S. Endo
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ON RAMIFICATION THEORY IN PROJECTIVE ORDERS, II

SHIZUO ENDO

Let R be a commutative ring and K be the total quotient ring of R. Let Σ be a separable K-algebra which is a finitely generated projective, faithful K-module and Λ be an R-order in Σ. We denote by $D_{\Lambda/R}$ the Dedekind different of Λ and by $N_{\Lambda/R}$ the Noetherian different of Λ.

The purpose of this paper is to give the following results, as a continuation to [2].

(I) For any projective R-order Λ in a separable K-algebra Σ, we have $\mathrm{trd}_{\Sigma/c(\Sigma)}(D_{\Lambda/R}) = N_{\Lambda/R}$.

(II) (Dedekind different theorem) Let R be a Noetherian normal domain with quotient field K. Let Σ be a separable K-algebra and Λ be a projective R-order in Σ. Then, for any prime ideal $\mathfrak{P}$ of Λ, the following conditions are equivalent:

(1) $D_{\Lambda/R} \not\subseteq \mathfrak{P}$.

(2) $[D_{\Lambda/R}]^2 \not\subseteq (\mathfrak{P} \cap c(\Lambda))\Lambda$.

(3) $\mathfrak{P}$ is unramified over R.

Here we denote the center of Λ by $c(\Lambda)$.

We remark that both (I) and (II) have been proved under some additional assumptions ([1], [2], [4], [5], [8], etc.).

Our notation and terminology used in this paper are the same as in [2].

1. Let Λ be an R-algebra. Now we regard $\mathrm{Hom}_R(\Lambda^*, \Lambda)$ as a left Λ^e-module by $[(\lambda \otimes \mu^\circ)\cdot h]\,(f) = \lambda h(\mu \cdot f)$ for $h \in \mathrm{Hom}_R(\Lambda^*, \Lambda)$ and $f \in \Lambda^*$. We define the Λ^e-homomorphism $\beta_{\Lambda/R} : \Lambda^e \to \mathrm{Hom}_R(\Lambda^*, \Lambda)$ by $\beta_{\Lambda/R}(\lambda \otimes \mu^\circ)\,(f) = \lambda f(\mu)$ for $f \in \Lambda^*$. Since $[\Lambda^e]^\Lambda = A_{\Lambda/R}$ and $\mathrm{Hom}_R(\Lambda^*, \Lambda)^\Lambda = \mathrm{Hom}_\Lambda({}_\Lambda\Lambda^*, {}_\Lambda\Lambda)$, we have $\beta_{\Lambda/R}(A_{\Lambda/R}) \subseteq \mathrm{Hom}_\Lambda({}_\Lambda\Lambda^*, {}_\Lambda\Lambda)$.

Suppose that Λ is a finitely generated projective R-module. Then $\beta_{\Lambda/R}$ is evidently an isomorphism and therefore $\beta_{\Lambda/R}(A_{\Lambda/R}) = \mathrm{Hom}_\Lambda({}_\Lambda\Lambda^*, {}_\Lambda\Lambda)$. Let

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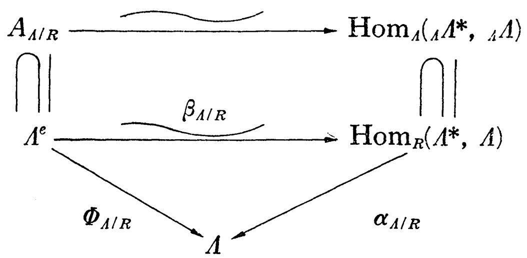
$\{f_i, \lambda_i\}_{1\leq i\leq m}$ be a dual basis of Λ over R and define $\alpha_{\Lambda/R} : \mathrm{Hom}_R(\Lambda^*, \Lambda) \to \Lambda$ by $\alpha_{\Lambda/R}(h) = \sum_i h(f_i)\lambda_i$ for $h \in \mathrm{Hom}_R(\Lambda^*, \Lambda)$. Then we can easily see that $\alpha_{\Lambda/R}$ does not depend on the choice of the dual basis of Λ, and we get the following commutative diagram:



Further suppose that Λ is a separable R-algebra which is a finitely generated projective, faithful R-module. Then we have $\Lambda^* = \Lambda \cdot \mathrm{trd}_{\Lambda/R}$ and so the homomorphism $\gamma_{\Lambda/R} : \mathrm{Hom}_\Lambda({}_\Lambda\Lambda^*, {}_\Lambda\Lambda) \to \Lambda$ defined by $\gamma_{\Lambda/R}(h) = h(\mathrm{trd}_{\Lambda/R})$ is an isomorphism.

