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OSCILLATION FUNCTION OF A MULTIPARAMETER GAUSSIAN PROCESS

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1. Introduction. It is our object in this paper to show that the recent results of K. Ito and M. Nisio [4] on the oscillation function of Gaussian processes on $[0,1]$ are valid for Gaussian processes with a general multi-parameter "time" set T. Except in extending Theorem 4 of [4] where we assume T to be the d-dimensional cube, in all other cases we allow T to be a separable metric space. Despite the generality of the time set, the proofs are achieved essentially using the method of the above mentioned authors. However, in Theorem 1 below we find the use of Lemma 6 of [5] more convenient than the approach via orthogonal expansions and Kolmogorov's zero-one law as is done in [4].

The extension of Ito and Nisio's results has been undertaken with a view to considerably enlarging their scope of application. One application is Theorem 5 which states that for certain multi-parameter (S,Γ) stationary Gaussian processes (see Section 5 for the definition) the oscillation function is either identically 0 or identically ∞ and consequently that Yu. K. Belyaev's 0–1 law [1] holds for these processes. As a corollary we show that the same conclusion holds for Gaussian stationary processes (often called homogeneous random fields) given on homogeneous spaces. The corollary to Theorem 5 has also been obtained by Eaves [2] using a method which seems to be based on Belyaev's original proof. Theorems 1–3 of [4] have been extended to the case where $T=[0,1]^d$ (though not to more general parameter sets) by Kawada [6].

Let T be a separable metric space and let Ω be the space of all real-valued functions T. Let $\mathscr{B}$ denote σ-algebra generated by the cylinder sets. Let P be a Gaussian probability measure on $(\Omega,\mathscr{B})$ such that the random

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variables $X(t, \omega) = \omega(t), \omega \in \Omega$, are jointly Gaussian, each with mean 0. We also assume $X(t)$ to be mean-continuous, i.e. for every $t \in T$

$$\lim_{s \to t} E\,|X(s) - X(t)|^2 = 0 \,. \tag{1.1}$$

We shall take $X(t)$ to be a separable process such that for each $\omega \in \Omega$ $X(t, \omega + m) = X(t, \omega) + m(t)$ for a continuous function $m \in \Omega$. To see that this can be done, we refer to [3, p. 151]. In the notation of [3] the way a separable version is constructed, either $X(t, \omega)$ is left alone or it must be modified to take an arbitrary value in a certain set $A(t, \omega)$; whenever such modification is necessary we simply pick the sup $A(t, \omega)$. The separable version chosen in this way satisfies the above property. Such a choice of a separable version is no restriction because the sample paths of any two separable versions of a stochastically continuous process almost surely have the same oscillation functions (Definition 1.1). This can be seen as follows: let Y_t and Z_t be separable versions of a stochastically continuous process $\{X_t, t \in T\}$. Since every countable dense subset D of T is a separating set for both Y_t and Z_t, there exists a null set N_1 such that if $\omega \notin N_1$, then for any open subset I of T,

$$\sup_{t \in I} Y_t(\omega) = \sup_{t \in I \cap D} Y_t(\omega)$$
$$\inf_{t \in I} Y_t(\omega) = \inf_{t \in I \cap D} Y_t(\omega) \,,$$

where the same holds for the Z_t version as well. There exists another null set N_2 such that if $\omega \notin N_2$ then $Y_t(\omega) = Z_t(\omega) = X_t(\omega)$ for all $t \in D$. Hence if $\omega \notin N_1 \cup N_2$ then it is clear from Remark 1 (after Definition 1.1) that $W_Y(t, \omega) = W_Z(t, \omega)$ for all t a.s.

Let $H(R)$ denote the reproducing kernel Hilbert space associated with this process, where $R = R(s, t)$ denotes the covariance (necessarily continuous in view of (1.1)), i.e.

$$R(s, t) = E(X(s)X(t)) \,. \tag{1.2}$$

It is well-known that under (1.1) $H(R)$ consists of continuous functions on T and the separability of T implies that $H(R)$ is a separable Hilbert space. In the following "a Gaussian process" will always mean "a Gaussian process with mean 0 and continuous covariance."

Before stating the main results we give the definition of the oscillation function.

