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ISOMETRIC IMMERSIONS OF CONSTANT MEAN CURVATURE AND TRIVIALITY OF THE NORMAL CONNECTION*

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0. Introduction. In a recent paper [2] Nomizu and Smyth have determined the hypersurfaces M^n of non-negative sectional curvature isometrically immersed in the Euclidean space $\boldsymbol{R}^{n+1}$ or the sphere S^{n+1} with constant mean curvature under the additional assumption that the scalar curvature of M^n is constant. This additional assumption is automatically satisfied if M^n is compact. In this paper we extend these results to codimension p isometric immersions. We determine the n-dimensional submanifolds M^n of non-negative sectional curvature isometrically immersed in the Euclidean Space $\boldsymbol{R}^{n+p}$ or the sphere $\boldsymbol{S}^{n+p}$ with constant mean curvature under the additional assumptions that M^n has constant scalar curvature and the curvature tensor of the connection in the normal bundle is zero. By constant mean curvature we mean that the mean curvature normal is parallel with respect to the connection in the normal bundle. The assumption that M^n has constant scalar curvature is automatically satisfied if M^n is compact. The assumption on the normal connection is automatically satisfied if $p = 2$ and the mean curvature normal is not zero.

We then investigate isometric immersions of space forms into space forms and obtain conditions that imply the vanishing of the curvature tensor of the connection in the normal bundle. We make some applications of these results and in particular determine the local nature of an isometric immersion of the sphere S^n into the Euclidean space $\boldsymbol{R}^{n+2}$ for $n \geq 4$.

1. Notation and some formulas of Riemannian geometry.

Let $\phi : M^n \to \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold

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$\tilde{M}^{n+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. For all local formulas and computations we may consider ψ as an imbedding and thus identify $x \in M^n$ with $\psi(x) \in \tilde{M}^{n+p}$. The tangent space $T_x(M^n)$ is identified with a subspace of $T_x(\tilde{M}^{n+p})$. The normal space $T_x^\perp$ is the subspace of $T_x(\tilde{M}^{n+p})$ consisting of all $X \in T_x(\tilde{M}^{n+p})$ which are orthogonal to $T_x(M^n)$ with respect to the Riemannian metric g. Let ∇ (respectively $\tilde{\nabla}$) denote the covariant differentiation in M^n (respectively $\tilde{M}^{n+p}$) and let D denote the covariant differentiation in the normal bundle. We will refer to ∇ as the tangential connection and to D as the normal connection.

To each $\xi \in T_x^\perp$ is associated a linear transformation of $T_x(M^n)$ in the following way. Extend ξ to a normal vector field defined in a neighborhood of x and define $-A_\xi X$ to be the tangential component of $\tilde{\nabla}_X \xi$ for $X \in T_x(M^n)$. $A_\xi X$ depends only on ξ at x and X. Given an orthonormal basis $\xi_1, \cdots, \xi_p$ of $T_x^\perp$, we write $A_\alpha = A_{\xi_\alpha}$ and call the A_α's the second fundamental forms associated with $\xi_1, \cdots, \xi_p$. If $\xi_1, \cdots, \xi_p$ are now orthonormal normal vector fields in a neighborhood of x, they determine normal connection forms $s_{\alpha\beta}$, in a neighborhood of x, by

$$D_X \xi_\alpha = \sum_\beta s_{\alpha\beta}(X) \xi_\beta$$

for X tangent to M^n.

Let R, $\tilde{R}$, and R^N be the curvature tensors for ∇, $\tilde{\nabla}$, and D, respectively, and S the Ricci tensor (of type 1-1) for M^n as defined in [1]. If $X, Y \in T_x(M^n)$ we let $X \wedge Y$ denote the skew symmetric endomorphism:

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Let X and Y be tangent to M^n and $\xi_1, \cdots, \xi_p$ orthonormal normal vector fields. We then have the following relationships (in this paper Greek indices run form 1 to p and Latin indices run from 1 to n, except when noted):

(1) $\tilde{\nabla}_X Y = \nabla_X Y + \sum_\alpha g(A_\alpha X, Y)\xi_\alpha$

(2) $g(A_\alpha X, Y) = g(X, A_\alpha Y)$

(3) $\tilde{\nabla}_X \xi_\alpha = -A_\alpha X + D_X \xi_\alpha = -A_\alpha X + \sum_\beta s_{\alpha\beta}(X)\xi_\beta$

