RAMIFICATION THEORY FOR EXTENSIONS
OF DEGREE p. II

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Introduction. Let k denote the quotient field of a complete discrete rank one valuation ring R of unequal characteristic and let p denote the characteristic of $\bar{R}$; assume that R contains a primitive p^{th} root of unity, so that the absolute ramification index e of R is a multiple of $p - 1$, and each Gallois extension $K \supset k$ of degree p may be obtained by the adjunction of a pth root.

The purpose of this paper is to assign to each Galois extension $K \supset k$ of degree p an integer f with $-1 \leq f \leq ep/p - 1$ from which the ramification-theoretic properties of $K \supset k$ can be determined. Specifically, f shall determine the unramified, wildly ramified, or fiercely ramified character of $K \supset k$, (see Thm. 1.11); moreover, the ramification number i of $K \supset k$ shall have an expression in terms of f, (see Prop. 2.1).

Let $U^{(i)}$ for $i \geq 0$ denote the usual filtration on the units of R, and let $U^{(-1)}$ denote the set of prime elements of R. In a recent publication, ([5]), the author has studied the ramification-theoretic properties of a Galois extension $K \supset k$ of degree p by constructing the integral closure S of R in K from a judiciously chosen element b of $U^{(x)}$ $(-1 \leq x \leq p)$ whose p^{th} root defines K. The method for computing S in [5] entails the construction of a chain of $g + 1$ ring extensions of R in S; the number g is unique for $K \supset k$, satisfies the inequality $0 \leq g \leq (e/p - 1) - 1$, and is called the conductor number of $K \supset k$.

The present paper makes use of the fact that R is an Eisenstein extension of an unramified complete discrete rank one valuation ring in order to determine an alternate method for the construction of the integral closure S. In Section 1 we associate to each Galois extension $K \supset k$ of degree p an integer f with $-1 \leq f \leq ep/p - 1$ called its absolute field exponent such that

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K may be obtained from k by the adjunction of a judiciously chosen element b of $U^{(f)}$. By computing S from such an element b of R, we prove the following main result.

THEOREM. *Let f denote the absolute field exponent of a Galois extension $K \supset k$ of degree p. Then*

i) *$K \supset k$ is wild if and only if f is relatively prime to p*

ii) *$K \supset k$ is fierce if and only if p divides f and $f < ep/p - 1$*

iii) *$K \supset k$ is unramified if and only if $f = ep/p - 1$.*

In Section 2 we relate the results of the present paper to those of [5]. More specifically, we relate the notions of absolute field exponent and conductor number by computing an expression for the ramification number of $K \supset k$ in terms of the absolute field exponent f, and then applying the results of Section 3 of [5].

The author's recent paper and her present paper provide a choice of two significantly different methods for computing the integral closures S of R in a Galois extension $K \supset k$ of degree p. Available information concerning an element b whose p^{th} root defines the extension determines the proper choice of method.

The following notation shall be used throughout the paper. The multiplicative group of units of a ring R shall be denoted by $U(R)$; the intermediate ring obtained by adjoining to R an element t of an overring of R shall be denoted by $R[t]$; and, the residue class field of a local ring R shall be denoted by $\bar{R}$.

Unless otherwise stated, R shall always denote a complete discrete rank one valuation ring of unequal characteristic which contains a primitive p^{th} root of unity where p denotes the characteristic of $\bar{R}$, and S shall denote the integral closure of R in a Galois extension K of degree p over the quotient field k of R; $\prod$ shall denote a prime element of R, $\prod$ a prime element of S, and e the absolute ramification index of R. The usual filtration on $U(R)$ shall be denoted by $U^{(i)}$ ($i \geq 0$) and $U^{(-1)}$ shall denote the set of prime elements of R.

In [5], the author has defined the quotient field extension of an extension of discrete rank one valuation rings to be fiercely ramified if the residue class field extension has a non-trivial inseparable part. For further details, the reader may refer to [5].

1. **The absolute field exponent.** Throughout this section $K \supset k$ shall always denote a pth root extension, where k is the quotient field of a complete discrete rank one valuation ring R of unequal characteristic containing a primitive p^{th} root of unity and p is the characteristic of $\bar{R}$, and S shall always denote the integral closure of R in K. The purpose of this section is to define for each Galois extension $K \supset k$ of degree p an integer f with $-1 \leq f \leq ep/p - 1$ called its absolute field exponent, and to establish a criterion for determining if $K \supset k$ is unramified, wild, or fierce in terms of its absolute field exponent.

In [5], the author has assigned to each such extension $K \supset k$ an integer x with $-1 \leq x \leq p$ called its field exponent; the notions of field exponent and absolute field exponent coincide in the case when k has absolute ramification index $p - 1$. We shall make use of results established in [5] in our study of the absolute field exponent.

The first three lemmas concern elements b whose p^{th} roots define the extension $K \supset k$. Lem. 1.1 follows at once from Prop. 1.3 of [5].