DEFINITION 1.1. Let

$$W_X(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{u,v \in S(t,\varepsilon)} |X(u,\omega) - X(v,\omega)| , \tag{1.3}$$

where, denoting by ρ the metric on T,

$$S(t,\varepsilon) = \{s \in T : \rho(s,t) < \varepsilon\} . \tag{1.4}$$

We apply the conventions $(+\infty) - (+\infty) = (-\infty) - (-\infty) = 0$. The separability of the process implies that the supremum in (1.3) need be taken only for u, v in a countable separating set, hence $W_X(t,\omega) = W(t,\omega)$ is measurable in ω for each $t \in T$. We shall suppress the subscript on W whenever only one process is involved and no confusion is possible.

Remark 1. Define

$$M_X(t,\omega) = M(t,\omega) = \lim_{\varepsilon \downarrow 0} \sup_{s \in S(t,\varepsilon)} X(s,\omega) ,$$
$$m_X(t,\omega) = m(t,\omega) = \lim_{\varepsilon \downarrow 0} \inf_{s \in S(t,\varepsilon)} X(s,\omega) .$$

Then it is easy to check that $W(t,\omega) = M(t,\omega) - m(t,\omega)$. This fact will be needed later.

We now state the main results. The underlying probability space will always be $(\Omega, \mathscr{B}_0, P)$ as explained above; $\mathscr{B}_0$ here denotes the P-completion of $\mathscr{B}$.

THEOREM 1. *Let T be a separable metric space and $\{X(t), t \in T\}$ a separable Gaussian process. Then there exists a function $\alpha(t) = \alpha_X(t)$, $t \in T$, which does not depend on ω, such that*

$$P\{\omega : W(t,\omega) = \alpha(t) \quad \text{for every } t \in T\} = 1 . \tag{1.5}$$

The function α has the following properties:

$$\alpha(t) \text{ is upper semicontinuous ,} \tag{1.6}$$

$$\{t : a \leq \alpha(t) < \infty\} \text{ is nowhere dense for every } a > 0 . \tag{1.7}$$

For T a separable metric space and f an extended real-valued function on T, define

$$\limsup_{s \to t} f(s) = \lim_{\varepsilon \downarrow 0} \sup_{s \in [u : 0 < \rho(u,t) < \varepsilon]} f(s) .$$

$\liminf_{s \to t} f(s)$ is defined similarly with sup replaced by inf. f is upper semi-continuous if for every real a the set $[t : f(t) \geq a]$ is closed, which is equivalent to the condition that $\limsup_{s\to t} f(s) \le f(t)$ for each $t \in T$.

Remark 2. Our definitions of $\limsup_{s\to t} \alpha(s)$ and $\liminf_{s\to t} \alpha(s)$ are different from those in [4]. It is asserted in [4], p. 213, that $W(t,\omega) = \limsup_{s\to t} x(s) - \liminf_{s\to t} x(s)$, which would be true in general only if one defines $\limsup_{s\to t} x(s) = \lim_{\epsilon\downarrow 0} \sup_{s\in[u:\rho(u,t)<\epsilon]} x(s)$ with a similar modification for lim inf. With our modified definitions Theorem 2 is slightly more interesting, because it then tells us that if $\alpha(t) > 0$, then the discontinuity at t is a.s. "non-removable". If the time set is linear then one can define 'one-sided' oscillation functions and it is not hard to conclude that if $\alpha(t) > 0$ for some t, then the discontinuity at t is a.s. oscillatory.

THEOREM 2. *Under the conditions of Theorem* 1, *for each* $t \in T$

$$P\left\{\limsup_{s\to t} X(s) = X(t) + \frac{1}{2}\alpha(t), \liminf_{s\to t} X(s) = X(t) - \frac{1}{2}\alpha(t)\right\} = 1\,. \tag{1.8}$$

THEOREM 3. *Under the conditions of Theorem* 1, *if, for some constant* $a, \alpha(t) \ge a > 0$ *on a dense subset* D *of an open set* $I \subset T$, *then*

$$P\{\limsup_{s\to t} X(s) = +\infty, \liminf_{s\to t} X(s) = -\infty \quad \text{for every } t \in I\} = 1\,. \tag{1.9}$$

For the next result we specialize the time set to be the unit cube in R^d.

THEOREM 4. *Let* $T = [0,1]^d$, *the d-dimensional unit cube. Then, given any function* $\alpha\colon T \to [0,\infty]$ *satisfying* (1.6) *and* (1.7), *there exists a separable Gaussian process* $\{X(t), t \in T\}$ *whose oscillation function is* α.

