = \tanh^{-1}\left(\left|\frac{a-b}{1-\bar{a}b}\right|\right) .$$

LEMMA L. *If a and b are in D with $\rho(a, b) = M$, then*

$$\tanh M \leq \frac{|a-b|}{1-|a|} \leq \frac{2\tanh M}{1-\tanh M} .$$

We show in section 4 that normal functions have neither non-essential chordal cluster values nor non-essential angular cluster values. On the other hand, in the last section we exhibit a normal function having non-essential cluster values at almost every $\zeta \in C$ relative to D.

2. Chordal Cluster Sets

For each $a \in D$ and each $M > 0$, we set

$$D(a, M) = \{z \in D : \rho(a, z) < M\}$$

and we denote the boundary of $D(a, M)$ by $\partial D(a, M)$.

LEMMA 1. *Suppose $\{a_n\}$ is a sequence of points lying on the chord χ at $\zeta \in C$ with $a_n \to \zeta$. Then, for each $M > 0$,*

$$\limsup_{r \to 0} \frac{m_1[\chi_r \cap \bigcup_{n=1}^{\infty} D(a_n, M)]}{m_1 \chi_r} > 0 .$$

Proof. Choose a number $M > 0$. Denote by b_n the point of $\chi \cap \partial D(a_n, M)$ that is furthest from ζ (in the Euclidean sense). Setting $r_n = |b_n - \zeta|$, we have

$$\frac{m_1[\chi_{r_n} \cap D(a_n, M)]}{m_1 \chi_{r_n}} > \frac{|a_n - b_n|}{|b_n - \zeta|} .$$

Since $\{b_n\} \subset \chi$, there exists a constant $K > 0$ with $(1 - |b_n|)/|b_n - \zeta| > K$

for all n. Also, by Lemma L, $|a_n - b_n|/(1 - |b_n|) \geq \tanh M$ for each n. Consequently

$$\liminf_{n \to \infty} \frac{m_1[\chi_{r_n} \cap D(a_n, M)]}{m_1 \chi_{r_n}} \geq K \tanh M ,$$

which clearly implies the conclusion of the lemma.

A sequence $\{a_n\} \subset D$ is said to be *close to* the sequence $\{b_n\} \subset D$ if $\rho(a_n, b_n) \to 0$ as $n \to \infty$.

THEOREM 1. *Let $f : D \to W$ be continuous, let χ be a chord at $\zeta \in C$, and suppose $w_0 \in C(f, \zeta, \chi) - C_e(f, \zeta, \chi)$. If $\{a_n\}$ is a sequence of points on χ with $a_n \to \zeta$ and $f(a_n) \to w_0$, then there exists a sequence $\{b_n\} \subset \chi$ which is close to a subsequence of $\{a_n\}$ and for which $\{f(b_n)\}$ converges to a point of $C_e(f, \zeta, \chi)$.*

Proof. Set $G_j = \{w \in W : \sigma(w, C_e(f, \zeta, \chi)) < 1/j\}$ and choose positive integers N and J such that $f(a_n) \notin \bar{G}_j$ (the closure of G_j) for $n > N$ and $j > J$. Set

$$M(n, j) = \max \{M : f(\chi \cap D(a_n, M)) \cap G_j = \phi\} ,$$

and suppose there exists a $j_0 > J$ for which

$$\limsup_{n \to \infty} M(n, j_0) = M_{j_0} > 0 .$$

This implies the existence of a subsequence $\{a'_n\}$ of $\{a_n\}$ such that $f(\chi \cap D(a'_n, M_{j_0}/2)) \cap G_{j_0} = \phi$ for each n. Then, by Lemma 1,

$$\limsup_{r \to 0} \frac{m_1[\chi_r \cap \bigcup_{n=1}^{\infty} D(a'_n, M_{j_0}/2)]}{m_1 \chi_r} > 0 ;$$

and hence

$$\lim_{r \to 0} \frac{m_1[\chi_r \cap f^{-1}(G_{j_0})]}{m_1 \chi_r} \neq 1$$

in violation of $C_e(f, \zeta, \chi) \subset G_{j_0}$. It follows that, for each $j > J$, $\lim_{n \to \infty} M(n, j) = 0$. Thus for each $j > J$ there exist an integer n_j and a point $b_j \in \chi$ for which $|a_{n_j} - \zeta| < 1/j$, $\rho(a_{n_j}, b_j) < 1/j$ and $f(b_j) \in G_j$. Then since each convergent subsequence of $\{f(b_j)\}$ converges to a point of $C_e(f, \zeta, \chi)$, the theorem is proved.