LEMMA 1.1. *If $K \supset k$ is a Galois extension of degree p, then $K = k(b^{1/p})$ for some element b in $U^{(-1)}$ or in $U^{(0)}$.*

LEMMA 1.2. i) *If b is in $U^{(-1)}$, then $k(b^{1/p}) \supset k$ is wild of degree p, and $k(b^{1/p}) \neq k(b_1^{1/p})$ for every element b_1 of $U^{(0)}$.*

ii) *If b is in $U^{(0)}$, b_1 is in $U^{(1)}$, and $k(b^{1/p}) = k(b_1^{1/p})$, then $\bar{b}$ has a pth root in $\bar{R}$.*

iii) *If b is in $U^{(0)}$ and $X^p - \bar{b}$ is irreducible over $\bar{R}$, then $k(b^{1/p}) \supset k$ is fierce of degree p.*

Proof. If b is in $U^{(-1)}$ then $b^{1/p}$ is a root of an Eisenstein polynomial of degree p, from which it follows that $K \supset k$ is wild of degree p. If b is in $U^{(0)}$ and $X^p - \bar{b}$ is irreducible over $\bar{R}$, then $\bar{S} \supset \bar{R}$ is purely inseparable of degree p. The remaining assertions are restatements of parts ii) and iii) of Lem. 1.5 of [5].

Remark 1.3. If b is in $U^{(-1)}$, then the integral closure S of R in $k(b^{1/p})$ is $S = R[b^{1/p}]$. If b is in $U^{(0)}$ and $X^p - \bar{b}$ is irreducible over $\bar{R}$, then the integral closure S of R in $k(b^{1/p})$ is $S = R[b^{1/p}]$.

The above expressions for the integral closure have been established in

Prop. 2.6 A of [5]; we shall make use of them in our study of the rami-
fication number of $K \supset k$ in Section 2.

Lemma 1.6 pertains to extensions $K \supset k$ of degree p obtained by the
adjunction of a pth root of an element b of R present in $U^{(1)}$. Recall that
the complete discrete rank one valuation ring R is an Eisenstein extension
of an unramified complete discrete rank one valuation ring R_0 (see Thm.
31.12 p. 111 of [3]). Let e denote the ramification index of the totally
ramified extension $R \supset R_0$; then $R = R_0[\pi]$ where π denotes a prime element
of R, $\{1, \pi, \cdots, \pi^{e-1}\}$ is a free basis for R over R_0, $\pi^e R = pR$, and $\bar{R} = \bar{R}_0$,
(see Thm. 1 p. 23 of [2]). Moreover, e is the absolute ramification index of
R.

Facts 1.4 and 1.5 shall be used in the proof of Lem. 1.6.

FACT 1.4. *If b_1 and b_2 are elements of $U(R)$ such that $b_1 \equiv b_2 \bmod \pi^{(ep/p-1)+1}R$,
then $k(b_1^{1/p}) = k(b_2^{1/p})$.*

Proof. Since b_1 and b_2 are in $U(R)$, the congruence $b_1 \equiv b_2 \bmod \pi^{(ep/p-1)+1}R$
implies that $b_1/b_2 \equiv 1 \bmod \pi^{(ep/p-1)+1}R$, from which it follows that b_1/b_2 has a
p^{th} root in R according to Lem. 1.2 of [5]. The fact that b_1 and b_2 differ
multiplicatively by a pth power from k implies that $k(b_1^{1/p}) = k(b_2^{1/p})$.

FACT 1.5. *If b is an element of R, then there exists an element c of R of the
form $c = \sum y_i \pi^i$ $(0 \le i \le ep/p - 1)$ with each y_i in $U(R_0) \cup \{0\}$ which satisfies the
congruence $c \equiv b \bmod \pi^{(ep/p-1)+1}R$. If b is in $U^{(1)}$, then c may be chosen so that
$y_0 = 1$.*

Proof. Since $\{1, \pi, \cdots, \pi^{e-1}\}$ is an R_0-module basis for R, the element
b may be written in the form $b = \sum b_i \pi^i$ $(0 \le i \le e - 1)$ with each b_i in R_0.
If each b_i is in $U(R_0) \cup \{0\}$, then $c = b$ satisfies the assertion. Otherwise,
we may consider the least positive integer h such that b_h is not in $U(R_0) \cup$
$\{0\}$. Since b_h is in pR_0, the element b satisfies the congruence $b \equiv \sum b_i \pi^i$
$\bmod \pi^{h+1}R$ $(0 \le i \le h - 1)$; we may consider therefore an element $\tilde{b}$ of R of
the form $\tilde{b} = \sum \tilde{b}_i \pi^i$ $(0 \le i \le ep/p - 1)$ with each $\tilde{b}_i$ in R_0, $\tilde{b}_i = b_i$ for $0 \le i$
$\le h - 1$, and $\tilde{b}_h = 0$, which satisfies the congruence $\tilde{b} \equiv b \bmod \pi^{(ep/p-1)+1}R$.
If each $\tilde{b}_i$ is in $U(R_0) \cup \{0\}$ for $0 \le i \le ep/p - 1$, then $c = \tilde{b}$ satisfies the as-
sertion. Otherwise, we may consider the least positive integer m such that
$\tilde{b}_m$ is not in $U(R_0) \cup \{0\}$. Observe that $h < m$, so that by proceeding in this
way we may obtain, after finitely many steps, an element which c satisfies
the statement of our assertion.