(4) $s_{\alpha\beta} + s_{\beta\alpha} = 0$

(5) $R(X, Y) = \tilde{c}(X \wedge Y) + \sum_\alpha A_\alpha X \wedge A_\alpha Y$——Gauss equation

(6) $(\nabla_X A_\alpha)Y - \sum_\beta s_{\alpha\beta}(X)A_\beta Y = (\nabla_Y A_\alpha)X - \sum_\beta s_{\alpha\beta}(Y)A_\beta X$

——Codazzi's equation

(7) $(\nabla_X s_{\alpha\beta})Y - (\nabla_Y s_{\alpha\beta})X = 2(ds_{\alpha\beta})(X,Y)$

$= X\cdot s_{\alpha\beta}(Y) - Y\cdot s_{\alpha\beta}(X) - s_{\alpha\beta}([X,Y])$

$= g([A_\alpha, A_\beta])X, Y) + \sum_\gamma \{s_{\alpha\gamma}(X)s_{\gamma\beta}(Y) - s_{\alpha\gamma}(Y)s_{\gamma\beta}(X)\}$ ——Ricci equation

(8) $R^N(X,Y)\xi_\alpha = \sum_\beta g([A_\alpha, A_\gamma]X, Y)\xi_\beta$

$= \sum_\beta \{2(ds_{\alpha\beta})(X,Y) + \sum_\gamma \{s_{\alpha\gamma}(Y)s_{\gamma\beta}(X) - s_{\alpha\gamma}(X)s_{\gamma\beta}(Y)\}\}\xi_\beta$

(9) $S = (n-1)\tilde{c}I + \sum_\alpha (tr\ A_\alpha)A_\alpha - \sum_\alpha A_\alpha^2$

($tr\ A_\alpha$ = trace $A_\alpha = \sum_i g(A_\alpha E_i, E_i)$, $\{E_i\}$ an orthonormal basis of $T_x(M^n)$; I = the identity transformation)

(10) $tr\ S = n(n-1)\tilde{c} + \sum_\alpha (tr\ A_\alpha)^2 - \sum_\alpha tr\ A_\alpha^2,$

where $tr\ S$ is the scalar curvature.

The mean curvature normal η is defined by

$$\eta = \sum_\alpha (tr\ A_\alpha)\xi_\alpha$$

where the *RHS* (right hand side) is independent of our choice of orthonormal basis of $T_x^\perp$. Note that (10) may be written as

(10′) $tr\ S = n(n-1)\tilde{c} + g(\eta,\eta) - \sum_\alpha tr\ A_\alpha^2.$

Let ∇^* denote the sum of the tangential and the normal connections. ∇^* is the connection in the Whitney sum of the tangent bundle of M^n and the normal bundle of M^n induced by ∇ and D; see proposition 6.3, pg. 82, of Volume I of [1]. Then, letting $\nabla_X^* A_\alpha$ denote $(\nabla_X^* A)_{\xi_\alpha}$, we have

(11) $\nabla_X^* A_\alpha = \nabla_X A_\alpha - \sum_\beta s_{\alpha\beta}(X)A_\beta$

and Codazzi's equation may be written as

(6′) $(\nabla_X^* A_\alpha)Y = (\nabla_Y^* A_\alpha)X.$

We note that (8) implies that $R^N = 0$ at x if and only if $A_\alpha A_\beta = A_\beta A_\alpha$ at x for all α, β; or, equivalently, the A_α's are simultaneously diagonalizable at x. Also, $R^N = 0$ everywhere if and only if for each $x \in M^n$ there exists orthonormal normal vector fields $\xi_1, \cdots, \xi_p$ defined in a neighborhood U of x such that $D\xi_\alpha = 0$ in U, i.e., $s_{\alpha\beta} = 0$ in U. If $R^N = 0$ at $x \in M^n$ we will say that the normal connection is trivial at x; if $R^N = 0$ for all $x \in M^N$ we will say that the normal connection is trivial.

Note that (10′) implies that $\sum_\alpha tr\, A_\alpha^2$ is independent of our choice of orthonormal basis of $T_x^\perp$.

For X, Y tangent to M^n, $K(X \wedge Y)$ will denote the sectional curvature in M^n of the plane spanned by X and Y. $\|T\|^2 = g(T, T)$ for any tensor T. Let $\boldsymbol{R}^k$ denote k-dimensional Euclidean space and $S^k(\tilde{c})$, $\tilde{c} > 0$, will denote the sphere in $\boldsymbol{R}^{k+1}$ of curvature $\tilde{c}$.