A chord χ at $\zeta \in C$ is called a *segment of Julia for a function* f *at* ζ provided f assumes all values of W except possibly two in each Stolz angle at ζ meeting χ.

COROLLARY 1. *Let* f *be a meromorphic function in* D, *and let* χ *be a chord at* $\zeta \in C$. *If* $C_e(f, \zeta, \chi) \neq C(f, \zeta, \chi)$, *then* χ *is a segment of Julia for* f *at* ζ.

Proof. Suppose $C_e(f, \zeta, \chi) \neq C(f, \zeta, \chi)$. Applying Theorem 1 we obtain sequences $\{a_n\}, \{b_n\} \subset \chi$ and distinct complex values α and β for which $\rho(a_n, b_n) \to 0$, $f(a_n) \to \alpha$ and $f(b_n) \to \beta$. According to Lappan [3, Theorem 4, p. 44], for each value $\delta \in W$ with perhaps two exceptions, there exists a sequence $\{z_k^\delta\}$ close to a subsequence of $\{a_n\}$ with $f(z_k^\delta) = \delta$ for each k. Then, for any Stolz angle Δ at ζ meeting χ, each of the sequences $\{z_k^\delta\}$ has a terminal subsequence that lies in Δ and the corollary is proved.

The *outer angular cluster set of* f *at* $\zeta \in C$ is the set

$$C_{\mathscr{A}}(f, \zeta) = \bigcup_{\Delta} C(f, \zeta, \Delta)$$

where Δ ranges over all Stolz angles at ζ. Since $C(f, \zeta, \Delta) = W$ for each Stolz angle Δ at ζ meeting a segment of Julia for f at ζ, we have the following result.

COROLLARY 2. *Let* f *be a meromorphic function in* D, *and let* ζ *be a point of* C. *If* $C_{\mathscr{A}}(f, \zeta) \neq W$, *then* $C_e(f, \zeta, \chi) = C(f, \zeta, \chi)$ *for every chord* χ *at* ζ.

3. Angular Cluster Sets

We start with a simple lemma.

LEMMA 2. *Let* a *be a point of* D, *let* χ *be the chord at* $\zeta \in C$ *that passes through* a, *and let* M *be a positive number. If* Q *represents either component of* $D(a, M) - \chi$, *then*

$$m_2 Q \geq K_M (1 - |a|)^2$$

where $K_M = [\arcsin(\operatorname{sech} M) - \tanh M \operatorname{sech} M] \tanh^2 M > 0$.

Proof. For $a = 0$, $m_2 Q = (\pi/2) \tanh^2 M > K_M$. Suppose $a \neq 0$. Let A denote the line segment joining a to the origin, and let χ^* be the chord of C that passes through a and is orthogonal to A. Then let T

be the component of $D(a, M) - \chi^*$ that is separated from the origin by χ^*. Clearly $m_2Q \geq m_2T$. Through elementary calculations we find that

$$m_2T = \tau[|a|, M](1 - |a|)^2$$

with

$$\tau[|a|, M] = \left(\frac{1 + |a|}{|a|}\right)^2 \left(\frac{\lambda}{1 - \lambda^2}\right)^2 [\arcsin(\sqrt{1 - \lambda^2}) - \lambda\sqrt{1 - \lambda^2}]$$

where $\lambda = |a| \tanh M$. Finally, it is easy to see that $\tau[|a|, M] \geq K_M$ and the proof is complete.

LEMMA 3. *Let $\{a_n\}$ be a sequence of points in the Stolz angle Δ at $\zeta \in C$ with $a_n \to \zeta$. Then, for each $M > 0$,*

$$\limsup_{r \to 0} \frac{m_2[\Delta_r \cap \bigcup_{n=1}^{\infty} D(a_n, M)]}{m_2\Delta_r} > 0 .$$