If b is in $U^{(1)}$, then $b = 1 + r\pi$ for some element r of R. By applying the first part of this fact to r, we may produce an element c of the desired form which satisfies the congruence $c \equiv b \bmod \pi^{(ep/p-1)+1}R$.

LEMMA 1.6. *Assume the notation introduced above. Let $K \supset k$ be an extension of degree p defined by $K = k(b^{1/p})$ for some element b of $U^{(1)}$. Then $K = k(b_1^{1/p})$ for some element b_1 of $U^{(1)}$ of the form $b_1 = 1 + \sum x_i \pi^i$ $(1 \le i \le ep/p - 1)$ where the x_i are elements of R_0 such that*

    i) *each x_i is in $U(R_0) \cup \{0\}$, and the x_i are not all zero*

    ii) *$X^p - \bar{x}_i$ is irreducible over $\bar{R}_0$ for each i divisible by p such that $i < ep/p - 1$ and $x_i \ne 0$.*

Proof. By combining Facts 1.4 and 1.5 we may consider an element c of R of the form $c = 1 + \sum y_i \pi^i$ $(1 \le i \le ep/p - 1)$ with the y_i in $U(R_0) \cup \{0\}$ such that $k(c^{1/p}) = k(b^{1/p})$. Observe that the y_i are not all zero; for if $y_i = 0$ for each i, then $k(c^{1/p}) = k$, which contradicts the assumption that $K \supset k$ has degree p. If $X^p - \bar{y}_i$ is irreducible over $\bar{R}_0$ for every i divisible by p and less than $ep/p - 1$ for which $y_i \ne 0$, then $b_1 = c$ satisfies the assertion of this lemma.

Otherwise, we may consider the least positive integer h divisible by p, less than $ep/p - 1$, for which $y_h \ne 0$ and $X^p - \bar{y}_h$ is reducible over $\bar{R}_0$. We proceed to show that c can be replaced by an element c_1 of $U^{(1)}$ of the form $c_1 = \sum r_i \pi^i$ $(1 \le i \le ep/p - 1)$, where the r_i are in $U(R_0) \cup \{0\}$ and are not all zero, such that for every $i \le h$ divisible by p for which $r_i \ne 0$, the polynomial $X^p - \bar{r}_i$ is irreducible over $\bar{R}_0$. Since $X^p - \bar{y}_h$ is reducible over $\bar{R}_0$, we may consider an element y of R_0 such that $\bar{y}^p = \bar{y}_h$, i.e. such that $y^p \equiv y_h \bmod pR$. Define the element $\tilde{c}$ of R by $\tilde{c} = c(1 - y\pi^{h/p})^p$, and observe that $k(\tilde{c}^{1/p}) \equiv k(c^{1/p})$ because $\tilde{c}$ and c differ multiplicatively by a pth power from k. By expanding $(1 - y\pi^{h/p})^p$ according to the binomial theorem, we obtain the congruence $\tilde{c} \equiv c(1 - y^p\pi^h) \bmod \pi^{e+(h/p)}R$ since $pR = \pi^e R$. It is easy to verify that $h + 1 \le e + h/p$ if and only if $h < ep/p - 1$. Therefore the fact that $h < ep/p - 1$ now implies that $\tilde{c} \equiv c - y^p\pi^h \bmod \pi^{h+1}R$, because c is in $U^{(1)}$. Since $c = 1 + \sum y_i \pi^i$ $(1 \le i \le ep/p - 1)$ and $y^p = y_h \bmod pR$, it now follows that $\tilde{c} \equiv 1 + \sum y_i \pi^i + (y_h - y^p)\pi^h \bmod \pi^{h+1}R$ $(1 \le i \le h - 1) \equiv 1 + \sum y_i \pi^i \bmod \pi^{h+1}R$ $(1 \le i \le h - 1)$. Now we may define he desired element c_1. According to the preceeding congruences we may write $\tilde{c}$ in the form $\tilde{c} = 1 + \sum r_1 \pi^i + r\pi^{h+1}$ $(1 \le i \le h)$ for some element r of R, where $r_i = y_i$