All manifolds, immersions, vector fields, and functions are assumed C^∞ unless otherwise stated.

2. Isometric immersions of constant mean curvature.

Let $\psi : M^n \to \tilde{M}^{n+p}(\tilde{c})$ be as in Section 1. Let $f = \sum_\alpha tr\, A_\alpha^2$.

Simons [3] has established a formula for the Laplacian of the second fundamental form of a submanifold in a Riemannian manifold and has made some applications to minimal hypersurfaces of spheres by means of the Laplacian of the function f above. Nomizu and Smyth [2] have obtained the same type of formula for the Laplacian of f for a hypersurface M^n immersed with constant mean curvature in a space of constant sectional curvature by a more direct route than Simons', and derived a new formula for the Laplacian of f involving the sectional curvatures of M^n. In Lemmas 1 and 2 below we extend the formulas of Nomizu and Smyth to codimension p.

LEMMA 1. *If $D\eta = 0$, then*

$$
\begin{aligned}
(12) \qquad \frac{1}{2}\Delta f = \tilde{c} n f - \tilde{c} \sum_\alpha (tr\, A_\alpha)^2 + \sum_{\alpha,\beta} tr[A_\alpha, A_\beta]^2 \\
+ \sum_{\alpha,\beta} (tr\, A_\alpha)(tr\, A_\alpha A_\beta^2) - \sum_{\alpha,\beta} (tr\, A_\alpha A_\beta)^2 + \sum_\alpha \|\nabla^* A_\alpha\|^2
\end{aligned}
$$

where Δ is the Laplacian.

LEMMA 2. *If in addition the normal connection is trivial and we let $\lambda_{i\alpha}$, $1 \leq i \leq n$, $1 \leq \alpha \leq p$, be the eigenvalues of A_α corresponding to eigenvectors E_i (recall $R^N = 0$ implies the A_α's are simultaneously diagonalizable), then*

(13) $$\frac{1}{2}\Delta f = \sum_{\alpha}\sum_{i<j}(\lambda_{i\alpha}-\lambda_{j\alpha})^2(\tilde{c}+\lambda_{i1}\lambda_{j1}+\cdots+\lambda_{ip}\lambda_{jp})+\sum_{\alpha}\|\nabla^* A_\alpha\|^2$$

where $\tilde{c}+\lambda_{i1}\lambda_{j1}+\cdots+\lambda_{ip}\lambda_{jp}=K(E_i\wedge E_j)$.

Proof of Lemma 1. Note that for X tangent to M^n

$$\begin{aligned} D_X\eta &= \sum_{\alpha}(X(tr\ A_\alpha))\xi_\alpha+\sum_{\alpha}(tr\ A_\alpha)D_X\xi_\alpha \\ &= \sum_{\alpha}(X(tr\ A_\alpha))\xi_\alpha-\sum_{\alpha}(\sum_{\beta}s_{\alpha\beta}(X)tr\ A_\beta)\xi_\alpha. \end{aligned}$$

Thus

(14) $$D\eta=0 \text{ if and only if } X(tr\ A_\alpha)-\sum_{\beta}s_{\alpha\beta}(X)tr\ A_\beta=0$$

for each α. Remark: $\tilde{\nabla}_X\eta=0$ for all $X\in T_x(M^n)$ if and only if $\eta=0$. Let $f_\alpha=tr\ A_\alpha^2$, then $\Delta f=\sum_{\alpha}\Delta f_\alpha$. If B is any tensor of type 1-1 on M^n, then for $F=tr\ B^2$ we have

(15) $$\frac{1}{2}\Delta F=tr((\Delta' B)B)+\|\nabla B\|^2,$$

where

$$(\Delta' B)(x)=\sum_{i}\nabla^2 B(;E_i;E_i),$$

$\{E_i\}$ an orthonormal basis of $T_x(M^n)$ and

$$\nabla^2 B(;Y;X)=\nabla_X(\nabla_Y B)-\nabla_{\nabla_X Y}B.$$

Let $K_\alpha(X,Y)=(\nabla^2 A_\alpha)(;Y;X)$. Then

(16) $$K_\alpha(X,Y)=K_\alpha(Y,X)+[R(X,Y),A_\alpha].$$

For $X,Y\in T_x(M^n)$ and an orthonormal basis $\{E_i\}$ of $T_x(M^n)$ extend X, Y, E_i to vector fields in a normal neighborhood of x by parallel translation along geodesics with respect to the connection in M^n. Let $\xi_1,\cdots,\xi_p$ be orthonormal normal vector fields defined in a neighborhood of x. Then

