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STABLE VECTOR BUNDLES ON ALGEBRAIC SURFACES

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Let k be an algebraically closed field, and X a nonsingular irreducible projective algebraic variety over k. These assumptions will remain fixed throughout this paper. We will consider a family of vector bundles on X of fixed rank r and fixed Chern classes (modulo numerical equivalence). Under what condition is this family a bounded family? When X is a curve, Atiyah [1] showed that it is so if all elements of this family are indecomposable. But when X is a surface, he showed also that this condition is not sufficient. We give the definition of an H-stable vector bundle on a variety X. This definition is a generalization of Mumford's definition on a curve. Under the condition that all elements of a family are H-stable of rank two on a surface X, we prove that the family is bounded. And we study H-stable bundles, when X is an abelian surface, the projective plane or a geometrically ruled surface.

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3. H-stable vector bundles on geometrically ruled surfaces
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1. H-stable vector bundles.

In this paper, we use the words vector bundles and locally free sheaf of finite rank interchangeably. Let F be a coherent sheaf on X. Under our hypothesis on X, we can define an invertible sheaf Inv (F) (first Chern class cf. [5]), i.e. let $E.$ be a finite resolution of F by locally free sheaves E_i. $\mathrm{Inv}(F) = \bigotimes_i (\dot{\wedge} E_i)^{(-1)^i}$ where $\dot{\wedge}$ denotes the highest exterior power. Then Inv (F) depends only on F, up to canonical isomorphism. Inv (F) has the following properties:

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PROPOSITION (1.1). i) $0 \to F_1 \to F_2 \to F_3 \to 0$ *is an exact sequence of coherent sheaves, then there is a canonical isomorphism* $\mathrm{Inv}(F_2) \simeq \mathrm{Inv}(F_1) \otimes \mathrm{Inv}(F_3)$.

ii) *If F is locally free, then* $\mathrm{Inv}(F)$ *is canonically isomorphic to* $\overset{\cdot}{\wedge} F$.

iii) *If F is torsion, then* $\mathrm{Inv}(F) = \mathcal{O}_X(D)$, *where D is a positive Cartier divisor. Moreover if* $\mathrm{codim}(\mathrm{Supp}(F)) \geqslant 2$, *then* $D = 0$.

iv) *If F is torsion-free, then* $\mathrm{Inv}(F)^{-1} = \mathrm{Inv}(F^*)$, *where F^* denotes* $\mathrm{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

Proof. For i), ii) and iii) see [5]. iv) follows from the following lemma.

LEMMA (1.2). *If* $0 \to F_1 \to F_2 \to F_3 \to 0$ *is an exact sequence of torsion-free sheaves, then* $\mathrm{Inv}(F_2^*) \simeq \mathrm{Inv}(F_1^*) \otimes \mathrm{Inv}(F_3^*)$.

Proof. We have an exact sequence $0 \to F_3^* \to F_2^* \to F_1^* \to \mathrm{Ext}^1_{\mathcal{O}_X}(F_3, \mathcal{O}_X)$. Since $\mathrm{codim}(\mathrm{Supp}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_3, \mathcal{O}_X))) \geqslant 2$ by our assumption, $\mathrm{Inv}(F_1^*/\mathrm{Im}(F_2^*)) = \mathcal{O}_X$ by iii).

If F is a coherent sheaf, then we can define the rank of F as the rank of F_ξ for a generic point ξ of X. We denote it by $r(F)$. We remark that F is torsion if and only if its rank is 0.

Let H be an ample line bundle on X. Let $s = \dim X$.

DEFINITION (1.3). A vector bundle E on X is *H-stable* (resp. *H-semi-stable*) if for every non-trivial, non-torsion, quotient coherent sheaf F of E, $d(E,H)/r(E) < d(F,H)/r(F)$ (resp. $\leqq$), where $d(F,H) = (\mathrm{Inv}(F), H^{s-1})$ and $(\ ,\ )$ is the intersection pairing.

DEFINITION (1.3)*. A vector bundle E on X is *H-stable* (resp. *H-semi-stable*) if for every non-zero coherent subsheaf G of E of rank $< r(E)$, $d(G,H)/r(G) < d(E,H)/r(E)$ (resp. $\leqq$).

It is obvious that (1.3) is equivalent to (1.3)*. And if a vector bundle E is H-stable, then for every non-zero coherent subsheaf G of E, $d(G,H)/r(G) \leqq d(E,H)/r(E)$. Indeed, if $r(G) = r(E)$, then E/G is torsion, which induces $d(G,H) \leqq d(E,H)$ by Prop. (1.1), iii).

Remark. In Definition (1.3), we may assume F is torsion-free. Indeed for any coherent sheaf E, let F be any torsion subsheaf of E, then $d(E,H) \geqslant d(E/F,H)$ by Prop. (1.1), iii).

PROPOSITION (1.4). i) *A line bundle is H-stable.*

ii) *A vector bundle is H-stable if and only if it is $H^{\otimes n}$-stable for any natural number n.*

iii) *If L is a line bundle, then a vector bundle E is H-stable if and only if $E \otimes L$ is H-stable.*

iv) *A vector bundle E is H-stable if and only if E^* is H-stable.*

v) *If E and F are two vector bundles, then $E \oplus F$ is never H-stable.*

vi) *If a vector bundle E of rank two is not H-semi-stable, then there is a unique torsion-free quotient sheaf F of rank one of E for which $d(F, H)$ is minimum.*

Proof. i), ii), iii) and v) are trivial. iv) follows from the above Remark and Prop. (1.1) iv). We show vi). In the same way as in the case of a curve, we can show that there exists a minimal H-degree quotient sheaf F of E of rank one. We may assume F is torsion-free. Let F, F' be such sheaves. Now we have an extension $0 \to G \to E \to F \to 0$. If the composition $G \to E \to F'$ is non-zero, then $d(F, H) = d(F', H) \geqq d(G, H) = d(E, H) - d(F, H)$ i.e. $d(F, H) \geqq (1/2)d(E, H)$. This contradicts our assumption. Hence $G \to E \to F'$ is zero, which induces $F \cong F'$.

DEFINITION (1.5) (Mumford [4]). A vector bundle E on a curve X is stable if and only if for every non-trivial quotient bundle F of E, $\deg(E)/r(E) < \deg(F)/r(F)$.

PROPOSITION (1.6). *Let X be a curve, and let E be a vector bundle on X. Then for any ample line bundle H, E is H-stable if and only if E is stable in the sense of Mumford.*

Proof. For any closed point $x \in X$, all torsion-free modules over the discrete valuation ring $\mathcal{O}_{X,x}$ are free.

PROPOSITION (1.7). *Let E, F be H-stable bundles, where $r = r(E) = r(F)$ and $d(E, H) = d(F, H)$. If $f: E \to F$ is a non-zero homomorphism, then f is an isomorphism.*

Proof. Put $G =$ Image of f. By definition, we have $d(E, H)/r(E) \leqq d(G, H)/r(G) \leqq d(F, H)/r(F)$, with strict inequalities holding unless $r(G) = r$. But by assumption, the two extreme sides are equal. Thus $r(G) = r = r(E)$, and we get $E \rightrightarrows G$, i.e. f is injective. Hence since $\bigwedge^r f : \bigwedge^r E$

$\to \overset{r}{\wedge} F$ is a non-zero homomorphism of line bundles and $d(\overset{r}{\wedge} E, H) = d(\overset{r}{\wedge} F, H)$, $\overset{r}{\wedge} f$ is an isomorphism, i.e. f is an isomorphism.

COROLLARY (1.8). *An H-stable vector bundle is simple.*

We say that a vector bundle E is simple if any global endomorphism of E is constant, i.e. $H^0(X, \operatorname{End}(E)) = k$.

Remark. 1) In Prop. (1.4), ii), iii) and iv), we may replace H-stability by H-semi-stability.

2) For any H-semi-stable vector bundle with $d(E, H) < 0, H^0(E) = 0$. Indeed, suppose there is a non-zero section $s \in H^0(E)$. Let F be the subsheaf of E generated by s. Then $F = \mathcal{O}_X$ and so $d(F, H) = 0$.