$(1 \leq i \leq h-1)$ and $r_h = 0$. An application of Fact 1.5 yields the existence of an element of the form $\sum r_i \pi^i$ $(h+1 \leq i \leq ep/p-1)$ with the r_i in $U(R_0)$ $\cup \{0\}$ which satisfies the congruence $\sum_{i=h+1}^{ep/p-1} r_i \pi^i \equiv r \pi^{h+1} \bmod \pi^{(ep/p-1)+1}R$. Define the element c_1 of $U^{(1)}$ by $c_1 = 1 + \sum r_i \pi^i$ $(1 \leq i \leq ep/p-1)$. Observe that $c_1 = \tilde{c} \bmod \pi^{(ep/p-1)+1}R$, so that $K = k(c_1^{1/p})$ according to Fact 1.4 because $K = k(\tilde{c}^{1/p})$. Since $r_h = 0$ and $r_i = y_i$ for $1 \leq i \leq h-1$, it is true that the polynomial $X^p - \bar{r}_i$ is irreducible over $\bar{R}_0$ for each i $(1 \leq i \leq h)$ divisible by p such that $r_i \neq 0$. An argument similar to the one at the beginning of the proof shows that the elements r_i $(1 \leq i \leq ep/p-1)$ are not all zero. If $X^p - \bar{r}_i$ is irreducible over $\bar{R}_0$ for every $i < ep/p-1$ which is divisible by p and for which $r_i \neq 0$, then $b_1 = c_1$ satisfies the assertion of this lemma.

Otherwise, we may consider the least positive integer m less than $ep/p-1$ and divisible by p such that $r_m \neq 0$ and $X^p - \bar{r}_m$ is reducible over $\bar{R}_0$; observe that $h < m$. By means of the same technique used above to produce c_1 from c, we may produce an element c_2 of $U^{(1)}$ of the form $c_2 = 1 + \sum \delta_i \pi^i$ $(1 \leq i \leq ep/p-1)$ such that $k(c_2^{1/p}) = k(c_1^{1/p})$, where the δ_i are in $U(R_0)$ $\cup \{0\}$ and are not all zero, and the polynomials $X^p - \bar{\delta}_i$ are irreducible over $\bar{R}_0$ for every $i \leq m$ divisible by p for which $\delta_i \neq 0$.

It follows from the inequality $h < m$, that by proceeding in this way we may finally obtain an element b_1 of $U^{(1)}$ which satisfies the assertion of this lemma.

DEFINITION. An element b of $U^{(1)}$ of the form $b = 1 + \sum x_i \pi^i$ $(1 \leq i \leq ep/p-1)$ with the x_i in R_0 is said to be in *normal form* if the x_i satisfy statements i) and ii) of Lem. 1.6.

The usefulness of Lem. 1.6 for the definition of the absolute field exponent motivates its name.

The following proposition concerning elements of $U^{(1)}$ shall be used to establish the main result (Thm. 1.11); its corollary (Lem. 1.9) shall be used to establish the uniqueness of the absolute field exponent in Prop. 1.10.

PROPOSITION 1.7. *Let* $b = 1 + \sum x_i \pi^i$ $(1 \leq i \leq ep/p-1)$ *denote an element of* $U^{(1)}$ *in normal form, and let* f *denote the least integer for which* $x_f \neq 0$; *let* $K = k(b^{1/p})$. *Then*

   i) $K \supset k$ *is wild of degree* p *if and only if* f *is relatively prime to* p

   ii) $K \supset k$ *is fierce of degree* p *if and only if* p *divides* f *and* $f < ep/p-1$

   iii) $K \supset k$ *is unramified if and only if* $f = ep/p-1$. *Moreover,* $K \supset k$ *is*

unramified of degree p if and only if $f = ep/p - 1$ and the polynomial $X^p + \bar{v}X - \bar{x}_f$ is irreducible over $\bar{R}_0$, where v is the element of $U(R)$ defined by $v\pi^e = p$.

Proof. Recall (see Prop. 1.1 of [5]) that a pth root β of an element b of $U^{(1)}$ satisfies an equality of the form $(\beta - 1)^p = (b - 1) + uv\pi^e(\beta - 1)$ where u is an element of the R-module $R(1, \beta, \cdots, \beta^{p-2})$ which satisfies the congruence $u \equiv -1 \bmod (p, \beta - 1)R[\beta]$, and v is the element of $U(R)$ defined by $v\pi^e = p$. The proceeding equality shall be used for establishing the asserted relationships between f and the ramification-theoretic character of $K \supset k$.