(17) $$\nabla X=\nabla Y=\nabla E_i=0 \quad \text{at} \quad x.$$

Because of (17) we have at x

$$K_\alpha(Y,X)=\nabla_Y(\nabla_X A_\alpha)-\nabla_{\nabla_Y X}A_\alpha=\nabla_Y(\nabla_X A_\alpha).$$

Similarly, at x we have

$$(18) \qquad K_\alpha(Y,X)Y = \nabla_Y\,((\nabla_X A_\alpha)Y)$$
$$= \nabla_Y\,((\nabla_Y A_\alpha)X) + \nabla_Y\,(\sum_\beta (s_{\alpha\beta}(X)A_\beta Y - s_{\alpha\beta}(Y)A_\beta X))$$

where the last line is obtained by using Codazzi's equation. Similarly, at x we have

$$K_\alpha(Y,Y)X = \nabla_Y\,((\nabla_Y A_\alpha)X)$$
$$= K_\alpha(X,Y)Y - [R(X,Y),\ A_\alpha]Y$$
$$- \nabla_Y\,(\sum_\beta (s_{\alpha\beta}(X)A_\beta Y - s_{\alpha\beta}(Y)A_\beta X))$$

where we have used (16) and (18) to get the last line. Thus, at x, we have

$$(19) \qquad \sum_i K_\alpha(E_i,E_i)X = \sum_i K_\alpha(X,E_i)E_i + \sum_i [R(E_i,X), A_\alpha]E_i$$
$$- \sum_i \nabla_{E_i}(\sum_\beta (s_{\alpha\beta}(X)A_\beta E_i - s_{\alpha\beta}(E_i)A_\beta X))$$

We compute the second term on the RHS of (19):

$$\sum_i R(E_i,X)A_\alpha E_i = \sum_i \tilde{c}g(A_\alpha E_i, X)E_i - \sum_i \tilde{c}g(A_\alpha E_i, E_i)X$$
$$+ \sum_{i,\beta} g(A_\alpha E_i, A_\beta X)A_\beta E_i - \sum_{i,\beta} g(A_\alpha E_i, A_\beta E_i)A_\beta X$$
$$= \sum_i \tilde{c}g(E_i, A_\alpha X)E_i - \sum_i \tilde{c}g(A_\alpha E_i, E_i)X$$
$$+ \sum_{i,\beta} A_\beta g(E_i, A_\alpha A_\beta X)E_i - \sum_{i,\beta} g(A_\beta A_\alpha E_i, E_i)A_\beta X$$
$$= \tilde{c}A_\alpha X - \tilde{c}(tr\ A_\alpha)X + \sum_\beta A_\beta A_\alpha A_\beta X - \sum_\beta (tr\ A_\beta A_\alpha)A_\beta X.$$

Similarly we can compute $\sum_i A_\alpha R(E_i,X)E_i$. We find

$$(20) \qquad \sum_i [R(E_i,X), A_\alpha]E_i = n\tilde{c}A_\alpha X - \tilde{c}(tr\ A_\alpha)X$$
$$+ \sum_\beta [A_\beta, A_\alpha A_\beta]X + \sum_\beta (tr\ A_\beta)A_\alpha A_\beta X - \sum_\beta (tr\ A_\beta A_\alpha)A_\beta X.$$

To compute the first term on the RHS of (19),

$$\sum_i K_\alpha(X,E_i)E_i = \sum_i \nabla_X\,((\nabla_{E_i}A_\alpha)E_i),$$

note that A_α symmetric implies that $\nabla_X A_\alpha$ is also symmetric. Thus for an arbitrary vector field Z on M^n, we have in a neighborhood of x

$$\sum_i g((\nabla_{E_i}A_\alpha)E_i, Z) = \sum_i g(E_i, (\nabla_{E_i}A_\alpha)Z)$$
$$= \sum_i g(E_i, (\nabla_Z A_\alpha)E_i) - \sum_{i,\beta} g(E_i, s_{\alpha\beta}(Z)A_\beta E_i)$$