LEMMA 1. *Let Λ be a separable R-algebra which is a finitely generated projective, faithful R-module. Then $\alpha_{\Lambda/R} \cdot \gamma_{\Lambda/R}^{-1} = trd_{\Lambda/c(\Lambda)}$, where $c(\Lambda)$ denotes the center of Λ.*

Proof. For any commutative R-algebra S, we have $\alpha_{S\otimes_R\Lambda/S} = I_S \otimes_R \alpha_{\Lambda/R}$, $\gamma_{S\otimes_R\Lambda/S} = I_S \otimes_R \gamma_{\Lambda/R}$ and $\mathrm{trd}_{S\otimes_R\Lambda/c(S\otimes_R\Lambda)} = I_S \otimes_R \mathrm{trd}_{\Lambda/c(\Lambda)}$. Therefore we see $\alpha_{\Lambda/R} \cdot \gamma_{\Lambda/R}^{-1} = \mathrm{trd}_{\Lambda/c(\Lambda)}$, if and only if, for any maximal ideal $\mathfrak{m}$ of R, $\alpha_{\Lambda_\mathfrak{m}/R_\mathfrak{m}} \cdot \gamma_{\Lambda_\mathfrak{m}/R_\mathfrak{m}}^{-1} = \mathrm{trd}_{\Lambda_\mathfrak{m}/c(\Lambda_\mathfrak{m})}$. Hence we may assume without loss of generality that R is a local ring. Furthermore, if S is a commutative R-faithful R-algebra and if $\alpha_{S\otimes\Lambda/S} \cdot \gamma_{S\otimes_R\Lambda/S}^{-1} = \mathrm{trd}_{S\otimes_R\Lambda/c(S\otimes_R\Lambda)}$, then $\alpha_{\Lambda/R} \cdot \gamma_{\Lambda/R}^{-1} = \mathrm{trd}_{\Lambda/c(\Lambda)}$. So we may further assume that R is a separably closed, Henselian local ring ([6]). Then Λ is of split type and we can write

$$c(\Lambda) = R_1 \oplus R_2 \oplus \cdots \oplus R_t, \quad R_i \cong R$$

and

$$\Lambda = M_{n_1}(R_1) \oplus M_{n_2}(R_2) \oplus \cdots \oplus M_{n_t}(R_t)$$

where each $M_{n_k}(R_k)$ denotes the total matric algebra of degree n_k over R_k. Also we put $1_R = e_1 + e_2 + \cdots + e_t$, $e_k \in R_k$.

For each k let $\{e_{ij}^{(k)}\}$ be the set of all matrix units of $M_{n_k}(R_k)$. Then we can easily see that $\{e_{ij}^{(k)}\, \mathrm{trd}_{M_{n_k}(R_k)/R_k}, e_{ji}^{(k)}\}_{1\leq i\leq n_k, 1\leq j\leq n_k}$ forms a dual basis of

$M_{n_k}(R_k)$ over R_k and, for any $\lambda_k \in M_{n_k}(R_k)$, $\mathrm{trd}_{M_{n_k}(R_k)/R_k}(\lambda_k) = \sum_{i,j} e_{ij}^{(k)} \lambda_k e_{ji}^{(k)}$. Furthermore we see that $\{e_{ij}^{(k)} \mathrm{trd}_{\Lambda/R}, e_{ji}^{(k)}\}_{1 \leq i \leq n_k, 1 \leq j \leq n_k, 1 \leq k \leq t}$ forms a dual basis of Λ over R. In fact, for any $\lambda = \lambda_1 + \cdots + \lambda_t$, $\lambda_k \in M_{n_k}(R_k)$, we have