2. H-stable vector bundles on algebraic surfaces

In this section X will be a non-singular projective surface and H will be an ample line bundle on X. Let K be the canonical line bundle on X. We begin with a trivial lemma.

LEMMA (2.1). *Let E be an H-semi-stable vector bundle on X. If the Euler-Poincaré characteristic $\chi(E)$ of E is positive and $d(E^* \otimes K, H) < 0$. then $H^0(E) \neq 0$.*

Proof. Since $E^* \otimes K$ is H-semi-stable, $H^0(E^* \otimes K) = 0$ by the last Remark in § 1. Hence $H^2(E) = 0$ by Serre duality. Hence $H^0(E) \neq 0$.

COROLLARY (2.2). *Let S be a set of H-semi-stable vector bundles of rank two on X with fixed Chern classes (modulo numerical equivalence). Then there is an integer n such that $H^0(E \otimes H^{\otimes n}) \neq 0$ for any $E \in S$.*

Proof. For any $E \in S, \chi(E \otimes H^{\otimes n})$ is the same polynomial in n of degree two. Since the coefficient of n^2 is $(H^2), \chi(E \otimes H^{\otimes n})$ is positive for sufficiently large n. On the other hand, $d((E \otimes H^{\otimes n})^* \otimes K, H) = -d(E, H) - 2n(H^2) + 2(K, H) < 0$, for sufficiently large n. Hence we have the desired result by Lemma (2.1).

COROLLARY (2.3). *Let S be as in Cor. (2.2). Then there are integers n_1, n_2 such that for any $E \in S$, there is a coherent subsheaf F of E of rank 1 such that $n_1 \leqq d(F, H) \leqq n_2$.*

Proof. Let n be an integer satisfying Cor. (2.2). So there is a coherent subsheaf of E of rank 1 such that $d(F \otimes H^{\otimes n}, H) \geqslant 0$. i.e. $d(F, H) \geqslant -n(H^2)$. On the other hand, $d(F, H) \leqq (1/2)d(E, H)$ by H-semi-stability of E.

We say that a set A of vector bundles on X is bounded if there exists an algebraic k-scheme T and a vector bundle V on $T \times_k X$ such that each $F \in A$ is of the form $V_t = V | t \times X$ for some closed point $t \in T$.

THEOREM (2.4). *Let X be a non-singular projective surface, H an ample line bundle on X, and S the set of all H-semi-stable vector bundles on X of rank two and fixed Chern classes (modulo numerical equivalence). Then S is bounded.*

Proof. By a theorem of Kleiman ([3] Th. 1.13), it is sufficient to show that there are integers m_1, m_2 such that for any $E \in S$, 1) $\dim_k H^0(E) \leqq m_1$ 2) there is a non-singular curve C such that $\mathcal{O}_X(C) = H$ and $\dim_k H^0(E \otimes \mathcal{O}_C) \leqq m_2$. We may assume H-degree is negative. Hence 1) follows from the last Remark in § 1. We now show 2). Let n_1, n_2 be the same as in Cor. (2.3). Put $n_i = d(E, H) - n_{i-2}$, $i = 3, 4$ and $t = \max(0, 2g - n_1, 2g - n_4)$, where $g = \chi(H^{-1}) - \chi(\mathcal{O}_X) + 1$. Let E be any vector bundle contained in S. There are torsion-free sheaves F_1, F_2 of rank 1 such that there is an exact sequence $0 \to F_1 \to E \to F_2 \to 0$, $n_1 \leqq d(F_1, H) \leqq n_2$. Hence $n_4 \leqq d(F_2, H) \leqq n_3$. Now F_i is locally free at any point outside a finite set Z of closed points. Hence there exists a non-singular curve C in H, disjoint from Z. Here the genus of C is g. So the restriction of F_i to C is a line bundle on C. Since $d(F_i \otimes H^{\otimes t} \otimes \mathcal{O}_C) = d(F_i, H) + t(H^2) \geqslant \min(n_1, n_4) + t \geqslant 2g$, $\dim_k H^0(F_i \otimes \mathcal{O}_C) \leqq \dim_k H^0(F_i \otimes H^{\otimes t} \otimes \mathcal{O}_C) \leqq t(H^2) + \max(n_2, n_3) - g + 1 = c$. Hence $\dim_k H^0(E \otimes \mathcal{O}_C) \leqq 2c$.

We now give another definition of H-stability of a vector bundle. First, we recall that for any non-zero global section s of a vector bundle E, there exists a surface Y and a morphism $f: Y \to X$ obtained by successive dilatations, and a sub-line bundle L of f^*E on Y and a global section t of L such that the inclusion $L \subset f^*E$ maps t to f^*s and f^*E/L is locally free. (cf. Schwarzenberger [10])

LEMMA (2.5). *Let φ be a homomorphism from a non-torsion coherent sheaf F to a vector bundle E such that* $\operatorname{codim}(\operatorname{Supp}(\ker \varphi)) \geqslant 2$. *Then there is a surface Y and a morphism $f: Y \to X$ obtained by successive dila-*

tations, and a vector subbundle G of f^*E on Y such that $f^*(\varphi)(f^*F) \subset G$ and $r(F) = r(G)$ (and f^*E/G is locally free).

Proof. We proceed by induction on $r = \operatorname{rank} E$. Suppose the lemma is true for all rank $< r = \operatorname{rank} E$. We may assume there is a non-torsion global section u of F. Let s be the global section of E corresponding to u. Let Y, f, L and t be as above. Then we have exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & f^*E & \longrightarrow & f^*E/L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (f^*\varphi)^{-1}(L) & \longrightarrow & f^*F & \longrightarrow & f^*F/(f^*\varphi)^{-1}(L) & \longrightarrow & 0
\end{array}$$

Now since u is not torsion, $r(f^*F/(f^*\varphi)^{-1}(L)) = r(F) - 1$. By induction, there exists a surface Y' and a morphism $f' : Y' \to Y$ obtained by successive dilatations and a vector subbundle G' of $f'^*(f^*E/L)$ on Y' such that $(f'^*f^*\varphi)(f^*F/(f^*\varphi)^{-1}(L)) \subset G'$ and $r(G') = r(F) - 1$ (and $f'^*(f^*E/L)/G'$ is locally free). Let G be the subbundle of f'^*f^*E with $G' = G/f'^*L$.

PROPOSITION (2.6). *A vector bundle E on a surface X is H-stable if and only if for any morphism $f : Y \to X$ obtained by successive dilatations and any non-trivial quotient bundle F of f^*E, $d(E,H)/r(E) < d(F, f^*H)/r(F)$.*

Proof. First, suppose E is H-stable. Let f, F be as in Prop. (2.6). We may assume H is a very ample line bundle. Now there exists a finite set Z of closed points such that f is an isomorphism on $X - Z$. Then we find a curve D such that $\mathcal{O}_X(D) = H$ and $Z \cap D$ is empty. Let G be the kernel of $f^*E \to F$. Since $\operatorname{Supp}(G/f^*f_*G) \cap f^*(D)$ is empty, $d(G, f^*H) = d(f^*f_*G, f^*H)$. On the other hand $d(f^*f_*G, f^*H)d = d(f_*G, H)$. Conversely let F be a non-zero subsheaf of E of rank $< \operatorname{rank} E$, and let Y and G be the same as in Lemma (2.5). Since f^*E/G is locally free, $d(G, f^*H)/r(G) < d(E, H)/r(E)$ by assumption. On the other hand, $r(G) = r(f^*F)$ by construction and $d(F, H) = d(f^*F, f^*H) \leqq d(G, f^*H)$ since the image of f^*F in f^*E is contained in G. Thus $d(F, H)/r(F) < d(E, H)/r(E)$, and E is H-stable.

From now on, we study vector bundles of rank two on a non-singular projective surface X. It is known (Schwarzenberger [10]) that for a vector bundle E of rank two on X there exists a morphism $f : Y \to X$ obtained by successive dilatations, line bundles L_1 and L_2 on X, and a positive exceptional line bundle M on Y (i.e. line bundle on Y associated with a non-negative linear combination of exceptional curves on Y) such that f^*E is given by an extension of the form

$$0 \longrightarrow f^*L_1 \otimes M \longrightarrow f^*E \longrightarrow f^*L_2 \otimes M^{-1} \longrightarrow 0$$

Conversely, for any morphism $f: Y \rightarrow X$ obtained by successive dilatations, a quotient line bundle of f^*E is always of the form $f^*L_2 \otimes M^{-1}$ where L_2 is a line bundle on X and M is a positive exceptional line bundle. (Schwarzenberger loc. cit.)