$$+ \sum_{i,\beta} g(E_i, s_{\alpha\beta}(E_i)A_\beta Z) \quad \text{(by Codazzi's equation)}$$
$$= Z \cdot tr\ A_\alpha - \sum_\beta s_{\alpha\beta}(Z) tr\ A_\beta + \sum_{i,\beta} g(s_{\alpha\beta}(E_i)A_\beta E_i, Z)$$
$$= \sum_{i,\beta} g(s_{\alpha\beta}(E_i)A_\beta E_i, Z)$$

by using (14)—— here, and the use of (21) in (22) to get (23), are the only places in the calculation where we use $D\gamma = 0$. Thus

$$\sum_i (\nabla_{E_i} A_\alpha)E_i = \sum_{i,\beta} s_{\alpha\beta}(E_i)A_\beta E_i. \tag{21}$$

Thus the sum of the first and third terms on the RHS of (19) is

$$\sum_{i,\beta} \nabla_X (s_{\alpha\beta}(E_i)A_\beta E_i) - \sum_{i,\beta} \nabla_{E_i}(s_{\alpha\beta}(X)A_\beta E_i - s_{\alpha\beta}(E_i)A_\beta X)$$

which again by (17), is

$$\begin{aligned} \sum_{i,\beta} \{ &(\nabla_X s_{\alpha\beta})(E_i)A_\beta E_i - (\nabla_{E_i} s_{\alpha\beta})(X)A_\beta E_i \\ &+ (\nabla_{E_i} s_{\alpha\beta})(F_i)A_\beta X + s_{\alpha\beta}(E_i)(\nabla_X A_\beta)E_i \\ &+ s_{\alpha\beta}(E_i)(\nabla_{E_i} A_\beta)X - s_{\alpha\beta}(X)(\nabla_{E_i} A_\beta)E_i \}. \end{aligned} \tag{22}$$

If we now use the Ricci equation for the first two terms of (22) and Codazzi's equation for $(\nabla_{E_i} A_\beta)X$ in the fourth term and (21) for the last term we find that (22) equals

$$\begin{aligned} &\sum_{i,\beta} g([A_\alpha, A_\beta]X, E_i)A_\beta E_i \\ &+ \sum_{i,\beta} (\nabla_{E_i} s_{\alpha\beta})(E_i)A_\beta X \\ &+ 2\sum_{i,\beta} s_{\alpha\beta}(E_i)(\nabla_{E_i} A_\beta)X \\ &- \sum_{i,\beta,\gamma} s_{\alpha\beta}(E_i)s_{\beta\gamma}(E_i)A_\gamma X \end{aligned} \tag{23}$$

Note: the first term in (23) is $\sum_\beta A_\beta[A_\alpha, A_\beta]X$. Thus

$$\begin{aligned} \Delta' A_\alpha = n\tilde{c}A_\alpha &- \tilde{c}(tr\ A_\alpha)I + \sum_\beta (tr\ A_\beta)A_\alpha A_\beta \\ &- \sum_\beta (tr\ A_\beta A_\alpha)A_\beta + \sum_\beta [A_\beta, A_\alpha A_\beta] \\ &+ \sum_\beta A_\beta[A_\alpha, A_\beta] + \sum_{i,\beta} (\nabla_{E_i} s_{\alpha\beta})(E_i)A_\beta \\ &+ 2\sum_{i,\beta} s_{\alpha\beta}(E_i)\nabla_{E_i} A_\beta - \sum_{i,\beta,\gamma} s_{\alpha\beta}(E_i)s_{\beta\gamma}(E_i)A_\gamma \end{aligned} \tag{24}$$

and

$$
\begin{aligned}
(25)\qquad \frac{1}{2}\Delta f &= \sum_\alpha tr((\Delta' A_\alpha)A_\alpha) + \sum_\alpha \|\nabla A_\alpha\|^2 \\
&= n\tilde{c}\sum_\alpha tr\ A_\alpha^2 - \tilde{c}\sum_\alpha (tr\ A_\alpha)^2 + \sum_{\alpha,\beta}(tr\ A_\beta)(tr\ A_\alpha A_\beta A_\alpha) \\
&\quad - \sum_{\alpha,\beta}(tr\ A_\beta A_\alpha)^2 + \sum_{\alpha,\beta} tr[A_\beta, A_\alpha A_\beta]A_\alpha \\
&\quad + \sum_{\alpha,\beta} tr\ A_\beta[A_\alpha, A_\beta]A_\alpha + \sum_{i,\alpha,\beta}(\nabla_{E_i}s_{\alpha\beta})(E_i) tr\ A_\beta A_\alpha \\
&\quad + 2\sum_{i,\alpha,\beta} s_{\alpha\beta}(E_i) tr(\nabla_{E_i}A_\beta)A_\alpha \\
&\quad - \sum_{i,\alpha,\beta,\gamma} s_{\alpha\beta}(E_i)s_{\beta\gamma}(E_i) tr\ A_\gamma A_\alpha + \sum_\alpha \|\nabla A_\alpha\|^2.
\end{aligned}
$$