Proof. Choose a number $M > 0$. Since $\{a_n\} \subset \Delta$, there exists a number $M^*(0 < M^* \leq M)$ such that the set $D(a_n, M^*) \cap (D - \Delta)$ is connected for each n. Let χ_n denote the chord at ζ that passes through the point a_n, and let D_n^1 and D_n^2 denote the components of $D(a_n, M^*) - \chi_n$ with $m_2D_n^1 \leq m_2D_n^2$. It is clear that, for each n, either $D_n^1 \subset \Delta$ or $D_n^2 \subset \Delta$. Denote by b_n the point of $\chi_n \cap \partial D(a_n, M^*)$ that is furthest from ζ (in the Euclidean sense), and set $r_n = |b_n - \zeta|$. If E_n denotes the region obtained by reflecting D_n^1 across χ_n, it is evident that

$$E_n \subset D_n^2 \cap \{z \in D : |z - \zeta| < r_n\}$$

and that $m_2E_n = m_2D_n^1$. Thus

$$m_2[\Delta_{r_n} \cap D(a_n, M^*)] \geq m_2D_n^1 .$$

Applying Lemma 2 we obtain

$$m_2[\Delta_{r_n} \cap D(a_n, M^*)] \geq K_{M^*}(1 - |a_n|)^2 .$$

Then, using α to denote the angular opening of Δ, we have

$$\frac{m_2[\Delta_{r_n} \cap D(a_n, M^*)]}{m_2\Delta_{r_n}} \geq (2\alpha^{-1}K_{M^*})\left(\frac{1 - |a_n|}{|b_n - \zeta|}\right)^2 .$$

It follows from Lemma L that there exists a constant $A > 0$ such that

$(1-|a_n|)/(1-|b_n|) \geq A$ for all n; and, since $\{b_n\} \subset \Delta$, there exists a constant $B > 0$ with $(1-|b_n|)/|b_n - \zeta| \geq B$ for all n. Hence

$$\liminf_{n\to\infty} \frac{m_2[\Delta_{r_n} \cap D(a_n, M^*)]}{m_2\Delta_{r_n}} \geq (AB)^2\alpha^{-1}K_{M^*} > 0 ,$$

and the conclusion of the lemma now follows.

We now give the analogue of Theorem 1 for Stolz angles.

THEOREM 2. *Let $f: D \to W$ be continuous, let Δ be a Stolz angle at $\zeta \in C$, and suppose $w_0 \in C(f, \zeta, \Delta) - C_e(f, \zeta, \Delta)$. If $\{a_n\}$ is a sequence of points in Δ with $a_n \to \zeta$ and $f(a_n) \to w_0$, then there exists a sequence $\{b_n\} \subset \Delta$ which is close to a subsequence of $\{a_n\}$ and for which $\{f(b_n)\}$ converges to a point of $C_e(f, \zeta, \Delta)$.*

The proof of Theorem 2 is obtained by using Lemma 3 in place of Lemma 1 and replacing χ by Δ in the proof of Theorem 1. Also the proofs of the following corollaries are similar to those of the corollaries of Theorem 1.

COROLLARY 1. *Let f be a meromorphic function in D, and let Δ be a Stolz angle at $\zeta \in C$. If $C_e(f, \zeta, \Delta) \neq C(f, \zeta, \Delta)$, then f assumes all values on W except possibly two in each Stolz angle Δ^* at ζ containing $\bar{\Delta}$.*

COROLLARY 2. *Let f be a meromorphic function in D, and let ζ be a point of C. If $C_{\mathscr{A}}(f, \zeta) \neq W$, then $C_e(f, \zeta, \Delta) = C(f, \zeta, \Delta)$ for each Stolz angle Δ at ζ.*

4. Applications to Normal Functions

Let $\mathscr{T}$ denote the collection of all one-one conformal mappings of D onto D. A continuous function $f: D \to W$ is said to be *normal* if the family of functions $\{f(T(z))\}_{T(z)\in\mathscr{T}}$ is normal in D in the sense of Montel.

THEOREM 3. *Suppose the continuous function $f: D \to W$ is normal. Then, for each $\zeta \in C$, (1) $C_e(f, \zeta, \chi) = C(f, \zeta, \chi)$ for each chord χ at ζ and (2) $C_e(f, \zeta, \Delta) = C(f, \zeta, \Delta)$ for each Stolz angle Δ at ζ.*

Proof. Assume $C_e(f, \zeta, \chi) \neq C(f, \zeta, \chi)$ for some $\zeta \in C$ and some chord χ at ζ. By Theorem 1 there exist sequences $\{a_n\}, \{b_n\} \subset \chi$ and distinct complex values α and β for which $\rho(a_n, b_n) \to 0$, $f(a_n) \to \alpha$ and $f(b_n) \to \beta$. According to Lappan [4, Theorem 2, p. 156], f is non-normal in violation of the hypothesis; and (1) is proved. The proof of (2) is similar.