Put $N(E) = c_1^2(E) - 4c_2(E)$, where $c_i(E)$ is the i-th Chern class of E. This integer is equal to $-c_2(\mathrm{End}(E))$. It has the following geometric meaning. Let L be a quotient line bundle of E, and p the canonical projection $P(E) \rightarrow X$. Then L defines a section s of p. Let Y denote $s(X)$. Then $(Y^3)_{P(E)} = N(E)$. Note that $N(E) = N(E \otimes L')$ for any line bundle L'.

PROPOSITION (2.7). *Let E be a vector bundle of rank two. If $N(E) > 0$, then E is H-stable if and only if E is H'-stable for any ample line bundle H' on X.*

Proof. By Prop. (2.6), E is H-stable if we have $(L_2 \otimes L_1^{-1}, H) > 0$ for any morphism $f: Y \rightarrow X$ obtained by successive dilatations and an extension

$$0 \longrightarrow f^*L_1 \otimes M \longrightarrow f^*E \longrightarrow f^*L_2 \otimes M^{-1} \longrightarrow 0$$

where L_1 and L_2 are line bundles on X, and M is a positive exceptional line bundle on Y. By our assumption, $N(E) = (L_2 \otimes L_1^{-1}, L_2 \otimes L_1^{-1}) + 4(M^2) > 0$. But by the negative definiteness of the intersection pairing on exceptional divisors, $(M^2) \leqq 0$, hence $(L_2 \otimes L_1^{-1}, L_2 \otimes L_1^{-1}) > 0$. We thus have the desired result by the Hodge index theorem ([6] Lecture 18).

DEFINITION (2.8). We say that a vector bundle E of rank two on X is *of trivial type* if there are line bundles L_1, L_2 on X with $H^0(L_2) = H^0(L_2^{-1}) = 0$, a morphism $f: Y \rightarrow X$ obtained by successive dilatations and a positive exceptional line bundle M on Y such that we have a non-trivial extension of line bundles $0 \rightarrow M \rightarrow f^*E_1 \rightarrow f^*L_2 \otimes M^{-1} \rightarrow 0$, where $E_1 = E \otimes L_1$.

PROPOSITION (2.9). *Let E be a vector bundle of rank two on X.*

Then E is simple if and only if E is either H-stable for an ample line bundle H or of trivial type.

Proof. If E is of trivial type, then by Oda's lemma [9], E is simple since $\mathrm{Hom}(M, f^*L_2 \otimes M^{-1}) = H^0(X, f^*L_2 \otimes M^{-2}) \subsetneq H^0(L_2) = 0$. If E is H-stable, then E is simple by Cor. (1.8). Assume E is simple and not H-stable. Therefore there are line bundles L_1 and L_2 on X, and a morphism $f: Y \to X$ obtained by successive dilatations and an extension of line bundles $0 \to M \to f^*E_1 \to f^*L_2 \otimes M^{-1} \to 0$, where $E_1 = E \otimes L_1, M$ is a positive exceptional line bundle and $d(E_1, H) \leqq 0$. Hence $d(L_2, H) \leqq 0$. Now we show $H^0(L_2) = 0$. Indeed, if $H^0(L_2) \neq 0$, then $L_2 = \mathcal{O}_X$ by $d(L_2, H) \leqq 0$. And since $H^0(\mathrm{Hom}(M^{-1}, M)) \neq 0, E$ is not simple. This contradicts our assumption. Since $\mathrm{Hom}(M, f^*L_2 \otimes M^{-1}) \subsetneq H^0(L_2) = 0$, $H^0(\mathrm{End}(E)) = H^0(\mathrm{End}(E_1)) = k \oplus H^0(f^*L_2^{-1} \otimes M^2) = k \oplus H^0(L_2^{-1})$ by Oda's lemma. Thus $H^0(L_2^{-1}) = 0$. i.e. E is of trivial type.

We now give a result about the cohomology of an H-semi-stable vector bundle.

PROPOSITION (2.10). *Let X be a surface and E an H-semi-stable vector bundle on X with $d(E, H) = 0$. Then $\dim_k H^0(E) \leqq \mathrm{rank}\, E$. And the equality holds if and only if E is free.*

Proof. If $H^0(E) \neq 0$, there is a morphism $f_1: X_1 \to X$ obtained by successive dilatations and a line bundle L_1 and a vector bundle E_1 on X_1 such that we have an extension $0 \to L_1 \to f_1^*E \to E_1 \to 0$ and $H^0(L_1) \neq 0$. Since $d(L_1, H) \leqq 0, L_1$ is a positive exceptional line bundle and hence $H^0(L_1) = k$, which induces $\dim_k H^0(E) \leqq \dim_k H^0(E_1) + 1$. Moreover if $H^0(E_1) \neq 0$, there is a morphism $f_2: X_2 \to X_1$ obtained by successive dilatations and a line bundle L_2 and a vector bundle E_2 on X_2 such that we have an extension $0 \to L_2 \to f_2^*E_1 \to E_2 \to 0$ and $H^0(L_2) \neq 0$. Let φ denote $f_1^*E \to E_1$. Since $0 \leqq d(L_2, H) = d(\varphi^{-1}(L_2), H) \leqq 0$, L_2 is a positive exceptional line bundle. Hence $\dim_k H^0(E) \leqq \dim_k H^0(E_2) + 2$. Continuing in this fashion we get $\dim_k H^0(E) \leqq \mathrm{rank}\, E$. If $\dim_k H^0(E) = \mathrm{rank}\, E = r$, then we can define E_i, L_i $(i = 1, 2, \cdots, r-1)$ inductively and $E_{r-1} = L_r$ is also a positive exceptional line bundle, i.e. L_i is a positive exceptional line bundle for $i = 1, 2, \cdots, r$. On the other hand, $L_1 \otimes L_2 \otimes \cdots \otimes L_r = \mathrm{Inv}(E)$, hence $L_i = \mathcal{O}_X$, $(i = 1, 2, \cdots, r)$, i.e. E is obtained by successive extensions of the structure sheaf $\mathcal{O}_X$, and $\dim_k H^0(E) = \operatorname{rank} E$, which implies that E is free.

3. H-stable vector bundles of rank two on geometrically ruled surfaces

Let C be a non-singular projective curve of genus g over an algebraically closed field k, V a vector bundle of rank two on C, and $\mathcal{O}_{\boldsymbol{P}(V)}(1)$ the tautological line bundle on $\boldsymbol{P}(V)$ (See EGA II. 4.1.1 for the definition of $\boldsymbol{P}(V)$). Then the Néron-Severi group of $\boldsymbol{P}(V)$ is $\boldsymbol{Z} \oplus \boldsymbol{Z}$, and is generated by the class d of $\mathcal{O}_{\boldsymbol{P}(V)}(1)$ and the class f of a fibre of $\boldsymbol{P}(V)$ over C. And $(d^2) = \deg V = a$. In case V is decomposable, put $V = M_1 \oplus M_2$, where M_1 and M_2 are line bundles on C with $\deg M_i = a_i, a_2 \geqq a_1$ and $a = a_1 + a_2$. Let p denote the canonical projection: $\boldsymbol{P}(V) \to C$. In this section, these assumptions will remain fixed.

PROPOSITION (3.1). *Let L be a line bundle on $\boldsymbol{P}(V)$, and let the class of L be $nd + mf$. Then L is ample, if one of the following conditions is satisfied:*

1.1) *If V is semi-stable and* char. $k = 0$, *then $n > 0$ and $na + 2m > 0$.*

1.2) *If V is semi-stable,* char. $k = p > 0$ *and $g \geqq 1$, then $n > 0$ and $na + 2m > (2n/p)(g - 1)$.*

2) *If V is indecomposable, then $n > 0$ and $na + 2m > 2n(g - 1)$.*

3.1) *If V is decomposable and either* char. $k = 0$ *or $g = 0$, then $n > 0$ and $na_1 + m > 0$.*

3.2) *If V is decomposable,* char. $k = p > 0$ *and $g \geqq 1$, then $n > 0$ and $na_1 + m > (n/p)(g - 1)$.*

Moreover, when V is semi-stable and either char. $k = 0$ or $g = 1$, then L is ample if and only if $n > 0$ and $na + 2m > 0$. And when V is decomposable and either char. $k = 0$ or $g = 0, 1$, then L is ample if and only if $n > 0$ and $na_1 + m > 0$.