First we shall prove that if f is relatively prime to p, then $K \supset k$ is wild of degree p by constructing a prime element $\prod$ of the integral closure S of R in K. By applying the division algorithm to f and p we may obtain (unique) integers q and t such that $f = qp + t$ where $0 \leq t < p$. Observe that $q \geq 0$ because $f \geq 1$, and that $0 < t$ because $(f, p) = 1$. The element θ of K defined by $\theta = (\beta - 1)\pi^q$ shall be useful for constructing $\prod$. We proceed to show that θ is a non-unit of S and that θ^p is in $\pi^t U(S)$. Consider the element x of $U(R)$ defined by $b - 1 = x\pi^f$. The definition of θ and the equality $(\beta - 1)^p = (b - 1) + uv\pi^e(\beta - 1)$ yield the equality $\theta^p = x\pi^t + uv^{e-qp+q}\theta$ by an easy computation. Observe that $e - qp + q \geq 1$. For, $e - qp + q \geq 1$ if and only if $q < e/p - 1$, which holds if and only if $qp < ep/p - 1$; therefore the inequalities $qp < f < ep/p - 1$ imply that $e - qp + q \geq 1$. The above expression for θ^p now shows that θ satisfies a monic polynomial with coefficients in S, from which it follows that θ is itself in S. Observe moreover that θ is a non-unit of S because $t \geq 1$ and $e - qp + q \geq 1$. In order to show that θ^p is in $\pi^t U(S)$, it suffices to show that $e - qp + q \geq t$ since θ is a non-unit of S. Now $e - qp + q \geq t$ if and only if $f \leq e + q$ if and only if $fp \leq ep + f - t$ if and only if $f \leq (ep/p - 1) - t/p - 1$. Since $0 < t/p - 1 \leq 1$, we now have that $t \leq e - qp + q$ if and only if $f \leq (ep/p - 1) - 1$ if and only if $f < ep/p - 1$. The assumption that $(f, p) = 1$ guarantees that $f < ep/p - 1$, and so we may conclude at last that $t \leq e - qp + q$. The equality $\theta^p = x\pi^t + uv\pi^{e-qp+q}\theta$, together with the inequality $e - qp + q \geq t$ and the fact that θ is a non-unit, now implies that θ^p is in $\pi^t U(S)$ because x is in $U(R)$. Now we may show that $K \supset k$ is wild of degree p by showing that $K \supset k$ has ramification index p. Since we have assumed that f and p are relatively prime, we may consider integers m and n such that $mp + nt = 1$. An easy computation shows that the element $\prod$ of K defined by $\prod = \theta^n \pi^m$ has the property that $\prod^p$ is in $\pi U(S)$, from which it follows

that $\prod$ is an element of S, that $K \supset k$ has ramification index p, and that $K \supset k$ is wild of degree p.

The next step is to show that if f is divisible by p and less than $ep/p-1$, then $K \supset k$ is fiercely ramified of degree p. Consider the element θ of K defined by $\theta = (\beta - 1)/\pi^q$ where $q = f/p$, and observe that $1 \leq q < e/p - 1$. In order to prove the assertion we shall show that θ is an element of S with the property that $\bar{R}(\bar{\theta}) \supset \bar{R}$ is purely inseparable of degree p. Let x denote the element of $U(R)$ defined by $b - 1 = x\pi^f$ and observe that $\bar{x} = \bar{x}_f$. The equality $(\beta - 1)^p = (b - 1) + uv\pi^e(\beta - 1)$ (see the beginning of the proof) together with the definition of θ implies that $\theta^p = x + uv\pi^{e-qp+q}\theta$, where $e - pq + q > 0$ because $q < e/p - 1$; therefore θ is in S because it satisfies a monic polynomial equation with coefficients in S. Since b is in normal form by assumption, the fact that p divides f implies that $X^p - \bar{x}_f$ is irreducible over $\bar{R}_0 = \bar{R}$. Therefore $\bar{R}(\bar{\theta}) \supset \bar{R}$ is purely inseparable of degree p because $\bar{\theta}^p = \bar{x} = \bar{x}_f$. We may now conclude that $\bar{S} = \bar{R}(\bar{\theta})$ and that $K \supset k$ is fierce of degree p. (Moreover, $S = R[\theta]$ according to part iii) of Lem. 2.4 of [5].)

We show finally that if $f = ep/p - 1$ then $K \supset k$ is unramified and we establish necessary and sufficient conditions for $K \supset k$ to have degree p. Consider the element θ defined by $\theta = (\beta - 1)/\pi^{e/p-1}$ and observe that $K = k(\theta)$. We shall show that θ is in S, and that $\bar{S} = \bar{R}(\bar{\theta})$ with $\bar{\theta}$ separable over $\bar{R}$. For convenience of notation let $x = x_f = x_{ep/p-1}$. Then the definition of θ together with the equality $(\beta - 1)^p = (b - 1) + uv\pi^e(\beta - 1)$ (see the beginning of the proof) implies that $\theta^p = x + uv\theta$. It follows from the definition of θ together with the fact that u is in the R-module $R(1, \beta, \cdots, \beta^{p-2})$ that u is in $R(1, \theta, \cdots, \theta^{p-2})$; therefore, the equality $\theta^p - uv\theta - x = 0$ gives rise to a monic polynomial $f(X)$ in $R[X]$ having θ as a root, from which it follows that θ is in S. Observe that $\bar{f}(X) = X^p + \bar{v}X - \bar{x}$ in $\bar{R}[X]$ because $\bar{u} = -\bar{1}$, and that $\bar{f}(X)$ is a separable polynomial because $\bar{f}'(X) = \bar{v} \neq \bar{0}$. We proceed to show that $[K : k] = p$ if and only if $\bar{f}(X)$ is irreducible over $\bar{R} = \bar{R}_0$, and that $K = k$ otherwise. If $\bar{f}(X)$ is reducible over $\bar{R}$, then $f(X)$ is reducible over R by Hensel's lemma because R is complete and $\bar{f}(X)$ is separable; the reducibility of $f(X)$ over R implies that $\deg_k \theta < p$ from which it follows that β is in k and that $K = k$. If, on the other hand, $\bar{f}(X)$ is irreducible over $\bar{R}$, then the separability of $\bar{f}(X)$ implies that $K \supset k$ is unramified of degree p (with $\bar{S} = \bar{R}(\bar{\theta})$ and $S = R[\theta]$) according to Prop. 1 p. 25 of [2].