$$\begin{aligned}\sum_k \sum_{i,j} \mathrm{trd}_{\Lambda/R}(e_{ij}^{(k)}\lambda) e_{ji}^{(k)} &= \sum_k \sum_{i,j} \mathrm{trd}_{\Lambda/R}(e_{ij}^{(k)}\lambda_k) e_{ji}^{(k)} \\ &= \sum_k \sum_{i,j} \mathrm{trd}_{M_{n_k}(R_k)/R_k}(e_{ij}^{(k)}\lambda_k) e_{ji}^{(k)} \\ &= \sum_k \lambda_k = \lambda,\end{aligned}$$

because $\mathrm{trd}_{\Lambda/R}(e_{ij}^{(k)}\lambda_k) e_k = \mathrm{trd}_{M_{n_k}(R_k)/R_k}(e_{ij}^{(k)}\lambda_k)$ and $e_k e_{ji}^{(k)} = e_{ji}^{(k)}$. Hence $\alpha_{\Lambda/R} \cdot \gamma_{\Lambda/R}^{-1}(\lambda) = \sum_k \sum_{i,j} e_{ij}^{(k)} \lambda e_{ji}^{(k)} = \sum_k \sum_{i,j} e_{ij}^{(k)} \lambda_k e_{ji}^{(k)} = \sum_k \mathrm{trd}_{M_{n_k}(R_k)/R_k}(\lambda_k) = \mathrm{trd}_{\Lambda/c(\Lambda)}(\lambda)$. Thus $\alpha_{\Lambda/R} \cdot \gamma_{\Lambda/R}^{-1} = \mathrm{trd}_{\Lambda/c(\Lambda)}$.

THEOREM 1. *Let R be a commutative ring and K be the total quotient ring of R. Let Σ be a separable K-algebra which is a finitely generated projective, faithful K-module. Then, for any R-order Λ in Σ, we have $N_{\Lambda/R} \subseteq \mathrm{trd}_{\Lambda/c(\Lambda)}(D_{\Lambda/R})$. Especially, if Λ is a projective R-order in Σ, $N_{\Lambda/R} = \mathrm{trd}_{\Lambda/c(\Lambda)}(D_{\Lambda/R})$ and $D_{\Lambda/R} \subseteq C_{\Lambda/c(\Lambda)}$.*

Proof. $\mathrm{Hom}_R(\Lambda^*, \Lambda)$ can be regarded naturally as the submodule of $\mathrm{Hom}_K(\Sigma^*, \Sigma)$. Then, by the definition of $D_{\Lambda/R}$, we have $\gamma_{\Sigma/K}(\mathrm{Hom}_\Lambda(\Lambda^*, \Lambda)) = D_{\Lambda/R}$. Hence we get the following commutative diagram:

$$\begin{array}{ccccc} & & D_{\Lambda/R} & \subseteq & \Sigma \\ & & \uparrow\wr & & \uparrow\wr\ \gamma_{\Sigma/K} \\ A_{\Lambda/R} & \longrightarrow & \mathrm{Hom}_\Lambda({}_\Lambda\Lambda^*, {}_\Lambda\Lambda) & \subseteq & \mathrm{Hom}_\Sigma({}_\Sigma\Sigma^*, {}_\Sigma\Sigma) \end{array}$$

$$\begin{array}{ccccccc} \Lambda^e & \underset{\sim}{\xrightarrow{\beta_{\Lambda/R}}} & \mathrm{Hom}_R(\Lambda^*, \Lambda) & \subseteq & \mathrm{Hom}_K(\Sigma^*, \Sigma) & \underset{\sim}{\xleftarrow{\beta_{\Sigma/K}}} & \Sigma^e \\ & \searrow{\scriptstyle \Phi_{\Lambda/R}} & & & & {\scriptstyle \alpha_{\Sigma/K}}\searrow & \swarrow{\scriptstyle \Phi_{\Sigma/K}} \\ & & \Lambda & & \subseteq & & \Sigma \end{array}$$