Proof is due essentially to Hartshorne ([2] Prop. (7.5)). He treated the case when the maximal degree of subline bundles of V is non-positive $a > 0$ and $n = 1, m = 0$, i.e. $L = \mathcal{O}_{\boldsymbol{P}(V)}(1)$. (In this case V is stable.)

COROLLARY (3.2). *There is a constant c depending on V such that a line bundle L on $\boldsymbol{P}(V)$, whose class is $nd + mf$, is ample if $n > 0$ and $m + nc > 0$.*

Remark. If L as above is ample, then $n > 0$ and $na + 2m > 0$. Indeed, $(L, f) = n$, $(L^2) = n(na + 2m)$.

Remark. If V is indecomposable and there is a non-trivial extension of line bundles $0 \to L_1 \to V \to L_2 \to 0$, then $\deg(L_2 \otimes L_1^{-1}) \geqslant 2 - 2g$. Indeed, since $H^1(\mathrm{Hom}(L_2, L_1)) \neq 0$, $H^0(L_2 \otimes L_1^* \otimes K_C) \neq 0$, where K_C denotes the canonical line bundle on C.

Proof of Proposition (3.1). Let D be any irreducible curve on $\boldsymbol{P}(V)$. Since $(L^2) = n(na + 2m) > 0$ in each case, it is sufficient, by Nakai's criterion, to show that $(D, L) > 0$. Let the class of D be $kd + hf$. Since $(D, f) \geqslant 0, k \geqslant 0$. If $k = 0$, then $h = 1$, since D is irreducible. So $(D, L) = n > 0$. If $k = 1$, then D is a section of $\boldsymbol{P}(V)$ over C, and we can write $\mathcal{O}_{\boldsymbol{P}(V)}(D) = \mathcal{O}_{\boldsymbol{P}(V)}(1) \otimes p^*(M)$ for a line bundle M on C of degree h. Then we have an exact sequence of sheaves on $\boldsymbol{P}(V)$: $0 \to \mathcal{O}_{\boldsymbol{P}(V)}(-D) \to \mathcal{O}_{\boldsymbol{P}(V)} \to \mathcal{O}_D \to 0$. Tensoring with $\mathcal{O}_{\boldsymbol{P}(V)}(1)$, we have $0 \to p^*(M^{-1}) \to \mathcal{O}_{\boldsymbol{P}(V)}(1) \to \mathcal{O}_D \otimes \mathcal{O}_{\boldsymbol{P}(V)}(1) \to 0$. We apply p_*. Note that $p_* p^*(M^{-1}) = M^{-1}, p_*(\mathcal{O}_{\boldsymbol{P}(V)}(1)) = V$, $R^1 p_* p^*(M^{-1}) = 0$, and $p_*(\mathcal{O}_D \otimes \mathcal{O}_{\boldsymbol{P}(V)}(1))$ is a line bundle on C, since D is a section of p. Thus we have an exact sequence of vector bundles on C:

$$0 \longrightarrow M^{-1} \longrightarrow V \longrightarrow p_*(\mathcal{O}_D \otimes \mathcal{O}_{\boldsymbol{P}(V)}(1)) \longrightarrow 0$$

Case 1) $d(M^{-1}) \leqq (1/2)d(V)$ i.e. $a + 2h \geqslant 0$.
Case 2) $d(p_*(\mathcal{O}_D \otimes \mathcal{O}_{\boldsymbol{P}(V)}(1))) - d(M^{-1}) \geqq 2 - 2g$ i.e. $a + 2h \geqq 2 - 2g$
Case 3) $d(M^{-1}) \leqq a_2 = \max(a_1, a_2)$ i.e. $a_2 + h \geqq 0$.

On the other hand, $(D, L) = na + hn + m$. Hence

Case 1) $(D, L) = (1/2)(na + 2m) + (1/2)n(a + 2h) > 0$.
Case 2) $(D, L) > n(g-1) - n(g-1) = 0$.
Case 3) $(D, L) = (na_1 + m) + n(a_2 + h) > 0$.

Therefore we may assume $k \geqq 2$. Since $K_{\boldsymbol{P}(V)} = \mathcal{O}_{\boldsymbol{P}(V)}(-2) \otimes p^*(K_C \otimes \mathrm{Inv}(V))$, the class of $K_{\boldsymbol{P}(V)}$ is $-2d + (2g - 2 + a)f$ (where $K_{\boldsymbol{P}(V)}$ and K_C are the canonical line bundles on $\boldsymbol{P}(V)$ and C respectively).

Suppose either char. $k = 0$ or $k < p$. Then we can apply the Hurwitz formula to the projection of D onto C, and find $2p_a(D) - 2 \geqq k(2g - 2)$. On the other hand, $2p_a(D) - 2 = (D, (D + K_{\boldsymbol{P}(V)})) = (k-1)(ka + 2h) + k(2g - 2)$. Combining these, we have $ka + 2h \geqq 0$, since $k \geqq 2$. $(D, L) = kna + nh + mk = (1/2)n(ka + 2h) + (1/2)k(na + 2m) > 0$.

Suppose char. $k = p \neq 0$, and $k \geqq p$. Then we have an inequality $2p_a(D) - 2 \geqq 2g - 2$. As above, we deduce $ka + 2h \geqq 2 - 2g$. Thus $(D, L) = (1/2)n(ka + 2h) + (1/2)k(na + 2m) \geqq n(1 - g) + (1/2)p(na + 2m)$.

If $g = 0$, then $(D, L) > 0$. In case (1.2), (2) and (3.2), we have $na + 2m > (2n/p)(g - 1)$, hence $(D, L) > 0$.

The first statement of the converse is trivial. Let V be decomposable and Y the image of the section associated with $V \to M_1 \to 0$. Then the class of Y is $d - a_2 f$. Hence $(Y, L) = na_1 + m > 0$. q.e.d.

LEMMA (3.3). *Let E be a vector bundle of rank two on $\boldsymbol{P}(V)$. Assume $N(E) \geqq 0$. Then E is H-stable if and only if E is H'-stable for any ample line bundle H' on $\boldsymbol{P}(V)$.*

Proof. By Prop. (2.6), E is H-stable if we have $(L_2 \otimes L_1^{-1}, H) > 0$ for any morphism $f: Y \to \boldsymbol{P}(V)$ obtained by successive dilatations and an extension of line bundles on Y

$$0 \longrightarrow f^*(L_1) \otimes M \longrightarrow f^*(E) \longrightarrow f^*(L_2) \otimes M^{-1} \longrightarrow 0$$

where L_1 and L_2 are line bundles on $\boldsymbol{P}(V)$, and M is a positive exceptional line bundle on Y. Let H be an ample line bundle on $\boldsymbol{P}(V)$ and let the class of H be $nd + mf$. Let the class of $L_2 \otimes L_1^{-1}$ be $kd + hf$. Then $(L_2 \otimes L_1^{-1}, H) = kna + nh + mk = (1/2)k(na + 2m) + (1/2)n(ka + 2h)$, and $N(E) = (L_2 \otimes L_1^{-1}, L_2 \otimes L_1^{-1}) + 4(M^2) = k(ka + 2h) + 4(M^2) \geqslant 0$. So $k(ka + 2h) \geqq -4(M^2) \geqq 0$. Now $n > 0$ and $na + 2m > 0$ by the ampleness of H. Hence $(L_2 \otimes L_1^{-1}, H) > 0$ if and only if either $k > 0$ and $ka + 2h \geqq 0$, or $k = 0$ and $ka + 2h > 0$.