We give the proof of Theorem 1 in section 2. The proofs of Theorems 2 and 3 are essentially the same as given in [4] and we omit them. Theorem 4 is proved in section 3.

The arguments given in [4] apply without change in certain places in our more general context; whenever that happens, we simply refer the reader to [4].

2. Proof of Theorem 1. Let F be a non-empty closed subset of T, where T is a separable metric space with metric ρ. For a positive integer n, let

$$F_n = \left\{ s \in T : \rho(s, F) < \frac{1}{n} \right\} . \tag{2.1}$$

Following Ito and Nisio [4] we define

$$W_X(F, \omega) = \lim_{n \uparrow \infty} \lim_{k \uparrow \infty} \sup_{\substack{u,v \in F_n \\ \rho(u,v) < 1/k}} |X(u, \omega) - X(v, \omega)| . \tag{2.2}$$

Again the $\mathscr{B}_0$-measurability of $W(F, \omega)$ follows from the assumed separability of the process.

The proof is based on the following lemmas. Lemmas 2.1 and 2.4 below are simple consequences of the definition of oscillation function. Their proofs are omitted.

LEMMA 2.1. *If $F = \{t\}$, then $W(F, \omega) = W(t, \omega)$ for each $\omega \in \Omega$, where $W(t, \omega)$ is given by* (1.3) *and $W(F, \omega)$ by* (2.2).

The following lemma is given in [5] as Lemma 6. Although this lemma is stated there only for a real-valued function g, the same proof works for extended real-valued functions as well.

LEMMA 2.2. *Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal system in $H(R)$ and g a $\mathscr{B}_0$-measurable extended real-valued function such that for each $\omega \in \Omega$ and every rational r*

$$g(\omega + re_j) = g(\omega) , \qquad j = 1, 2, \cdots ,$$

then

$$g(\omega) = \text{constant a.s. } (P) .$$

LEMMA 2.3. *For each closed subset F of T, there exists a constant $\alpha(F)$ such that*

$$P\{W(F, \omega) = \alpha(F)\} = 1 .$$

Proof. As already remarked, $W(F, \omega)$ is $\mathscr{B}_0$-measurable. Recall that our separable process $X(t, \omega)$ has the property that $X(t, \omega + m) = X(t, \omega) + m(t)$ for a continuous function $m \in \Omega$. Hence if $m \in \Omega$ is a continuous function, then $W(F, \omega + m) = W(F, \omega)$ for each $\omega \in \Omega$. Since $H(R)$ consists of continuous functions, it follows that $g = W(F, \cdot)$ satisfies the hypotheses of Lemma 2.2, which implies the conclusion of this lemma.

Let S be a countable subset of T which is dense in T, and let $\mathscr{U}$ consist of sets $S(s, 1/n)$, $s \in S$, $n = 1, 2, \cdots$, where $S(s, 1/n) = \{t \in T : \rho(s, t) < 1/n\}$.

Then $\mathscr{U}$ is a countable basis for the metric topology of T. If $t \in T$, there exists a sequence $\{V_j\}$ of sets in $\mathscr{U}$ such that $V_j \supset \overline{V}_{j+1}$ and $\bigcap_{j=1}^{\infty} \overline{V}_j = \{t\}$, where $\overline{V}_j$ denotes the closure of V_j. First we observe that it follows from Lemma 2.3 that

$$P\{W(\overline{V}, \omega) = \alpha(\overline{V}) \quad \text{for } V \in \mathscr{U}\} = 1 . \tag{2.3}$$

The following lemma and (2.3) imply (1.5).

LEMMA 2.4. *For each $t \in T$, there exist $V_j \in \mathscr{U}$, $j = 1, 2, \cdots$, such that $t \in \overline{V}_j$ for each j, and $\lim_{j\to\infty} W(\overline{V}_j, \omega) = W(t, \omega)$ for all $\omega \in \Omega$.*

(1.6) *and* (1.7) *are proved the same way as in* [4].