Since $\beta_{\Lambda/R}(A_{\Lambda/R}) \subseteq \mathrm{Hom}_\Lambda({}_\Lambda\Lambda^*, \Lambda) = \gamma_{\Sigma/K}^{-1}(D_{\Lambda/R})$, $N_{\Lambda/R} = \Phi_{\Lambda/R}(A_{\Lambda/R}) = \alpha_{\Sigma/K} \cdot \beta_{\Lambda/R}(A_{\Lambda/R}) \subseteq \alpha_{\Sigma/K} \cdot \gamma_{\Sigma/K}^{-1}(D_{\Lambda/R})$. By Lemma 1, $\alpha_{\Sigma/K} \cdot \gamma_{\Sigma/K}^{-1} = \mathrm{trd}_{\Sigma/c(\Sigma)}$ and so $N_{\Lambda/R} \subseteq \mathrm{trd}_{\Sigma/c(\Sigma)}(D_{\Lambda/R})$. Especially, if Λ is a projective R-order in Σ, we have $\beta_{\Lambda/R}(A_{\Lambda/R}) = \gamma_{\Sigma/K}^{-1}(D_{\Lambda/R})$. Hence we obtain $N_{\Lambda/R} = \mathrm{trd}_{\Sigma/c(\Sigma)}(D_{\Lambda/R})$. Further, from this it follows directly that $D_{\Lambda/R} \subseteq C_{\Lambda/c(\Lambda)}$. Thus our proof is completed.

2. In the rest of this paper we assume that R is a Noetherian normal domain, in order to simplify our description. We should remark that the following results can be proved under weaker assumptions (cf. [2]).

Let K be the quotient field of R and Σ be a separable K-algebra. Then, for any R-order Λ in Σ, we have $\mathrm{trd}_{\Sigma/K}(\Lambda) \subseteq R$ and $\mathrm{t}_{c(\Sigma)/K}(c(\Lambda)) \subseteq R$ and so $D_{\Lambda/R} \subseteq \Lambda \subseteq C_{\Lambda/R}$. If Λ is a projective R-order in Σ, then the discriminant $d_{\Lambda/R}$ of Λ is a projective ideal of R.

LEMMA 2. *Let R be a Henselian normal local domain with maximal ideal $\mathfrak{p}$ and K be the quotient field of R. Let L be a commutative separable K-algebra and S be a subring of L containing R which is integral over R and such that $KS = L$. Let $\mathfrak{q}$ be the Jacobson radical of S. Then we have $\mathrm{t}_{L/K}(\mathfrak{q}) \subseteq \mathfrak{p}$, where $\mathrm{t}_{L/K}$ denote the trace of L over K.*

Proof. Let $\bar{S}$ be the derived normal ring of S in L and $\mathfrak{q}$ be the Jacobson radical of $\bar{S}$. Then $\bar{\mathfrak{q}} \cap S = \mathfrak{q}$, and therefore we may assume that S is integrally closed. Since R is Henselian, we can write $S = S_1 \oplus S_2 \oplus \cdots \oplus S_t$ where each S_i is a Henselian normal local domain. Let $\mathfrak{q}_i$ be the maximal ideal of S_i and L_i be the quotient field of S_i. Then we have $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2 + \cdots + \mathfrak{q}_t$ and $\mathrm{t}_{L/K}(\mathfrak{q}) = \sum_{i=1}^{t} \mathrm{t}_{L_i/K}(\mathfrak{q}_i)$. Hence we may further suppose that S is a Henselian normal local domain with maximal ideal $\mathfrak{q}$.

Let F be a Galois extension of K containing L and T be the derived normal ring of S in F. Then T is also a Henselian normal local domain and we see $\sigma(T) = T$ for any $\sigma \in \mathrm{Gal}(F/K)$. Denoting by $\mathfrak{q}'$ the maximal ideal of T, we have $\sigma(\mathfrak{q}') = \mathfrak{q}'$ for any $\sigma \in \mathrm{Gal}(F/K)$. From this it follows immediately that $\mathrm{t}_{L/K}(\mathfrak{q}) \subseteq \mathfrak{q}' \cap R = \mathfrak{p}$.

We give, as a generalization of [2], (2.8), ii),

PROPOSITION 2. *Let R be a Noetherian normal domain with quotient field K and Σ be a separable K-algebra. Then, for a projective R-order Λ in Σ, the following conditions are equivalent:*

(1) $C_{\Lambda/R} = \Lambda$.

(2) $D_{\Lambda/R} = \Lambda$.

(3) $d_{\Lambda/R} = R$.