PROPOSITION (3.4). *Let E be a stable vector bundle of rank two on C. Then p^*E is H-stable for any ample line bundle H on $\boldsymbol{P}(V)$. (In this case $N(p^*E) = 0$.)*

Proof. Let H be an ample line bundle whose class is $d + sf$, where s is large enough. We remark $a + 2s > 0$. By Lemma (3.3), it is enough to show the Proposition for this H. Put $m = \deg(E)$. Then the class $c_1(p^*E)$ is mf and $c_2(p^*E)$ is zero. By Prop. (2.6), E is H-stable if we have $(L_2 \otimes L_1^{-1}, H) > 0$ for any morphism $f: Y \to \boldsymbol{P}(V)$ obtained by successive dilatations and an extension of line bundles on Y

$$(*) \qquad 0 \longrightarrow f^*L_1 \otimes M \longrightarrow f^*p^*E \longrightarrow f^*L_2 \otimes M^{-1} \longrightarrow 0$$

where L_1 and L_2 are line bundles on $\boldsymbol{P}(V)$ and M is a positive exceptional line bundle on Y. We wish to show that $d(L_1, H) < (1/2)d(p^*E, H)$, i.e.

$2ka + 2ks + 2h - m < 0$, where the class of L_1 is $kd + hf$. On the other hand, $0 = N(p^*E) = (L_2 \otimes L_1^{-1}, L_2 \otimes L_1^{-1}) + 4(M^2) = 4k(ka + 2h - m) + 4(M^2)$. So $-4(M^2) = 4k(ka + 2h - m) \geqq 0$. Now if we restrict (*) to a fibre f of $\boldsymbol{P}(V)$ over C, we have an exact sequence $0 \to \mathcal{O}_f(k) \to \mathcal{O}_f \oplus \mathcal{O}_f \to \mathcal{O}_f(-k) \to 0$, where $f \cong \boldsymbol{P}^1$, and hence $k \leqq 0$. If $k < 0$, then $ka + 2h - m \leqq 0$ and hence $2ka + 2ks + 2h - m = k(a + 2s) + ka + 2h - m < 0$. If $k = 0$, then $(M^2) = 0$ and hence $M = \mathcal{O}_Y$. Therefore the above extension is of the following form: $0 \to p^*L_1' \to p^*E \to p^*L_2' \to 0$, where L_1' and L_2' are line bundles on C such that $L_1 = p^*L_1'$ and $L_2 = p^*L_2'$. Apply p_*. Then we have an exact sequence $0 \to L_1' \to E \to L_2' \to 0$. By our assumption, $h < (1/2)m$. Hence $2ka + 2ks + 2h - m = 2h - m < 0$.

PROPOSITION (3.5). *There is no vector bundle E of rank two on $\boldsymbol{P}(V)$ with the first Chern class $c_1(E) = \mathcal{O}_{\boldsymbol{P}(V)}(-1) \otimes p^*(L)$ for some line bundle L on C such that E is H-stable for every ample line bundle H on $\boldsymbol{P}(V)$.*

Proof. Suppose there exists such a vector bundle E. Let m be the degree of L. Then the class of $c_1(E)$ is $-d + mf$. We may assume m is sufficiently large. Put $b = N(E)$. Let H be an ample line bundle on $\boldsymbol{P}(V)$ whose class is $d + sf$. Then the Euler Poincaré characteristic $\chi(E)$ of E is equal to $(1/4)(b - a + 2m) + 1 - g$ and $d(E^* \otimes K, H) = 4g - 4 - a - m - 3s$. Hence we may assume $\chi(E) > 0$ and $d(E^* \otimes K, H) < 0$. So $H^0(E) \neq 0$ by Lemma (2.1). Therefore there is a morphism $f: Y \to \boldsymbol{P}(V)$ obtained by successive dilatations and an extension of line bundles on Y, $0 \to f^*L_1 \otimes M \to f^*E \to f^*L_2 \otimes M^{-1} \to 0$, where L_1 and L_2 are line bundles on $\boldsymbol{P}(V), H^0(L_1) \neq 0$ and M is a positive exceptional line bundle. Let the class of L_1 be $kd + hf$. For large enough n, any line bundle $H_{1,n}$ whose class is $d + nf$ is ample by Cor. (3.2). By $H^0(L_1) \neq 0$, we have $d(L_1, H_{1,n}) \geqq 0$ i.e. $ka + h + kn \geqslant 0$ for large enough n. So $k \geqslant 0$. On the other hand, by our assumption, $d(L_1, H_{1,n}) \leqq (1/2)d(E, H_{1,n})$ i.e. $(n + a)(-1 - 2k) + m - 2h \geqq 0$ for large enough n. So $k \leqq -1/2$. This is a contradiction.

PROPOSITION (3.6). *Let E be a vector bundle on $\boldsymbol{P}(V)$ of rank two with the first Chern class $c_1(E) = p^*L$ for some line bundle L on C and $N(E) \geq 0$. If E is H-stable for an ample line bundle H, then there is a stable vector bundle F on C such that $E = p^*F$. (It follows that $N(E) = 0$.)*

Proof. Put $m = d(L)$ and $b = N(E)$. And let $H_{1,n}$ be the same as in Prop. (3.5). By Lemma (3.3) we may assume $H = H_{1,n}$. Then $\chi(E) = m + (1/4)b + 2 - 2g$ and $d(E^* \otimes K, H) = -2a + 4g - 4 - 4n - m$. By the same argument as in Prop. (3.5), we have an exact sequence $0 \to f^*L_1 \otimes M \to f^*E \to f^*L_2 \otimes M^{-1} \to 0$ where f, L_1, L_2, M are the same as before. By $H^0(L_1) \neq 0$, we have $d(L_1, H_{1,n}) \geq 0$ i.e. $ka + h + kn \geq 0$ for large enough n. So $k \geq 0$. On the other hand, by our assumption, $d(L_1, H_{1,n}) \leqq (1/2)d(E, H_{1,n})$ i.e. $2\text{m} - ka - h - kn \geq 0$ for large enough n. So $k \leq 0$. Hence $k = 0$ and $0 \leq h < (1/2)m$. Now since $N(E) = 4(M^2) \geq 0$, we conclude that $M = \mathcal{O}_Y, N(E) = 0$ and the above extension is of the following form: $0 \to p^*L_1' \to E \to p^*L_2' \to 0$, where L_1', L_2' are line bundles on C. This extension defines an element of $H^1(\mathrm{Hom}(p^*L_2', p^*L_1'))$. On the other hand, $H^1(\mathrm{Hom}(L_2', L_1')) \rightleftarrows H^1(\mathrm{Hom}(p^*L_2', p^*L_1'))$ (canonically). Hence $E = p^*F$ for some vector bundle F on C which is an extension of L_2' by L_1'. It is obvious that F is stable.

THEOREM (3.7). *Let H be an ample line bundle on $\boldsymbol{P}(V)$.*

1) *There is no H-stable bundle E of rank two on $\boldsymbol{P}(V)$ with $N(E) > 0$.*

2) *A vector bundle E of rank two on $\boldsymbol{P}(V)$ is H-stable with $N(E) = 0$ if and only if there is a stable vector bundle F of rank two on C and a line bundle L on $\boldsymbol{P}(V)$ such that $E = p^*F \otimes L$.*

3) *Let E be a vector bundle of rank two on $\boldsymbol{P}(V)$ with $N(E) < 0$, and let the first Chern class of E be $kd + hf$ where k is odd. If E is H-stable, then there exists an ample line bundle H' on $\boldsymbol{P}(V)$ such that E is not H'-stable.*

Proof. Tensoring E with a suitable line bundle $\mathcal{O}_{\boldsymbol{P}(V)}(n)$, we may assume $c_1(E) = kd + hf$ with $k = 0$ or 1. The statement is obtained from Lemma (3.3), Prop. (3.4), Prop. (3.5) and Prop. (3.6).