By properties of the trace the first six terms of the *RHS* of (25) reduce to the first five terms of the *RHS* of (12). Since $s_{\alpha\beta} + s_{\beta\alpha} = 0$, the seventh term of the *RHS* of (25) is zero. And the sum of the last three terms of the *RHS* of (25) is $\sum_\alpha \|\nabla^* A_\alpha\|^2$ as is easily seen from (11) and

$$\|\nabla^* A_\alpha\|^2 = \sum_i tr(\nabla^*_{E_i}A_\alpha)(\nabla^*_{E_i}A_\alpha).$$

Proof of Lemma 2. Nomizu and Smyth [2] have shown that for any $n\times n$ symmetric matrix A with eigenvalues $\lambda_1, \cdots, \lambda_n$ that

$$
\begin{aligned}
&\tilde{c}ntr\ A^2 - \tilde{c}(tr\ A)^2 - (tr\ A^2)^2 + tr\ A\ tr\ A^3 \\
&= \sum_{i<j}(\lambda_i - \lambda_j)^2(\tilde{c} + \lambda_i\lambda_j)
\end{aligned}
$$

To prove Lemma 2 it suffices to show that for any $n\times n$ commuting symmetric matrices A and B with eigenvalues λ_i and μ_i respectively that

$$
\begin{aligned}
(26)\qquad &tr\ A\ tr\ B^2A + tr\ B\ tr\ A^2B - (tr\ AB)^2 - (tr\ BA)^2 \\
&= \sum_{i<j}(\lambda_i - \lambda_j)^2\mu_i\mu_j + \sum_{i<j}(\mu_i - \mu_j)^2\lambda_i\lambda_j
\end{aligned}
$$

Equation (26) is proved by simultaneously diagonalizing A and B and calculating the *LHS* (left hand side).

LEMMA 3. *Suppose M^n has non-negative sectional curvatures, $D\eta = 0$, and the normal connection is trivial. If M^n has constant scalar curvature or M^n is compact, then $\Delta f = 0$ and $\|\nabla^* A_\alpha\|^2 = 0$. If M^n is compact then M^n has constant scalar curvature.*

Proof. If M^n has constant scalar curvature, then since $g(\eta, \eta)$ is constant (10′) implies that f is constant; hence $\Delta f=0$. Lemma 2 implies that $\Delta f \geqq 0$.

Hence, if M^n is compact, then $\Delta f = 0$ (cf. page 338, volume II of [1]) and f is constant. Since $g(\eta, \eta)$ is constant, (10′) implies that M^n has constant scalar curvature. Since all the terms on the RHS of (13) are non-negative and $\Delta f = 0$ we conclude that $\|\nabla^* A_\alpha\|^2 = 0$.

LEMMA 4. *If the assumption of Lemma* 3 *are satisfied and* $\tilde{M}^{n+p} = \boldsymbol{R}^{n+p}$, *then for each* $x \in M^n$ *there exist orthonormal normal vector fields* $\xi_1, \cdots, \xi_p$ *defined in a neighborhood* U *of* x *such that*

(*a*) $D\xi_\alpha = 0$ *in* U, *i.e.*, $s_{\alpha\beta} = 0$ *in* U

(*b*)
$$A_\alpha = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \lambda_\alpha I_{m_\alpha} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

where I_{m_α} *is the* $m_\alpha \times m_\alpha$ *identity matrix and the zero matrix in the upper left hand corner is of degree* $m_1 + \cdots + m_{\alpha-1}$ *and the* A_α*'s are expressed with respect to their common orthonormal eigenvectors* $E_1, \cdots, E_n$. *Note that* $A_\alpha = 0$ *if* $m_1 + \cdots + m_{\alpha-1} = n$ *and we may assume that* $A_\alpha = 0$ *implies that* $A_\beta = 0$ *for* $\beta > \alpha$

(*c*) *Each* λ_α *is constant in* U.