(4) $N_{\Lambda/R} = c(\Lambda)$, *i.e., Λ is separable over R.*

Proof. The equivalences of (1), (2) and (3) are evident and the implication (4) $\Longrightarrow$ (1) has been shown (e.g. [2]). Hence we have only to prove (1) $\Longrightarrow$ (4). Clearly it suffices to prove this in case R is a local domain. The

Henselization $\hat{R}$ of R is also normal and we can easily see $\hat{R}\otimes_R C_{\Lambda/R} = C_{\hat{R}\otimes\Lambda/\hat{R}}$, $\hat{R}\otimes_R D_{\Lambda/R} = D_{\hat{R}\otimes\Lambda/\hat{R}}$, $\hat{R}\otimes_R d_{\Lambda/R} = d_{\hat{R}\otimes\Lambda/\hat{R}}$ and $\hat{R}\otimes_R N_{\Lambda/R} = N_{\hat{R}\otimes\Lambda/\hat{R}}$. Therefore we may assume that R is a Henselian normal local domain. However, in this case, we can write $c(\Lambda) = S_1 \oplus S_2 \oplus \cdots \oplus S_t$ where each S_i is a Henselian local ring, and, putting $\Lambda_i = S_i \otimes_{c(\Lambda)} \Lambda$ for each i, we have $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_t$, $C_{\Lambda/R} = C_{\Lambda_1/R} \oplus \cdots \oplus C_{\Lambda_t/R}$, $D_{\Lambda/R} = D_{\Lambda_1/R} \oplus \cdots \oplus D_{\Lambda_t/R}$, etc.. Hence we may further suppose that $c(\Lambda)$ is also a Henselian local ring.

Now suppose (1) (equivalently (2) and (3)). Then, by Theorem 1, we have $\mathrm{trd}_{\Sigma/c(\Sigma)}(\Lambda) = \mathrm{trd}_{\Sigma/c(\Sigma)}(D_{\Lambda/R}) = N_{\Lambda/R}$. However, since Λ is a projective R-order in Σ, $\mathrm{trd}_{\Sigma/K}(\Lambda) = \mathrm{trd}_{\Sigma/K}(C_{\Lambda/R}) = R$. Accordingly we get $\mathrm{t}_{c(\Sigma)/K}(N_{\Lambda/R}) = R$. Let $\mathfrak{p}$ be the maximal ideal of R and $\mathfrak{q}$ be the maximal ideal of $c(\Lambda)$. By Lemma 2, then, we have $\mathrm{t}_{c(\Sigma)/K}(\mathfrak{q}) \subseteq \mathfrak{p}$. If $N_{\Lambda/R} \neq c(\Lambda)$, then $\mathrm{t}_{c(\Sigma)/K}(N_{\Lambda/R}) \subseteq \mathfrak{p}$, which is a contradiction. Thus we must have $N_{\Lambda/R} = c(\Lambda)$. This completes the proof of (1) $\Longrightarrow$ (4).

COROLLARY 1. *Let Λ be a projective R-order in a separable K-algebra Σ. Then any minimal prime divisor of $N_{\Lambda/R}$ in $c(\Lambda)$ is of height* 1 *in $c(\Lambda)$.*

Proof. Let $\mathfrak{q}$ be a minimal prime divisor of $N_{\Lambda/R}$ in $c(\Lambda)$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. By localizing and Henselizing R at $\mathfrak{p}$ as in the proof of Proposition 2, we may suppose that R is a Henselian normal local domain with maximal ideal $\mathfrak{p}$ and that $c(\Lambda)$ is a Henselian local ring with maximal ideal $\mathfrak{q}$. Then $N_{\Lambda/R}$ can be considered as a $\mathfrak{q}$-primary ideal of $c(\Lambda)$. If we suppose $\mathrm{height}_{c(\Lambda)}\mathfrak{q} > 1$, then, for any prime ideal $\mathfrak{p}'$ of height 1 in R, $N_{\Lambda_{\mathfrak{p}'}/R_{\mathfrak{p}'}} = (N_{\Lambda/R})_{\mathfrak{p}'} = c(\Lambda_{\mathfrak{p}'})$, and so $d_{\Lambda_{\mathfrak{p}'}/R_{\mathfrak{p}'}} = R_{\mathfrak{p}'}$ by Proposition 2. However, $d_{\Lambda/R}$ is an unmixed ideal of height 1 in R, because it is R-projective. Hence $d_{\Lambda/R} = R$. Again, by Proposition 2, we obtain $N_{\Lambda/R} = c(\Lambda)$, which contradicts the fact that $N_{\Lambda/R}$ is $\mathfrak{q}$-primary. Thus $\mathfrak{q}$ is of height 1 in $c(\Lambda)$.