We now give an example of Th. (3.7). 3). First

LEMMA (3.8). *Let X be a non-singular projective surface. Let L be a line bundle on X and let H be an ample line bundle on X. Suppose the extension $0 \to \mathcal{O}_X \to E \to L \to 0$ does not split and $d(L, H) = 1$. Then E is H-stable.*

Proof. First, remark $(1/2)d(E, H) = 1/2$. Suppose we are given a morphism $f: Y \to X$ obtained by successive dilatations and a surjective morphism $f^*E \to f^*L_1 \otimes M^{-1}$, where L_1 is a line bundle on X and M is a positive exceptional line bundle on Y. If $\mathcal{O}_Y \to f^*E \to f^*L_1 \otimes M^{-1}$ is zero, then $L = L_1$ and $M = \mathcal{O}_Y$. If not, then $0 \neq H^0(f^*L_1 \otimes M^{-1}) \subset H^0(L_1)$. Hence $d(L_1, H) \geq 0$. Then if $d(L_1, H) = 0$, then $L_1 = \mathcal{O}_X$ and $H^0(M^{-1}) \neq 0$, and so $M = \mathcal{O}_Y$. Therefore the above extension splits. Hence $d(L_1, H) \geqslant 1$.

PROPOSITION (3.9). *Assume $a + 2m > 2g$ if V is indecomposable, and $a_1 + m > g$ if V is decomposable. Denote by $H_{1,m}$ an ample line bundle on $\boldsymbol{P}(V)$ whose class is $d + mf$. Let M be a line bundle on C of degree $a + m + 1$. Put $L = \mathcal{O}_{\boldsymbol{P}(V)}(-1) \otimes p^*M$ and $s = \dim_k H^1(L^{-1}) - 1$. (In this case $s = a + 2m + 2g - 1 \geq 4g$.) If $0 \to \mathcal{O}_{\boldsymbol{P}(V)} \to E \to L \to 0$ is a non-trivial extension, then E is $H_{1,m}$-stable and is not $H_{1,n}$-stable for any ample line bundle $H_{1,n}$ with $n \geq m + 1$. We also have $N(E) = -a - 2 - 2m$, $H^0(E) = k$, $\dim_k H^1(E) = g$, $H^2(E) = 0$, $H^2(\mathrm{End}(E)) = 0$ and $\dim_k H^1(\mathrm{End}(E)) = s + 2g$. Let $\xi \neq \xi'$ be elements in $P(H^1(L^{-1}))$, and let E_ξ and $E_{\xi'}$ be vector bundles on $\boldsymbol{P}(V)$ corresponding to the extension classes ξ and ξ' respectively as above. Then $E_\xi \not\simeq E_{\xi'}$.*

Proof. First, we calculate $\dim_k H^1(L^{-1})$. $H^1(L^{-1}) = H^1(\boldsymbol{P}(V), \mathcal{O}_{\boldsymbol{P}(V)}(1) \otimes p^*M^{-1}) = H^1(C, V \otimes M^{-1})$. By duality, $\dim_k H^1(L^{-1}) = \dim_k H^0(C, V^* \otimes M \otimes K_C)$, where K_C denotes the canonical line bundle on C. In case V is indecomposable, let (L_1, L_2) be a maximal splitting of $V^* \otimes M \otimes K_C$. By the result of Atiyah [1] to the effect that $2g \geq d(L_2) - d(L_1) \geq -2g + 2$, we conclude $d(L_i) \geq (1/2)(6g - 3) - g > 2g - 2$, since $d(V^* \otimes M \otimes K_C) = a + 2m + 4g - 2 \geqslant 6g - 3$ by our assumption. Hence $H^1(L_i) = 0$. In case V is decomposable, we equally have $H^1(V^* \otimes M \otimes K_C) = 0$ since $d(M_i^* \otimes M \otimes K_C) = a - a_i + m + 2g - 1 > 2g - 2$. Therefore $\dim_k H^1(L^{-1}) = s + 1$. By Lemma (3.8), E is $H_{1,m}$-stable since $d(L, H_{1,m}) = 1$. On the other hand since $d(L, H_{1,n}) \leq 0$ for $n \geq m + 1$, E is not $H_{1,n}$-stable. Now since $H^i(\boldsymbol{P}(V), L) = 0$ for $i = 0, 1$ and 2, $H^i(E) \simeq H^i(\mathcal{O}_{\boldsymbol{P}(V)})$. We now show $H^2(\mathrm{End}(E)) = 0$. Since $0 \to \mathcal{O}_{\boldsymbol{P}(V)} \to E \to L \to 0$, we have an exact sequence $0 \to E^* \to \mathrm{End}(E) \to L \otimes E^* \to 0$ by tensoring it with E^*. On the other hand, since $E^* \otimes K_{\boldsymbol{P}(V)}$ and $E^* \otimes K_{\boldsymbol{P}(V)} \otimes L$ are $H_{1,m}$-stable bundles with negative $H_{1,m}$-degree, $H^0(E^* \otimes K_{\boldsymbol{P}(V)}) = 0$ and $H^0(E^* \otimes K_{\boldsymbol{P}(V)} \otimes L) = 0$. Hence $\dim_k H^2(\mathrm{End}(E)) = \dim_k H^0(\mathrm{End}(E) \otimes K_{\boldsymbol{P}(V)}) = 0$. So we can calculate $\dim_k H^1(\mathrm{End}(E))$, since E is simple. The last statement follows from $H^0(E) = k$.

We remark the following fact: Let M_1 and M_2 be line bundles on C of degree 0, and let N_1 and N_2 be line bundles on C of degree $a + m + 1$.

If a vector bundle E on $\boldsymbol{P}(V)$ is an extension of $\mathcal{O}_{\boldsymbol{P}(V)}(-1)\otimes p^*N_1$ by p^*M_1 which is also an extension of $\mathcal{O}_{\boldsymbol{P}(V)}(-1)\otimes p^*N_2$ by p^*M_2, then $M_1 = M_2$ and $N_1 = N_2$. Indeed we may assume $M_1 = \mathcal{O}_{\boldsymbol{P}(V)}$. Since $k = H^0(\mathcal{O}_{\boldsymbol{P}(V)}) = H^0(E) = H^0(M_2)$ and $d(M_2) = 0$, so $M_2 = \mathcal{O}_{\boldsymbol{P}(V)}$, and hence $N_1 = N_2$.

Hence we can say that there is an algebraic family S of simple vector bundles on $\boldsymbol{P}(V)$ parametrized by $J\times J\times \boldsymbol{P}^s$, in which isomorphic ones appear only once, and for any E contained in S, $\dim_k H^1(\mathrm{End}(E))$ = the dimension of $J\times J\times \boldsymbol{P}^s$. Here J is the Jacobian variety of C and $\boldsymbol{P}^s$ is the s-dimensional projective space.

Conversely,

PROPOSITION (3.10). *Assume* $a_1 + m > 0$. *Let C be the projective line and E a vector bundle of rank two on $\boldsymbol{P}(V)$ with* $N(E) = -a-2-2m$ *whose first Chern class is* $kd + hf$, *where k is odd. Then there is a line bundle L' on $\boldsymbol{P}(V)$ such that* $E' = E\otimes L'$ *is the extension of L by* $\mathcal{O}_{\boldsymbol{P}(V)}$ *where L is of the same type as in Prop. (3.9) i.e. there is a line bundle M on C of degree* $a+m+1$ *such that* $L = \mathcal{O}_{\boldsymbol{P}(V)}(-1)\otimes p^*M$.

Proof. Tensoring E with a suitable line bundle, we may assume the class of $c_1(E)$ is $-d+bf$. Moreover we may assume it is $-d+(a+m+1)f$. Indeed if $b\equiv a+m \pmod 2$, then $N(E) = c_1^2(E) - 4c_2(E) \equiv -a-2m \pmod 4$. This contradicts our assumption. Then $c_2(E) = 0$. $\chi(E) = 1$ and $d(E^*\otimes K_{\boldsymbol{P}(V)}, H_{1,m}) < 0$. Hence $H^0(E)\neq 0$ by Lemma (2.1). On the other hand, since $d(E, H_{1,m}) = 1$, we have a morphism $f\colon Y\to \boldsymbol{P}(V)$ obtained by successive dilatations and an exact sequence $0\to M\to f^*E \to f^*(\mathrm{Inv}\, E))\otimes M^{-1}\to 0$, where M is a positive exceptional line bundle on Y. Now since $0 = c_2(E) = -(M^2)$, we get $M = \mathcal{O}_Y$.