The converse of Theorem 3 is not true, as the following theorem shows.

THEOREM 4. *There exists a non-normal continuous function* $f: D \to W$ *such that, for each* $\zeta \in C$, (1) $C_e(f, \zeta, \chi) = C(f, \zeta, \chi)$ *for each chord* χ *at* ζ *and* (2) $C_e(f, \zeta, \Delta) = C(f, \zeta, \Delta)$ *for each Stolz angle* Δ *at* ζ.

Proof. Define the sets

$$A = \{z \in D: |z - 3/4| \leq 1/4\}$$

and

$$B = \{z \in D: |z - 1/2| \geq 1/2\} .$$

Let $\{a_n\}$ and $\{b_n\}$ be disjoint sequences of points in $D - (A \cup B)$ with $a_n \to 1$ and $\rho(a_n, b_n) \to 0$. Define the continuous function F on $A \cup B \cup \{a_n\} \cup \{b_n\}$ by

$$F(z) = \begin{cases} w_1 & \text{for } z \in A \cup B \cup \{a_n\} \\ w_2 & \text{for } z \in \{b_n\} \end{cases}$$

where w_1 and w_2 are distinct points of W. The function F can be extended to a continuous function $f: D \to W$. It follows from the result of Lappan cited in the proof of Theorem 3 that f is non-normal. Furthermore, for each $\zeta \in C$,

$$C(f, \zeta, S) = \{w_1\} = C_e(f, \zeta, S)$$

where S denotes an arbitrary Stolz angle or chord at ζ. Hence the theorem is proved.

We now show that Theorem 3 is not true if the condition that f is normal is removed, even if f is assumed to be holomorphic.

THEOREM 5. *The function*

$$F(z) = \prod_{j=1}^{\infty} \left\{1 - \left(\frac{z}{1 - n_j^{-1}}\right)^{n_j^2}\right\} \qquad (n_j = 3^j)$$

is holomorphic in D *and has the properties*: (1) *for nearly every* $\zeta \in C$, $C_e(F, \zeta, \rho_\zeta) \neq C(F, \zeta, \rho_\zeta)$ *where* ρ_ζ *denotes the chord at* ζ *which forms a diameter of* C, *and* (2) *for almost every* $\zeta \in C, C_e(F, \zeta, \Delta) \neq C(F, \zeta, \Delta)$ *where* Δ *denotes an arbitrary Stolz angle at* ζ.

Proof. Bagemihl and Seidel [1] have shown that $F(z)$ is holomorphic in D and that $F(z) \to \infty$ as $|z| \to 1$ through a region Ω which is described as follows: for $j = 1, 2, 3, \cdots$ and $\nu = 0, 1, \cdots, n_j^2 - 1$ set

$$z_{j\nu} = (1 - n_j^{-1})e^{2\pi i\nu/n_j^2}$$

and

$$\Gamma_{j\nu} = \{z : |z - z_{j\nu}| \leq r_j\}$$

where $r_j = 1/n_j^4$. Then Ω is the region obtained by deleting all the disks $\Gamma_{j\nu}$ from D.

Since each point of C is a limit point of the set $\{z_{j\nu}\}$ of zeros of $F(z), 0 \in C(F, \zeta)$ for every $\zeta \in C$; and it follows from a theorem of Collingwood [6, p. 66] that $0 \in C(F, \zeta, \rho_\zeta)$ for nearly every $\zeta \in C$. Also, as a consequence of the uniqueness theorem of Lusin and Privaloff [6, p. 72] and Plessner's theorem [6, p. 70], for almost every $\zeta \in C, C(F, \zeta, \Delta) = W$ for each Stolz angle Δ at ζ. That $F(z)$ has properties (1) and (2) now follows from the next two lemmas.