The above observations combine to establish the truthfulness of the proposition.

Observe that the equation $X^p + \bar{v}X - \bar{x} = \bar{0}$ of Prop. 1.7 is essentially an Artin-Schreier equation (see p. 80 of [4]). For, consider the elements v, v_1, and v_0 defined by $v\pi^e = p$, $v_1\pi^{e/p-1} = \zeta - 1$, and $v_0(\zeta - 1)^{p-1} = p$ where ζ denotes as usual a primitive p^{th} root of unity. An easy computation shows that $v = v_1{}^{p-1}v_0$, so that $\bar{v} = -\bar{v}_1{}^{p-1}$ because $\bar{v}_0 = -\bar{1}$ (see p. 158 of [1]). The change of variable $Y = X/\bar{v}_1$ yields the Artin-Schreier equation $Y^p - Y - \bar{x}/\bar{v}_1{}^p = \bar{0}$.

The following expressions for the integral closure S of R in K follow at once from the proof of Prop. 1.7.

Remark 1.8. Let $b = 1 + \sum x_i\pi^i$ denote an element of $U^{(f)} - U^{(f+1)}$ ($1 \leq f \leq ep/p - 1$) in normal form. Consider the unique integers q and t for which $f = qp + t$ with $0 \leq t < p$, and define $\theta = (\beta - 1)/\pi^q$.

i) If f is relatively prime to p, then $S = R[\prod]$ where $\prod = \theta^n\pi^m$ for integers m and n satisfying $mp + nt = 1$.

ii) If p divides f, then $S = R[\theta]$.

LEMMA 1.9. *Consider elements b_1 and b_2 of $U^{(1)}$ in normal form, where b_1 is in $U^{(f_1)} - U^{(f_1+1)}$ and b_2 is in $U^{(f_2)} - U^{(f_2+1)}$. If $k(b_1{}^{1/p}) = k(b_2{}^{1/p})$, then $f_1 = f_2$.*

Proof. Since $k(b_1{}^{1/p}) = k(b_2{}^{1/p})$ by hypothesis, an application of Prop. 1.7 shows that f_1 and f_2 are both relatively prime to p, are both divisible by p and less than $ep/p - 1$, or are both equal to $ep/p - 1$.

Consider an equality $k(b_1{}^{1/p}) = k(b_2{}^{1/p})$ with f_1 and f_2 relatively prime to p. We shall show that $f_1 = f_2$ by contradiction. Assume that $f_1 < f_2$. Since $k(b_1{}^{1/p}) = k(b_2{}^{1/p})$, we may consider an element c of k such that $b_1 = c^p b_2{}^n$ for some integer n relatively prime to p, (see Lem. 3 p. 90 of [2]). Observe that c^p is in $U^{(f_1)} - U^{(f_1+1)}$ because b_1 is in $U^{(f_1)} - U^{(f_1+1)}$, $b_2{}^n$ is in $U^{(f_2)}$, and $f_1 < f_2$, so that $k(c) \supset k$ is wild of degree p according to Prop. 1.7. This contradiction shows that $f_1 = f_2$.

Now consider an equality $k(b_1{}^{1/p}) = k(b_2{}^{1/p})$ with f_1 and f_2 divisible by p and less than $ep/p - 1$, and assume that $f_1 < f_2$. Once again we consider an element c in k such that $b_1 = c^p b_2{}^n$ for some integer n relatively prime to p. Since $b_2{}^n$ is in $U^{(f_2)}$, we have that $c^p \equiv b_1 \bmod \pi^{f_2}R$ from which it follows that c^p is of the form $c^p = 1 + y\pi^{f_1}$ with $\bar{y} = \bar{x}_{f_1}$ because $f_1 < f_2$.

The irreducibility of $X^p - \bar{y}$ over $\bar{R}$ now implies that $k(c) \supset k$ is fierce of degree p by Prop. 1.7. This contradiction shows that $f_1 = f_2$, and this completes the proof.

The following proposition follows at once from the four lemmas established above.