Proof. Since the normal connection is trivial there exist orthonormal normal vector fields $\xi_1, \cdots, \xi_p$ defined in a neighborhood U of x such that $D\xi_\alpha = 0$ in U. With such a choice of $\xi_1, \cdots, \xi_p$ we have $\nabla^*_X A_\alpha = \nabla_X A_\alpha$ for X tangent to M^n. By Lemma 3, $\|\nabla^* A_\alpha\| = 0$ and thus $\|\nabla A_\alpha\| = 0$. Hence the eigenvalues of A_α are constant. If $\xi'_\beta = \sum_\alpha O_{\alpha\beta}\xi_\alpha$, $[O_{\alpha\beta}]$ an orthogonal matrix with constant entries, then $D\xi'_\beta = 0$ in U and $A'_\beta = \sum O_{\alpha\beta} A_\alpha$. In what follows we will begin with any $\xi_1, \cdots, \xi_p$ such that $D\xi_\alpha = 0$ in U and show that there exists an orthogonal matrix $[O_{\alpha\beta}]$ with constant entries such that the second fundamental forms A'_β with respect to $\xi'_\beta = \sum_\alpha O_{\alpha\beta}\xi_\alpha$ have the desired property (b). The claim is clearly true if all the $A'_\alpha s = 0$ at x (and therefore by constancy of the eigenvalues $A_\alpha = 0$ in a neighborhood of x). If this is not the case we distinguish three cases:

(i) all sectional curvatures of $M^n > 0$ at x,

(ii) all sectional curvatures of $M^n = 0$ at x,

(iii) at least one non-zero sectional curvature at x and at least one sectional curvature that is zero at x.

Suppose $\xi_1, \cdots, \xi_p$ and U have been chosen such that (a) is satisfied; thus each A_α has constant eigenvalues in U.

Case (i): Lemmas 2 and 3 imply that $A_\alpha = \lambda_\alpha I$. We may assume $\lambda_1 \neq 0$. Let

$$\xi_1' = (\sum_\alpha \lambda_\alpha \xi_\alpha)/(\sum_\alpha \lambda_\alpha^2)^{1/2}$$

and

$$\bar{\xi}_\beta = (\lambda_1 \xi_\beta - \lambda_\beta \xi_1)/(\lambda_1^2 + \lambda_\beta^2)^{1/2}$$

for $\beta > 1$. Then $A_1' = \lambda I$, $\lambda \neq 0$, and $\bar{A}_\beta = 0$ for $\beta > 1$, $\bar{\xi}_\beta \perp \xi_1'$. Use the Gram-Schmidt orthogonalization process on $\bar{\xi}_2, \cdots, \bar{\xi}_p$ to obtain $\bar{\xi}_2', \cdots, \bar{\xi}_p'$. Then $A_\beta' = 0$ for $\beta > 1$.

Case (ii): Let $A_\alpha = \begin{pmatrix} \lambda_{1\alpha} & & \\ & \ddots & \\ & & \lambda_{n\alpha} \end{pmatrix}$

when expressed with respect to the common eigenvectors $E_1, \cdots, E_n$ of the A_α's. We may assume $\lambda_{11} \neq 0$. Let

$$\xi_1' = (\sum_\alpha \lambda_{1\alpha} \xi_\alpha)/(\sum_\alpha \lambda_{1\alpha}^2)^{1/2},$$

$$\bar{\xi}_\alpha = (\lambda_{11} \xi_\alpha - \lambda_{1\alpha} \xi_1)/(\lambda_{1\alpha}^2 + \lambda_{11}^2)^{1/2} \text{ for } \alpha > 1.$$