3. Proof of Theorem 4. $T = [0,1]^d = [0,1] \times [0,1] \times \cdots \times [0,1]$. $t = (t^1, t^2, \cdots, t^d)$ will denote an element of T. I with or without subscript will denote a linear subinterval of $[0,1]$ of positive length; J with or without subscript will denote a subinterval of T, by which we mean a set of the form $\{t \in T : t^1 \in I_1, \cdots, t^d \in I_d\}$. J^0 will denote the interior of J, $\bar{J}$ the closure of J. On some probability space $(\Omega, \mathscr{B}, P)$ we can define two independent Gaussian processes $B(t), S(t), t \in T$, where

$$S(t) = Y(t^1) , \qquad t = (t^1, \cdots, t^d) ; \tag{3.1}$$

$Y(t^1)$, $0 \le t^1 \le 1$, being a stationary Gaussian process with $EY(t^1) = 0$, $EY^2(t^1) = 1$ and $\alpha(t, Y) = +\infty$, the existence of which was proved by Belyaev [1]. Here $\alpha(t, Y)$ denotes the oscillation function (independent of ω) of the Y-process. We shall use this notation throughout this section. $S(t)$ thus defined is Gaussian with $ES(t) = 0$, $ES^2(t) = 1$ for $t \in T$; it is mean-continuous and $\alpha(t, S) = +\infty$. For $B(t)$ we take the Wiener process with d-dimensional time parameter, defined in [8]. It is a Gaussian process with continuous paths. $B(0) = 0, EB(t) = 0$. $E(B(s)B(t)) = \prod_{i=1}^{d} [\min(s^i, t^i)]$. It is shown in [8] that the process satisfies the law of the iterated logarithm given in (3.2) below. For $t, u \in T$, by $t > u$ we mean $t^1 > u^1, \cdots, t^d > u^d$. Then $B(t)$ satisfies

$$\limsup_{\substack{t \downarrow u \\ t > u}} \frac{B(t-u)}{\left[2^d \prod_{i=1}^{d} (t^i - u^i) \log\log\left(\prod_{i=1}^{d} (t^i - u^i)\right)^{-1}\right]^{1/2}} = 1 \tag{3.2}$$

As in [4] we use $\|X\|^2 = E(X^2)$ for a random variable X. Let L be the $\|\ \|$-closure of finite linear combinations of $B(t), t \in T$, and $S(t), t \in T$. Given

$\alpha(t), t \in T$, satisfying (1.6) and (1.7) we shall construct a Gaussian process $X(t) \in L$ such that $\alpha(t, X) = \alpha(t)$. The lemmas that follow are necessary modifications of the lemmas in section 6 in [4]. Recall that by a Gaussian process we mean one with mean 0 and continuous covariance.

LEMMA 3.1. *Let J be a subinterval of T and $\varepsilon > 0$. Then there exists a Gaussian process $X(t) \in L, t \in T$, such that*

(a) $X(t, \omega) = 0$ *for* $t \in T - J^0$,
(b) $\alpha(t, X) = \infty$ *for* $t \in J$,
(c) $\|X(t)\| \leq \varepsilon$ *for* $t \in J$.

Proof. Let $J = \{t : t^1 \in I_1, \cdots, t^d \in I_d\}$. Take $f(t) = f_1(t^1) \cdots f_d(t^d)$, where $f_i(t^i)$ is continuous on $[0, 1]$, $0 < f_i(t^i) < \varepsilon$ on I_i^0, and $f_i(t^i) = 0$ elsewhere on $[0, 1]$. Then $0 < f(t) < \varepsilon$ on J^0 and $f(t) = 0$ on $T - J^0$. f is continuous on T. Define

$$X(t, \omega) = f(t) S(t, \omega) . \tag{3.3}$$

(a), (b), (c) are now evident.

LEMMA 3.2. *Given $0 < a < \infty$, $\varepsilon > 0$, and J a subinterval of T, there exists a Gaussian process $X(t) \in L, t \in T$, such that*

(a) $X(t, \omega)$ *has continuous paths,*
(b) $X(t, \omega) = 0$ *for* $t \in T - J^0$,
(c) $\|X(t)\| < \varepsilon$ *for* $t \in T$,
(d) $P[|\sup_J X(t) - a| > \varepsilon] < \varepsilon$.

We denote such a process by $X(t; I, a, \varepsilon)$ as in [4].