Putting all these results together we have

THEOREM (3.11). *Let V be* $\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}(a)$ *on the projective line* $\boldsymbol{P}^1$ *with* $a\geq 0$, *and let* $p\colon \boldsymbol{P}(V)\to\boldsymbol{P}^1$ *be the canonical projection. (Then for positive* m, $H_{1,m} = \mathcal{O}_{\boldsymbol{P}(V)}(1)\otimes p^*(\mathcal{O}_{\boldsymbol{P}^1}(m))$ *is ample.) Let S be the set of all* $H_{1,m}$*-stable vector bundles E on $\boldsymbol{P}(V)$ of rank two with the first Chern class* $c_1(E) = \mathcal{O}_{\boldsymbol{P}(V)}(-1)\otimes p^*(\mathcal{O}_{\boldsymbol{P}^1}(a+m+1))$ *and the second Chern class* $c_2(E) = 0$. *Then there is a bijective map φ from S to $\boldsymbol{P}^s$ and a vector bundle $\mathcal{V}$ on* $\boldsymbol{P}^s\times_k\boldsymbol{P}(V)$ *such that for any* $E\in S$, E = *the restriction of $\mathcal{V}$ to* $\varphi(E)\times\boldsymbol{P}(V)$, *and* $\dim_k H^1(\mathrm{End}(E)) = s$. *Here* $s = a+2m-1$ *and* $\boldsymbol{P}^s$ *is the s-dimensional projective space.*

4. Simple vector bundles of rank two on the projective plane $\boldsymbol{P}^2$

Let E be a vector bundle on $\boldsymbol{P}^2$ of rank two. If E is simple, by the Riemann-Roch theorem, $N(E) = c_1^2(E) - 4c_2(E) = \dim_k H^0(\mathrm{End}(E)) - \dim_k H^1(\mathrm{End}(E)) + \dim_k H^0(\mathrm{End}(E) \otimes K_{\boldsymbol{P}^2}) - 4\chi(\mathcal{O}_{\boldsymbol{P}^2}) \leqq -2$, since $\mathrm{End}(E)$ is self-dual and the canonical bundle $K_{\boldsymbol{P}^2}$ of $\boldsymbol{P}^2$ is a sheaf of ideals. ([10] Th. 10) On the other hand, $N(E) \equiv 0$ or $1 \pmod 4$ according as c_1 is even or odd. We know that for any negative $n \equiv 0$ or $1 \pmod 4$ except for $n = -4$, there is a simple vector bundle E of rank two on $\boldsymbol{P}^2$ with $N(E) = n$. (See [11]. The result in p. 637 is false for $n = -4$ as we see below.)

PROPOSITION (4.1) (Schwarzenberger [11]). *Let E be a vector bundle on $\boldsymbol{P}^2$ of rank two with the first Chern class $c_1(E) = \mathcal{O}_{\boldsymbol{P}^2}(n)$. Put $m = \min\{k \mid H^0(E \otimes \mathcal{O}_{\boldsymbol{P}^2}(k)) \neq 0\}$. Then the following conditions are equivalent*; (i) *E is simple* (ii) *E is $\mathcal{O}_{\boldsymbol{P}^2}(1)$-stable* (iii) $2m + n > 0$.

Proof. It is obvious that (ii) is equivalent to (iii) by definition. Since there is no line bundle L on $\boldsymbol{P}^2$ with $H^0(L) = H^0(L^{-1}) = 0$, (i) is equivalent to (ii) by Prop. (2.9).

COROLLARY (4.2). *The set of all simple vector bundles on $\boldsymbol{P}^2$ of rank two with the fixed Chern classes is bounded.*

Proof. It is obvious by Th. 2.4 and Prop. 4.1.

Let E_0 be the kernel of the canonical surjection $\mathcal{O}_{\boldsymbol{P}^2}^{\oplus 3} \to \mathcal{O}_{\boldsymbol{P}}(1)$. i.e. $E_0 = \Omega_{\boldsymbol{P}^2}^1(1)$. Then E_0 is simple of rank two and with $N(E_0) = -3$. Indeed, since $c_1(E_0) = -1$ and $c_2(E_0) = 1, E_0$ is not an extension of line bundles. We now show E_0^* is $\mathcal{O}_{\boldsymbol{P}^2}(1)$-stable. Suppose we are given a morphism $f: X \to \boldsymbol{P}^2$ obtained by successive dilatations and a surjection $E_0^* \to f^*\mathcal{O}_{\boldsymbol{P}^2}(k) \otimes M^{-1}$, where M is a positive exceptional line bundle. By the definition of E_0, we have $\mathcal{O}_{\boldsymbol{P}^2}^{\oplus 3} \to E_0^* \to 0$. Hence there is a non-zero homomorphism $\mathcal{O}_{\boldsymbol{P}^2} \to f^*\mathcal{O}_{\boldsymbol{P}^2}(k) \otimes M^{-1}$, and so $k \geq 0$. If $k = 0$, then $M = \mathcal{O}_X$. This contradicts the fact that E_0 is not an extension of line bundles on $\boldsymbol{P}^2$. Therefore $k \geq 1$. On the other hand, $c_1(E_0^*) = 1$. Thus E_0^* is $\mathcal{O}_{\boldsymbol{P}^2}(1)$-stable.

PROPOSITION (4.3). 1) *There is no simple vector bundle E of rank two on $\boldsymbol{P}^2$ with $N(E) = -4$.* 2) *Let E be a simple vector bundle E of rank two on $\boldsymbol{P}^2$ with $N(E) = -3$. Then $E = \Omega_{\boldsymbol{P}^2}^1(n)$ for some n.*

Proof. 1) Let E be a vector bundle of rank two on $\boldsymbol{P}^2$ with $N(E) = -4$. We may assume $c_1(E) = 0$, and so $c_2(E) = 1$. Then since $\chi(E) = 1$ and $c_1(E^* \otimes K_{\boldsymbol{P}^2}) < 0$, E is not $\mathcal{O}_{\boldsymbol{P}^2}(1)$-stable by Lemma (2.1) and hence not simple. 2) Put $V = \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$. The surface $X = \boldsymbol{P}(V)$ has a unique exceptional curve D of the first kind. The contraction of D is $\boldsymbol{P}^2$. Now we consider the problem on X. Let E be a simple vector bundle on X of rank two with $N(E) = -3$. Put $c_1(E) = kd + hf$. By $N(E) = -3$, k is odd and h is even. So we may assume $k = -1$ and $h = 2$, and then $c_2(E) = 0$. Therefore since $\chi(E) = 1, d(E^* \otimes K_X, H_{1,1}) < 0$ and $d(E, H_{1,1}) = 0$, so E is not $H_{1,1}$-stable. Hence we have a morphism $f: Y \to X$ obtained by successive dilatations and an extension of line bundle on $Y: 0 \to f^*L_1 \otimes M \to f^*E \to f^*L_2 \otimes M^{-1} \to 0$, where L_1 and L_2 are line bundles on X and M is a positive exceptional line bundle on Y with $d(L_1, H_{1,1}) \geq 0$. Let the class of L_1 be $nd + mf$. Since E is simple, $H^0(L_1 \otimes L_2^{-1}) = 0$ and $H^0(L_1^{-1} \otimes L_2) = 0$ by the same argument as in Prop. (2.9). And $0 = c_2(E) = -4(M^2) + (L_2 \otimes L_1^{-1}, L_2 \otimes L_1^{-1})$. These relations are equivalent to the following: ① $2n + m \geq 0$. ② either $n \geq 0$ and $n + m \leqq 0$ or $n \geq -1$ and $n + m \leqq 2$. ③ $-(M^2) = n^2 + 2nm + m - n \geq 0$. Only $n = 0$ and $m = 0$ satisfies these relations, and so $M = \mathcal{O}_Y$. Hence the above extension is of the form: $0 \to \mathcal{O}_X \to E \to \mathcal{O}_X(-d + 2f) \to 0$. Since $\dim_k H^1(\mathcal{O}_X(d - 2f)) = 1$, the above non-trivial extension is unique. (It is obvious that the extension bundle is simple by Oda's lemma.)