LEMMA 4. *$C_e(F, \zeta, \chi) = \{\infty\}$ for every $\zeta \in C$ and every chord χ at ζ.*

Proof. Let ζ be a point of C and let χ be a chord at ζ. For $1/n_{k+1} \leq r \leq 1/n_k$ we have

$$\frac{m_1[\chi_r \cap (D - \Omega)]}{m_1\chi_r} \leq n_{k+1} \sum_{j=k}^{\infty} \frac{2}{n_j^4}$$
$$\leq 6 \sum_{j=k}^{\infty} \frac{1}{n_j^3} \to 0 \quad \text{as} \quad k \to \infty .$$

Hence

$$\lim_{r \to 0} \frac{m_1[\chi_r \cap (D - \Omega)]}{m_1\chi_r} = 0 ,$$

which implies $C_e(F, \zeta, \chi) = \{\infty\}$.

LEMMA 5. *$C_e(F, \zeta, \Delta) = \{\infty\}$ for every $\zeta \in C$ and every Stolz angle Δ at ζ.*

Proof. Choose a point $\zeta \in C$ and let Δ be a Stolz angle at ζ with angular opening α. For $1/n_{k+1} \leq r \leq 1/n_k$ we have

$$\frac{m_2[\Delta_r \cap (D - \Omega)]}{m_2\Delta_r} \leq 2\alpha^{-1}n_{k+1}^2 \sum_{j=k}^{\infty} n_j^2[\pi(1/n_j^4)^2]$$
$$\leq 18\alpha^{-1}\pi \sum_{j=k}^{\infty} \frac{1}{n_j^4} \to 0 \quad \text{as} \quad k \to \infty .$$

Consequently

$$\lim_{r\to 0} \frac{m_2[\Delta_r \cap (D - \Omega)]}{m_2\Delta_r} = 0 ,$$

and so $C_e(F, \zeta, \Delta) = \{\infty\}$.

5. Total Cluster Sets

If the analogue of Theorem 1 (or Theorem 2) for $C(f, \zeta)$ and $C_e(f, \zeta)$ were true, it would easily follow that these two sets are always equal for normal functions. Thus the next theorem shows that no such analogue exists. (In this section m denotes linear measure.)

THEOREM 6. *There exists a bounded holomorphic function g in D for which*

$$m\{\zeta \in C : C_e(g, \zeta) = C(g, \zeta)\} = 0 .$$

To establish the existence of such a function, we make use of the following portion of a result of Goffman and Sledd [2, Theorem 2]. (We note that their proof is for real-valued functions in the upper half plane, but only slight modifications are needed to obtain a proof for complex-valued functions in D.)

THEOREM GS. *For each $\zeta \in C$ let ρ_ζ denote the chord at ζ which forms a diameter of C. If $f : D \to W$ is measurable, then*

$$m\{\zeta \in C : C_e(f, \zeta) \subset C_e(f, \zeta, \rho_\zeta)\} = 2\pi .$$

Proof of Theorem 6. It follows from a theorem of Littlewood [5, p. 172] that there exists a bounded holomorphic function g in D for which there exists a subset E of C with $mE = 2\pi$ and $C(g, \zeta, \rho_\zeta) \neq C(g, \zeta)$ for each $\zeta \in E$. By Theorem GS there exists a subset F of C with $mF = 2\pi$ and $C_e(g, \zeta) \subset C_e(g, \zeta, \rho_\zeta)$ for each $\zeta \in F$. Then in view of Theorem 3 we have

$$C_e(g, \zeta) \subset C_e(g, \zeta, \rho_\zeta) = C(g, \zeta, \rho_\zeta) \subsetneq C(g, \zeta)$$

for each $\zeta \in E \cap F$, and the theorem is proved.

References

[1] F. Bagemihl and W. Seidel, *Some boundary properties of analytic functions,* Math. Z., **61** (1954), 186–199.

[2] Casper Goffman and W. T. Sledd, *Essential Cluster Sets,* J. London Math. Soc., (2), **1** (1969), 295–302.

[3] Peter Lappan, *Some sequential properties of normal and non-normal functions with applications to automorphic functions,* Comm. Math. Univ. Sancti Pauli, **12** (1964), 41–57.

[4] ——, *Some results on harmonic normal functions,* Math. Z., **90** (1965), 155–159.

[5] J. E. Littlewood, *Mathematical Notes* (4) : *On a theorem of Fatou,* London Math. J., **2** (1947), 172–176.

[6] K. Noshiro, *Cluster sets,* Berlin, 1960.

[7] D. C. Rung, *Behavior of holomorphic functions in the unit disk on arcs of positive hperbolyic diameter,* J. Math. Kyoto Univ., **8-3** (1968), 417–464.

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