Proof. In order to construct the process $X(t, \omega)$ of the lemma we proceed as follows. Define $Z(t, \omega)$ as follows:

$$Z(t, \omega) = 0 \quad \text{for } t \in T - J^0,$$
$$= \frac{aB(t - u, \omega)}{\left\{2^d \prod_{i=1}^d (t^i - u^i) \log \log \left(\prod_{i=1}^d (t^i - u^i)\right)^{-1}\right\}^{1/2}}, \quad t \in J^0,$$

where $u = (u^1, \cdots, u^d)$, $u^i =$ lower end point of I_i $(J = I_1 \times I_2 \times \cdots \times I_d)$. Such u will be called the lowest vertex of J. For $t \in J^0$

$$E(Z^2(t)) = \frac{a^2}{2^d \log \log \left(\prod_{i=1}^d (t^i - u^i)\right)^{-1}} \longrightarrow 0 \tag{3.4}$$

as $t \downarrow u$. Also by (3.2),

$$P\,[\limsup_{t \downarrow u} Z(t) = a] = 1\ . \tag{3.5}$$

By (3.4) and (3.5) we can find an interval $J_1 \subset J$ with lowest vertex u such that for $t \in J_1$,

$$E(Z^2(t)) < \varepsilon^2\ , \tag{3.6}$$

$$P\,[\sup_{t \in J_1^0} Z(t) < a + \varepsilon] > 1 - \frac{\varepsilon}{2}\ , \tag{3.7}$$

$$P\,[\sup_{t \in J_1^0} Z(t) > a - \varepsilon] > 1 - \frac{\varepsilon}{2}\ . \tag{3.8}$$

Since $Z(t)$ is continuous on J_1^0, by (3.8) there exists an interval J_2 such that $\bar{J}_2 \subset J_1^0$ and

$$P\,[\sup_{t \in J_2} Z(t) > a - \varepsilon] > 1 - \frac{\varepsilon}{2}\ . \tag{3.9}$$

Let f be a continuous function on T which equals 1 on $\bar{J}_2$, vanishes on the complement of J_1^0 and lies between 0 and 1 on $J_1^0 - \bar{J}_2$. Let $X(t) = f(t)Z(t)$. $X(t)$ satisfies all the requirements of the lemma.

LEMMA 3.3. *Suppose α_1, α_2 satisfy* (1.6) *and the following condition*:

$$\{t : \alpha_i(t) > 0\}\ , \qquad i = 1, 2, \text{ is nowhere dense}\ . \tag{3.10}$$

Then for any Gaussian process $X_1(t)$ with $\alpha(t_1, X_1) = \alpha_1(t)$ and $\varepsilon > 0$, there exists a Gaussian process $X_2(t) \in L, t \in T$, satisfying:

(a) $\alpha(t, X_1 + X_2) = \alpha_1(t) + \alpha_2(t)$,
(b) $\|X_2(t)\| < \varepsilon$,
(c) $P\,[\sup_{t \in T} |X_2(t)| > \sup_{t \in T} \alpha_2(t)] < \varepsilon$.

Proof. The details of the proof are identical with those in [4, Lemma 6.3]. Our notation differs only in the use of X in place of x. The linear intervals in that proof should be interpreted as intervals in T. The counter-parts of all the basic facts that are needed there (especially Theorem 2 and Lemma 3.2) have already been established.

To finish the proof of Theorem 4 we again proceed as in [4], p. 221. First we assume

$$\{t : \alpha(t) \geq c\} \text{ is nowhere dense for } c > 0\ ; \tag{3.11}$$

which is stronger than (1.7) but weaker than (3.10). A Gaussian process $X(t) \in L$ is then constructed satisfying (1.6) and (3.11) by the same argument as given on pages 221–222 [4]. The argument for removing the assumption (3.11) needs a little modification and we now explain this part. Let $T_\infty = \{t \in T : \alpha(t) = \infty\}$, and $T_c = \{t \in T : \alpha(t) \geq c\}$ for $c > 0$. T_c is a closed set and it is the disjoint union of T_∞ and the nowhere dense set $\{t \in T : c \leq \alpha(t) < \infty\}$. Hence, if T_c is not a nowhere dense set for some $c > 0$, then T_∞ must have non-empty interior and $T_\infty^0 = T_c^0$ for each $c > 0$.