PROPOSITION 1.10. *Let k denote the quotient field of a complete discrete rank one valuation ring R containing a primitive p^{th} root of unity, where $p = $ char $\bar{R}$, and consider a Galois extension $K \supset k$ of degree p. Then there exists a unique integer f $(-1 \leq f \leq ep/p - 1)$ such that $K \supset k$ is one of the following forms:*

i) $K = k(b^{1/p})$ *for some element b of $U^{(f)}$ with $f = -1$*

ii) $K = k(b^{1/p})$ *for some element b of $U^{(f)}$ with $f = 0$ for which $X^p - \bar{b}$ is irreducible over $\bar{R}$*

iii) $K = k(b^{1/p})$ *for some element b of $U^{(f)} - U^{(f+1)}$ in normal form, (where $1 \leq f \leq ep/p - 1$).*

DEFINITION. The unique integer f satisfying $-1 \leq f \leq ep/p - 1$ defined for each Galois extension $K \supset k$ of degree p by Prop. 1.10 is called the *absolute field exponent* of $K \supset k$ and is denoted by $f(K/k)$.

The following theorem has now been established.

THEOREM 1.11. *Let $f = f(K/k)$ denote the absolute field exponent of a Galois extension $K \supset k$ of degree p. Then*

i) $K \supset k$ *is wildly ramified if and only if f is relatively prime to p*

ii) $K \supset k$ *is fiercely ramified if and only if p divides f and $f < ep/p - 1$*

iii) $K \supset k$ *is unramified if and only if $f = ep/p - 1$.*

We terminate this section with some observations concerning the relationship between the field exponent $x = x(K/k)$ (see Section 1 of [5]) and the absolute field exponent $f = f(K/k)$ of a Galois extension $K \supset k$ of degree p. These observations follow at once from the definitions of x and f. Recall that $-1 \leq x \leq p$ and that $-1 \leq f \leq ep/p - 1$.

Remark 1.12. Let x denote the field exponent and f the absolute field exponent of a Galois extension $K \supset k$ of degree p.

i)   If $e = p - 1$, then $x = f$.

ii)   If $-1 \leq f \leq p$, then $x = f$.

iii) If $-1 \le x \le p-1$, then $x = f$.

2. The ramification number, the absolute field exponent, and the conductor number.

As usual, $K \supset k$ denotes a Galois extension of degree p where k is the quotient field of a complete discrete rank one valuation ring R which contains a primitive p^{th} root of unity and whose residue class field has characteristic p. In Section 2 of [5], the author has assigned to each such extension $K \supset k$ an integer g with $0 \le g \le (e/p - 1) - 1$ called the conductor number of $K \supset k$. Prop. 3.1 of [5] presents expressions for the ramification number i of $K \supset k$ in terms of its conductor number g.

The purpose of this section is to determine the relationships between the absolute field exponent of an extension and its ramification and conductor numbers.

PROPOSITION 2.1. *Let f denote the absolute field exponent of a Galois extension $K \supset k$ of degree p, and let i denote the ramification number of $K \supset k$.*

i) *If $f = -1$, then $i = ep/p - 1$.*

ii) *If p divides f, then $i = (e/p - 1) - f/p - 1$.*

iii) *If $f > 0$ and $(f, p) = 1$, then $i = (ep/p - 1) - f$.*

Proof. Let $x = x(K/k)$ denote the field exponent of $K \supset k$ (see Section 1 of [5]). If $f = -1$, then $x = -1$ (see Remark 1.12). According to part ii) of Prop. 3.1 of [5], $i = ep/p - 1$ when $x = -1$, and this proves statement i). (Or, the reader may refer to Exer. 4 p. 79 of [4]).

To prove statement ii) we first consider the case when p divides f and $f < ep/p - 1$; in this case $K \supset k$ is fierce and the integral closure S of R in K is given by $S = R[\theta]$ where $\theta = (\beta - 1)/\pi^q$ and $q = f/p$ (see Prop. 1.7 and Remark 1.8). Consider some primitive p^{th} root of unity ζ and let σ denote the element of the Galois group $G(K/k)$ defined by $\sigma(\beta) = \zeta\beta$; observe that σ is in the ith ramification group G_i of $K \supset k$ if and only if $\sigma(\theta) \equiv \theta \bmod \pi^{i+1}S$. An easy computation shows that $\sigma((\beta - 1)/\pi^q) \equiv (\beta - 1)/\pi^q \bmod \pi^{i+1}S$ if and only if the element $\zeta - 1$ is in $\pi^{i+q+1}S$, which in turn holds if and only if $\pi^{e/p-1}$ is in $\pi^{i+q+1}S$ because $\zeta - 1$ is in $\pi^{e/p-1}U(S)$. The fact that $\pi^{e/p-1}$ is in $\pi^{i+q+1}S$ if and only if $i \le (e/p - 1) - q - 1$ shows that σ is in G_i if and only if $i \le (e/p - 1) - q - 1$.