Let Λ be an R-algebra and $\mathfrak{P}$ be a prime ideal of Λ. Let us put $\mathfrak{p} = \mathfrak{P} \cap R$ and $\mathfrak{q} = \mathfrak{P} \cap c(\Lambda)$. We say that $\mathfrak{P}$ is unramified over R if $\Lambda_{\mathfrak{p}}/\mathfrak{P}\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\mathfrak{P}\Lambda_{\mathfrak{q}} = \mathfrak{p}\Lambda_{\mathfrak{q}}$.

COROLLARY 2 (*Discriminant theorem*). *Let R be a Noetherian normal domain with quotient field K and Σ be a separable K-algebra. Let Λ be a projective R-order in Σ. Then, for any prime ideal $\mathfrak{p}$ of R, the following conditions are equivalent:*

(1) $d_{\Lambda/R} \not\subseteq \mathfrak{p}$.

(2) *Any prime ideal* $\mathfrak{P}$ *of* Λ *such that* $\mathfrak{p} = \mathfrak{P} \cap R$ *is unramified over* R.

Proof. The condition (1) is equivalent to the condition that $d_{\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}} = R_{\mathfrak{p}}$. By Proposition 2 this is also equivalent to the condition that $\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}$, i.e., to the condition (2).

We now prove our main theorem in this paper. It should be remarked that this is not included in [2], (3.6).

THEOREM 3 (*Dedekind different theorem*). *Let* R *be a Noetherian normal domain and* K *be the quotient field of* R. *Let* Σ *be a separable* K*-algebra and* Λ *be a projective* R*-order in* Σ. *Then, for any prime ideal* $\mathfrak{P}$ *of* Λ, *the following conditions are equivalent:*

(1) $D_{\Lambda/R} \not\subseteq \mathfrak{P}$.

(2) $[D_{\Lambda/R}]^2 \not\subseteq (\mathfrak{P} \cap c(\Lambda))\Lambda$.

(3) $\mathfrak{P}$ *is unramified over* R.

Proof. The implication (1) $\Longrightarrow$ (2) is obvious. Therefore it is sufficient to prove (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). We put $\mathfrak{p} = \mathfrak{P} \cap R$ and $\mathfrak{q} = \mathfrak{P} \cap c(\Lambda)$. Then we may assume that R is a Henselian normal local domain with maximal ideal $\mathfrak{p}$ and that $c(\Lambda)$ is a Henselian local ring with maximal ideal $\mathfrak{q}$.

(3) $\Longrightarrow$ (1): Suppose that $\mathfrak{P}$ is unramified over R. By virtue of [2], (3.2), we have $N_{\Lambda/R} \not\subseteq \mathfrak{q}$ and so $N_{\Lambda/R} = c(\Lambda)$. According to Proposition 2, then, $D_{\Lambda/R} = \Lambda$, and therefore $D_{\Lambda/R} \not\subseteq \mathfrak{P}$.

(2) $\Longrightarrow$ (3): Suppose that $\mathfrak{P}$ is ramified over R. Again, by [2], (3.2), we have $N_{\Lambda/R} \subseteq \mathfrak{q}$. Now we shall prove $[D_{\Lambda/R}]^2 \subseteq \mathfrak{q}\Lambda$. In order to prove this we may further assume that $\mathfrak{q}$ is a minimal prime divisor of $N_{\Lambda/R}$. Therefore, by Corollary 1 to Proposition 2, we may assume that $\text{height}_{c(\Lambda)}\mathfrak{q} = \text{height}_R\mathfrak{p} = 1$. Now, by Theorem 1 and Lemma 2, we have $\text{trd}_{\Sigma/K}(D_{\Lambda/R}) \subseteq \mathfrak{p}$. Since R is a discrete rank one valuation ring, we easily see $\mathfrak{p}^{-1}D_{\Lambda/R} \subseteq C_{\Lambda/R}$. Consequently we get $[D_{\Lambda/R}]^2 \subseteq \mathfrak{p}\Lambda \subseteq \mathfrak{q}\Lambda$. This proves (2) $\Longrightarrow$ (3). q.e.d.

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McMaster University and
Tokyo University of Education