Define

$$\begin{aligned} \beta(t) &= 0 \qquad \text{for } t \in T_\infty \\ &= \alpha(t) \qquad \text{elsewhere .} \end{aligned}$$

Then $\beta(t)$ satisfies (1.6) and for every $c > 0$, $[t \in T : \beta(t) \geq c]$ is nowhere dense. Hence there exists a Gaussian process $Y(t) \in L, t \in T$, such that $\alpha(t, Y) = \beta(t), t \in T$. There exists a sequence of disjoint closed intervals J_n, $n \geq 1$, such that $\overline{\bigcup_{n=1}^\infty J_n} = T_\infty$. Let $Y_n(t), n \geq 1$ be Gaussian processes such that $\alpha(t, Y_n) = \infty$ on J_n, $Y_n(t) = 0$ on $T - J_n^0$ and $\|Y_n(t)\| \leq 2^{-n}$. Such processes exist by Lemma 2.1. We define

$$X(t) = Y(t) + \sum_{n=1}^{\infty} Y_n(t) \,. \tag{3.12}$$

Then

$$\begin{aligned} X(t) &= Y_n(t) + Y(t) \,, \quad t \in J_n^0 \,, \qquad n = 1, 2, \cdots \\ &= Y(t) \,, \qquad \text{elsewhere .} \end{aligned}$$

Since each $Y(t), Y_n(t)$ is continuous in the mean and $\|Y_n(t)\| \leq 2^{-n}$, the continuity in the mean of $X(t)$ follows. It remains to show that $\alpha(t, X) = \alpha(t)$. $X(t) = Y(t)$ on $T - \overline{\bigcup_{n=1}^\infty J_n}$, which is an open set, hence

$$\alpha(t, X) = \alpha(t, Y) = \beta(t) = \alpha(t) \,, \qquad t \in T - \overline{\bigcup_{n=1}^{\infty} J_n} \,.$$

Also, $X(t) = Y(t) + Y_n(t)$ on $J_n^0, n \geq 1, Y(t)$ is continuous, hence

$$\alpha(t, X) = \alpha(t, Y_n) = \infty \qquad \text{on } t \in J_n^0 \,.$$

If $t \in \overline{\bigcup_{n=1}^\infty J_n} - \bigcup_{n=1}^\infty J_n^0$, then t is an accumulation point of $\bigcup_{n=1}^\infty J_n^0$ and so $\alpha(t, X) \geq \limsup_{\substack{s \to t \\ s \in \bigcup_n J_n^0}} \alpha(s, X) = \infty$, which finishes the proof.

4. Belyaev's alternative

Let S be a separable metric space with metric ρ and Γ a set of transformations taking S into S. We assume the following conditions to be satisfied by S and the elements $\gamma \in \Gamma$:

(Γ–1) Each $\gamma \in \Gamma$ is a one-to-one bicontinuous mapping of S onto S.

(Γ–2) Given $s, t \in S$, there exists a $\gamma \in \Gamma$ such that $t = \gamma s$.

DEFINITION 4.1. Let $T \subseteq S$, and let $\{X_t, t \in T\}$ be a separable Gaussian process with mean zero and continuous covariance. We say that $\{X_t, t \in T\}$ is (S, Γ)-stationary if for any $t_1, \cdots, t_n$ in T and $\gamma \in \Gamma, \gamma t_1, \cdots, \gamma t_n$ are also in T, then the joint distribution of $X_{t_1}, \cdots, X_{t_n}$ is the same as that of $X_{\gamma t_1}, \cdots, X_{\gamma t_n}$.

Theorem 5 below generalizes Belyaev's alternative to (S, Γ)-stationary Gaussian processes. All topological concepts will be relative to the topolgy of S. If $s \in S$, then $S(s, \varepsilon) = \{t \in S : \rho(s, t) < \varepsilon\}$.

THEOREM 5. *Let $T \subseteq S$ and let $\{X_t, t \in T\}$ be an (S, Γ)-stationary Gaussian process. Let S, T satisfy the following conditions:*

(S–1) *S has no isolated points.*

(T–1) *T^0, the set of interior points of T, is dense in T.*

Then the oscillation function $\alpha(t)$ of the process $\{X_t, t \in T\}$ is either identically 0 or identically ∞ on T.

Remark. The assumption (S–1) does not imply any loss of generality and is made only to exclude an uninteresting situation for which the conclusion of the theorem holds trivially. For, if S has an isolated point s (by this we mean that $S(s, \varepsilon) = \{s\}$ for some $\varepsilon > 0$) then it is easy to verify from (Γ–1) and (Γ–2) that every point of S is isolated. When this happen Belyaev's result is true trivially for any subset T of S since then $\alpha(t) \equiv 0$. Hence we need consider only the case where S has no isolated points. It should be observed that (S–1) and (T–1) imply that T has no isolated points. This assumption is crucial, because, otherwise, one could easily give an example to show that the alternative does not hold. Indeed, let $(X_t, t \in [0, \infty))$ be the stationary (under translations) process of Belyaev [1] for which $\alpha(t) \equiv \infty$. If we restrict the process to the time set $[0, 1] \cup \{2\} \cup [3, \infty)$, then it is clear that the process is still stationary in our sense, but α for this process equals ∞ on the set $[0, 1] \cup [3, \infty)$ and $\alpha(2) = 0$.