We now give an example of a family of simple vector bundles of rank two on $\boldsymbol{P}^2$. Let x_1, x_2, x_3 be closed points of $\boldsymbol{P}^2$ in general position, and let f be the blowing up: $X \to \boldsymbol{P}^2$ whose center consists of x_1, x_2 and x_3. Put $L = f^*(\mathcal{O}_{\boldsymbol{P}^2}(-1)) \otimes \mathcal{O}_X(C_1 + C_2 + C_3)$, where $C_i = f^{-1}(x_i)$. It is easy to see that $\dim_k H^1(L^{\otimes 2}) = 3$, $H^2(L^{\otimes 2} \otimes \mathcal{O}_X(-C_i)) = 0$, $H^0(L) = 0$, $H^0(L^{-1}) = 0$ and $H^0(L^{\otimes -2}) = 0$. We have an exact sequence $0 \to \mathcal{O}_X(-C_i) \to \mathcal{O}_X \to \mathcal{O}_{C_i} \to 0$, which induces $k^{\oplus 3} = H^1(L^{\otimes 2}) \to H^1(C_i, \mathcal{O}_{C_i}(-2)) = k \to H^2(L^{\otimes 2} \otimes \mathcal{O}_X(-C_i)) = 0$. Consider an extension $0 \to L \to E' \to L^{-1} \to 0$. By Schwarzenberger [10], E' is of the form f^*E for some vector bundle E on $\boldsymbol{P}^2$ if and only if $E' \otimes \mathcal{O}_{C_i} = \mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}$, $i = 1, 2, 3$. Hence there is a non-empty Zariski open subset U of $\boldsymbol{P}^2$ and a vector bundle $\mathscr{V}$ of rank two on $U \times \boldsymbol{P}^2$ such that for any $u \in U$, the restriction of $\mathscr{V}$ to $u \times \boldsymbol{P}^2$ is a simple vector bundle of rank two on $\boldsymbol{P}^2$ with the first Chern class $= \mathcal{O}_{\boldsymbol{P}^2}$ and the second Chern class $= 2$, and isomorphic vector bundles appear only once. Indeed, let E' be f^*E for some vector bundle E on $\boldsymbol{P}^2$. It is easy to see that

$H^0(E \otimes \mathcal{O}_{P^2}(1)) \neq 0$. On the other hand, $H^0(E) = 0$ by the above fact. Hence E is simple by Cor. (2.10), iii). From $H^0(L^{\otimes -2}) = 0$, we can see that isomorphic vector bundles appear only once.

Remark. Conversely, let E be a simple vector bundle of rank two on P^2 with the first Chern class $= \mathcal{O}_{P^2}$ and the second Chern class $= 2$. Then there is a morphism $f: X \to P^2$ obtained by successive dilatations and a positive exceptional line bundle M on X such that $0 \to f^*(\mathcal{O}_{P^2}(-1)) \otimes M \to f^*E \to f^*(\mathcal{O}_{P^2}(1)) \otimes M^{-1} \to 0$, where $-(M^2) = 3$. Indeed, by Lemma (2.1), $H^0(E \otimes \mathcal{O}_{P^2}(1)) \neq 0$ since $\chi(E \otimes \mathcal{O}_{P^2}(1)) > 0$ and $d((E \otimes \mathcal{O}_{P^2}(1))^* \otimes K, \mathcal{O}_{P^2}(1)) < 0$. On the other hand, $H^0(E) = 0$. Hence we have the desired result.

When X is $P^1 \times P^1$, we have the almost same results as when X is the projective plane P^2. For example, 1) there is no simple vector bundle E of rank two on X with $N(E) = -2$. 2) Let E be a vector bundle of rank two on X with $N(E) = -4$. E is simple if and only if E is $H_{1,1}$-stable, or $H_{2,1}$-stable, or $H_{1,2}$-stable. Hence a set of such simple bundles is bounded etc.

On the other hand, it was shown by Schwarzenberger [11] that for any even negative integer $n \neq -2$, there is a simple vector bundle E on X of rank two with $N(E) = n$. (His statement is false for $n = -2$. We can prove there is no simple vector bundle E of rank two on X with $N(E) = -2$ as Prop. (4.3) (i).) Note that if E is a simple vector bundle of rank two on X, then $N(E)$ is an even negative integer.

5. Stable vector bundles of rank two on abelian surfaces

In this section, X will be an abelian surface over k. When E is a simple bundle of rank two on X, by the Riemann-Roch theorem, $N(E) = c_1^2(E) - 4c_2(E) = 2 \dim_k H^0(\mathrm{End}(E)) - \dim_k H^1(\mathrm{End}(E)) = 2 - \dim_k H^1(\mathrm{End}(E)) \leqq 2$, since $\mathrm{End}(E)$ is self-dual and the canonical bundle of X is trivial. When char. $k \neq 2$, $\dim_k H^1(\mathrm{End}(E)) \geqslant \dim_k H^1(\mathcal{O}_X) = 2$, since $\mathcal{O}_X \to \mathrm{End}(E)$ splits. Hence $N(E) \leqq 0$ when char. $k \neq 2$ and E is simple.

PROPOSITION (5.1). *Let X be an abelian surface and E a vector bundle of rank two with $N(E) = 0$ on X. Then E is simple if and only if E is H-stable for an ample line bundle H on X.*

Proof. We use freely results about the cohomology of a line bundle on an abelian variety. (See [7] and [8]). Assume E is of trivial type. As above there is a non-trivial extension $0 \to M \to f^*E \to f^*L_2 \otimes M^{-1} \to 0$ with $H^0(L_2) = H^0(L_2^{-1}) = 0$. Therefore we have the following three possibilities:

(Case 1) L_2 is non-degenerate of index 1, i.e. $(L_2^2) < 0$.

(Case 2) L_2 is not isomorphic to $\mathcal{O}_X$, but algebraically equivalent to $\mathcal{O}_X$.

(Case 3) L_2 is degenerate, but not algebraically equivalent to $\mathcal{O}_X$, with $L_2 \otimes \mathcal{O}_K \not\simeq \mathcal{O}_K$ where K is the component of the zero of the kernel of $\wedge(L_2)$. In cases 2 and 3 we have $M = \mathcal{O}_X$, since by assumption $(L_2^2) = 0$ and $0 = N(E) = 4(M^2) + (L_2^2)$. The extension is thus of the form, $0 \to \mathcal{O}_X \to E_1 \to L_2 \to 0$. But since $H^1(L_2^{-1}) = 0$, $E_1 = \mathcal{O}_X \oplus L_2$, contradicting the assumption that E_1 is simple. In case 1, $N(E) = 4(M^2) + (L_2^2) < 4(M^2) \leqq 0$. This contradicts $N(E) = 0$.

PROPOSITION (5.2). *Let X be an abelian surface and let E be a vector bundle of rank two on X with $N(E) = 0$. Then E is H-semi-stable if and only if E is either simple or is of the form $E' \otimes L$, where we have an extension $0 \to \mathcal{O}_X \to E' \to \mathcal{O}_X \to 0$ and L is a line bundle.*

Proof. The condition is clearly sufficient. To show that it is necessary, let E be H-semi-stable and not simple. By Prop. (5.1), E is not H-stable. Hence we have a morphism $f: Y \to X$ obtained by successive dilatations, line bundles L_1 and L_2 on X and a positive exceptional line bundle M on Y such that there is an exact sequence $0 \to f^*L_1 \otimes M \to f^*E \to f^*L_2 \otimes M^{-1} \to 0$ with $d(L_1, H) = d(L_2, H)$. If $H^0(L_2 \otimes L_1^{-1}) = 0$, then $H^0(\mathrm{End}(E)) = k \oplus H^0(L_1 \otimes L_2^{-1})$ by Oda's lemma and hence $L_1 \simeq L_2$. This is a contradiction. Therefore $H^0(L_1 \otimes L_2^{-1}) \neq 0$, and so $L_1 = L_2$. Since $N(E) = 4(M^2) = 0$, $M = \mathcal{O}_X$.

Remark. Let X be an abelian surface over the field of complex numbers and E a vector bundle of rank two with $N(E) = 0$ on X. Then Oda [9] has proved that E is simple if and only if E is obtained as the direct image of a line bundle under an isogeny of a special type. And also he has shown that there is a vector bundle E of rank two on an abelian surface with $N(E) = 0$, which is not H-semi-stable but indecomposable. On the other hand, it is well known [1] that any indecomposable vector bundle on an elliptic curve is semi-stable and the fact corresponding to Prop. (2.12) holds.

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