In the case when $f = ep/p - 1$, the extension $K \supset k$ is unramified according to Prop. 1.7. It is well known that the ramification number of an

unramified extension is -1. The observation that $(e/p-1) - f/p - 1 = -1$ when $f = ep/p - 1$ completes the proof of statement ii).

In the case when $f > 0$ and $(f, p) = 1$, the extension $K \supset k$ is wild according to Prop. 1.7. Let $\theta = (\beta - 1)/\pi^q$ where q is defined by $f = qp + t$ with $0 \le t < p$. Recall (Remark 1.8) that the integral closure S of R in K is given by $S = R[\prod]$ where $\prod = \theta^n \pi^m$ for integers m and n satisfying $mp + nt = 1$. Once again let ζ denote a primitive p^{th} root of unity and σ the element of $G(K/k)$ for which $\sigma(\beta) = \zeta\beta$. Observe that σ is in the ith ramification group G_i of $K \supset k$ if and only if $\sigma(\prod)/\prod \equiv 1 \bmod \prod^i S$. By substituting $((\beta - 1)/\pi^q)^n \pi^m$ for $\prod$ one can obtain the equality $\sigma(\prod)/\prod = (\sigma(\beta - 1)/(\beta - 1))^n$, so that σ is in G_i if and only if $(\sigma(\beta - 1)/(\beta - 1))^n \equiv 1 \bmod \prod^i S$. We proceed to show that $(\sigma(\beta - 1)/(\beta - 1))^n - 1$ is in $\prod^{(ep/p-1)-f} U(S)$. First observe that $\beta - 1$ is in $\prod^f U(S)$. For, θ is in $\prod^t U(S)$ because θ^p is in $\pi^t U(S)$, (see paragraph two of the proof of Prop. 1.7), and so the definition $\beta - 1 = \theta\pi^q$ implies that $\beta - 1$ is in $\prod^{qp+t} U(S) = \prod^f U(S)$. Since $\sigma(\beta - 1) - (\beta - 1)$ is in $\prod^{ep/p-1} U(S)$ and $\beta - 1$ is in $\prod^f U(S)$, we have that $\sigma(\beta - 1)/(\beta - 1) - 1$ is in $\prod^{(ep/p-1)-f} U(S)$. Therefore $(\sigma(\beta - 1)/(\beta - 1))^n - 1$ is in $\prod^{(ep/p-1)-f} U(S)$ because n is relatively prime to p. The above observations combine to give us that σ is in G_i if and only if $\prod^{(ep/p-1)-f}$ is in $\prod^i S$, i.e. σ is in G_i if and only if $i \le (ep/p - 1) - f$, and this completes the proof of part iii).

It remains to study the relationship between the absolute field exponent f and the conductor number g. For this the following definition is useful.

DEFINITION. Let f denote the absolute field exponent of a Galois extension $K \supset k$ of degree p. If $f \ge 0$, then the *quotient number* q and the *remainder number* t of $K \supset k$ are the unique integers q and t such that $f = qp + t$ with $0 \le t < p$; if $f = -1$, we define $q = 0$ and $t = 1$.

PROPOSITION 2.2. *Let q denote quotient number and g the conductor number of a Galois extension $K \supset k$ of degree p. If $K \supset k$ is unramified then $q = g + 1$. Otherwise, $q = g$.*

Proof. If $K \supset k$ is unramified, then $f = ep/p - 1$ (Thm. 1.11) and so $q = e/p - 1$. On the other hand, $g = (e/p - 1) - 1$ (Cor. 2.7 of [5]); therefore $q = e/p - 1 = g + 1$ in the unramified case.

We shall make use of Prop. 2.1 to prove that $q = g$ when $K \supset k$ is fiercely ramified or wildly ramified.

If $K \supset k$ is fierce, then the ramification number i of $K \supset k$ is given on

the one hand by $i = (e/p - 1) - q - 1$ (Prop. 2.1), and on the other hand by $i = (e/p - 1) - g - 1$ (Prop. 3.1 of [5]), from which it follows that $q = g$.

Now let x denote the field exponent of a wildly ramified extension $K \supset k$. If $f = -1$, then $q = 0 = g$. For, $x = -1$ when $f = -1$ (Remark 1.12) so that $g = 0$ (see p. 155 of [5]); and $q = 0$ when $f = -1$ according to the above definition of q. If $f \neq -1$, then $i = (ep/p - 1) - f$ by Prop. 2.1; and, the fact that $x \neq -1$ when $f \neq -1$ (Remark 1.12) implies that $i = (ep/p - 1) - gp - h$ where $1 \leq h \leq p - 1$ (Prop. 3.1 of [5]). The equalities $i = (ep/p - 1) - qp - t$ and $i = (ep/p - 1) - gp - h$ together with the inequalities $1 \leq h, t \leq p - 1$ imply that $q = g$ (and $h = t$).

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