Proof of Theorem 5. We will first show that α is constant on the set T^0. Let $\{r_i\}$ be a countable dense subset of T which is a separability set for the separable process $\{X_t, t \in T\}$. Let $s, t \in T^0$ be arbitrary but fixed. Let $\gamma \in \Gamma$ such that $t = \gamma s$. By (T–1) and the continuity of γ there exists $\varepsilon > 0$ such that $S(s, \varepsilon) \subset T, S(t, \varepsilon) \subset T$, and $\gamma(S(s, \varepsilon)) \subset T$. We have

$$(4.1) \qquad W(s, \omega) = \lim_{n \uparrow \infty} \sup_{u,v \in S(s,1/n) \cap \{r_i\}} |X(u, \omega) - X(v, \omega)| ,$$

$$(4.2) \qquad W(t, \omega) = \lim_{n \uparrow \infty} \sup_{u,v \in S(\gamma s,1/n) \cap \{r_i\}} |X(u, \omega) - X(v, \omega)| ,$$

where we can restrict the sup over $\{r_i\}$ by separability of the process. Since γ is one-to-one bicontinuous, there exist strictly increasing infinite sequences of positive integers $\{n'\}$ and $\{n''\}$ such that

$$S\left(\gamma s, \frac{1}{n'}\right) \subset \gamma\left(S\left(s, \frac{1}{n}\right)\right) \subset S\left(\gamma s, \frac{1}{n''}\right) \subset T$$

for all n sufficiently large, and $n' \to \infty, n'' \to \infty$ as $n \to \infty$. Hence we have

$$(4.3) \qquad W(t, \omega) = \lim_{n \uparrow \infty} \sup_{u,v \in \gamma(S(s,1/n)) \cap \{r_i\}} |X(u, \omega) - X(v, \omega)| .$$

Since the set $\{r_i\}_{i=1}^{\infty}$ is dense in $S(r, 1/n)$, $\{\gamma r_i\}_{i=1}^{\infty}$ is dense in $\gamma(S(s, 1/n))$ by the continuity of γ. If $r_i \in S(s, 1/n)$, then by the stochastic continuity of the process there exists a sequence $\{\gamma r_{k'}\}$ tending to r_i such that $X(\gamma r_{k'}) \to X(r_i)$ a.s., where the null set depends on r_i. Since there are only a countable number of r_i, there exists a null set N such that if $\omega \notin N$ then

$$(4.4) \qquad W(t, \omega) = \lim_{n \uparrow \infty} \sup_{u,v \in \gamma(S(s,1/n)) \cap \{\gamma r_i\}} |X(u, \omega) - X(v, \omega)| .$$

Using the stationarity of the process we conclude from (4.1) and (4.4) that $W(s, \omega)$ and $W(t, \omega)$ have the same distribution. On the other hand $W(s, \omega) = \alpha(s)$ a.s., $W(t, \omega) = \alpha(t)$ a.s. by Theorem 1. Hence $\alpha(s) = \alpha(t)$. Thus $\alpha(t)$ is constant on the set T^0.

To finish the proof of the theorem first let $\alpha(t) = a > 0$ for $t \in T^0$. By Theorem 3 then $\alpha(t) = \infty$ on T^0. Since T^0 is dense in T, the upper semi-continuity of α implies that $\alpha(t) = \infty$ for $t \in T$. Let us now assume that $\alpha(t) = 0$ for $t \in T^0$. Let $s \in T$, $s \notin T^0$. We have to show that $\alpha(s) = 0$ for such s. We pick a point $t \in T^0$. Then

$$W(s, \omega) = \lim_{n \uparrow \infty} \sup_{u,v \in S(s,1/n) \cap T \cap \{r_i\}} |X(u, \omega) - X(v, \omega)| . \tag{4.5}$$

$$W(t, \omega) = \lim_{n \uparrow \infty} \sup_{u,v \in S(\gamma s,1/n) \cap T \cap \{r_i\}} |X(u, \omega) - X(v, \omega)| , \tag{4.6}$$

where $t = \gamma s$ for some $\gamma \in \Gamma$ by (Γ–2). Since $\gamma s \in T^0$, for n sufficiently large $S(\gamma s, 1/n) \subset T$, hence

$$W(t, \omega) = \lim_{n \uparrow \infty} \sup_{u,v \in S(\gamma s,1/n) \cap \{r_i\}} |X(u, \omega) - X(v, \omega)| .$$

Since $S(\gamma s, 1/n) \subset T$ for all sufficiently large n, the continuity of γ implies $\gamma(S(s, 1/n)) \subset T$, for all sufficiently large n. The argument that we gave before applies for $W(t, \omega)$ and (4.4) is valid. Since α is 0 on T^0 we have by (4.4) and the fact that $t \in T^0$,

$$\begin{aligned} 0 = W(t, \omega) &= \lim_{n \uparrow \infty} \sup_{u,v \in \gamma(S(s,1/n)) \cap \{\gamma r_i\}} |X(u, \omega) - X(v, \omega)| \\ &\geq \lim_{n \uparrow \infty} \sup_{u,v \in \gamma(S(s,1/n) \cap T) \cap \{\gamma r_i\}} |X(u, \omega) - X(v, \omega)| \\ &= W'(t, \omega) , \qquad \text{say} . \end{aligned}$$

Now using stationarity we conclude from (4.5) that $W(s, \omega)$ and $W'(t, \omega)$ have the same distribution. Since $W(t', \omega) = 0$ independently of ω, we have $\alpha(s) = 0$. This establishes the theorem.

An important application of Theorem 5 is the extension of Belyaev's zero one law to Gaussian homogeneous random fields on homogeneous spaces. Let T be a (left) homogeneous space under the action of a group G (see [7], p. 128). A Gaussian process X_t with $t \in T$ is a homogeneous random field on T if it is (T, G) stationary in the sense of Definition 5.1.

COROLLARY TO THEOREM 5. *Let T be a homogeneous space under G and assume further that as a topological space T is separable and metrizable. If $\{X_t\}$ is a separable, zero mean and mean continuous Gaussian homogeneous random field on T then its oscillation function $\alpha(t)$ is either identically 0 or identically ∞ on T. In particular, almost every sample function of the process is either continuous or unbounded over every open neighborhood in T.*

Proof. The result follows immediately as a consequence of Theorem 5 once we verify assumptions (Γ–1) and (Γ–2). This we do taking $\Gamma =$

G and identifying S with T. The very definition of a homogeneous space yields (Γ-1) and (Γ-2): The latter is the property of "transitivity" of the group G acting on T. That each map $g: S$ to S in 1-1 and onto is now obvious. The bicontinuity follows from the fact (again from the definition of a homogeneous space) that for every g in G the mapping $t \to gt$ is a homeomorphism of T ([7], p. 126). We may assume without loss of generality that T has no isolated points for otherwise, recalling the remarks following Theorem 5 it follows that every point of T is isolated and the Belyaev alternative holds trivially in the sense that $\alpha(t) \equiv 0$. If T has no isolated points, since T is itself a homogeneous space, condition (T-1) of Theorem 5 is satisfied and the conclusion concerning $\alpha(t)$ follows. The second assertion of the theorem is now obvious.

We conclude with two examples.

EXAMPLE 1. Let $T = \boldsymbol{R}^d$ and G the group of all proper rigid motions of $\boldsymbol{R}^d$. Then (T, G) stationarity is invariance of the joint distribution of $(X_{t_1}, \cdots, X_{t_n})$ with respect to Euclidean displacements. $\boldsymbol{R}^d$, in this case is regarded as the homogeneous space G/K, K being the subgroup consisting of proper rotations about 0 (the origin of $\boldsymbol{R}^d$). The requirements of metrizability and separability of G/K are obviously satisfied.

EXAMPLE 2. $T = S^{d-1}$, the $(d-1)$-sphere in $\boldsymbol{R}^d$ $(d > 1)$, and $G = SO(d)$, the group of rotations. (T, G) stationarity has the obvious meaning. S^{d-1} is viewed here as the homogeneous space G/K where K is the subgroup of G of rotations which leave fixed an arbitrary point (say the north pole) of S^{d-1}.

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