Again, $\bar{\xi}_\alpha \perp \xi_1'$. Use the Gram-Schmidt orthogonalization process on $\bar{\xi}_2, \cdots, \bar{\xi}_p$ to obtain $\xi_2', \cdots, \xi_p'$. Then, for $\alpha \geq 2$,

$$A_\alpha' = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & * & & \\ \vdots & & \ddots & \\ 0 & & & * \end{array}\right)$$

and $\lambda_{11}' \neq 0$. Thus we may assume that $\lambda_{1\alpha} = 0$ for $\alpha > 1$, $\lambda_{11} \neq 0$. Since $0 = K(E_1 \wedge E_j) = \sum_\alpha \lambda_{1\alpha} \lambda_{j\alpha} = \lambda_{11} \lambda_{j1}$ for $j > 1$, we have $\lambda_{j1} = 0$ for $j > 1$. If one of the A_α's, for $\alpha \geq 2$, is not zero we may assume that it is A_2 and apply the above argument to $\xi_2, \cdots, \xi_p$ and $A_2, \cdots, A_p$ restricted to the span $\{E_2, \cdots, E_n\}$. We obtain $\lambda_{j2} = 0$ for $j > 2$ and $\lambda_{2\alpha} = 0$ for $\alpha > 2$. It is now clear that an induction argument will work.

Case (iii): Order $E_1, \cdots, E_n$ so that $K(E_1 \wedge E_l) > 0$ for $2 \leq l \leq m_1$, and $K(E_1 \wedge E_l) = 0$ for $l > m_1$. Then Lemmas 2 and 3 imply that $\lambda_{1\alpha} = \lambda_{l\alpha}$ for $1 \leq l \leq m_1$. Define ξ'_α as in case (ii). We see that we may assume that $\lambda_{l\alpha} = 0$ for $1 \leq l \leq m_1$, $2 \leq \alpha \leq p$. Then $K(E_1 \wedge E_l) = \lambda_{11}\lambda_{l1} = 0$ for $l > m_1$ and thus $\lambda_{l1} = 0$ for $l > m_1$. If $K(E_i \wedge E_j) \neq 0$ for some $i, j > m_1$ we repeat the above argument applied to $\xi_2, \cdots, \xi_p$ and $A_2, \cdots, A_p$ restricted to the span $\{E_{m_1+1}, \cdots, E_n\}$. If $K(E_i \wedge E_j) = 0$ for all $i, j > m_1$ we apply the argument of case (ii) to $\xi_2, \cdots, \xi_p$ and $A_2, \cdots, A_p$ restricted to the span $\{E_{m_1+1}, \cdots, E_n\}$. In either case we obtain the desired form for A_1 and A_2. It is clear that an induction argument will work.

Lemma 5. *Let M^n be isometrically immersed in S^{n+p} such that*

(a) $f = \sum_\alpha tr\ A_\alpha^2$ *is constant on* M^n

(b) $D\eta = 0$, *and*

(c) *the normal connection is trivial.*

Then, if we consider S^{n+p} as isometrically immersed in $\boldsymbol{R}^{n+p+1}$, conditions (a), (b), and (c) are also satisfied. (Of course f, η, and the normal connection are now taken with respect to M^n immersed in $\boldsymbol{R}^{n+p+1}$).

Proof. Let ξ be the inward normal on S^{n+p} and let $\xi_1, \cdots, \xi_p$ be orthonormal normal vectors to M^n but tangent to S^{n+p}. Let A_α be the corresponding second fundamental forms for M^n immersed in S^{n+p}. Let A' and A'_α be the second fundamental forms for ξ and ξ_α for M^n considered as immersed in $\boldsymbol{R}^{n+p+1}$. Let D (respectively D') be the covariant differentiation in the normal bundle for M^n immersed in S^{n+p} (respectively $\boldsymbol{R}^{n+p+1}$) Then it is easy to show that $A'_\alpha = A_\alpha$, $A' = I$, $D\xi_\alpha = D'\xi_\alpha$, and $D'\xi = 0$, from which the conclusion readily follows.

Consider the following example. Let $M^{n_i} = S^{n_i}\left(\frac{1}{r_i^2}\right)$ be isometrically immersed in $\boldsymbol{R}^{n_i+1}$ by ϕ_i for $i = 1, \cdots, l-1$. For $n_i = 1$ we assume $\phi_i(S^1)$ is a circle; for $n_i \geq 2$, ϕ_i is unique up to an isometry of $\boldsymbol{R}^{n_i+1}$. Let ξ_i be the inward normal to M^{n_i}. Let $M^{n_l} = \boldsymbol{R}^{n_l}$ and let $\boldsymbol{R}^{n_l}$ be isometrically immersed in $\boldsymbol{R}^{n_l+p+1-l}$ such that the image is of the form $S^1\left(\frac{1}{r_1^2}\right) \times \cdots \times S^1\left(\frac{1}{r_t^2}\right) \times \boldsymbol{R}^{n_l-t}$, where each $S^1\left(\frac{1}{r_k^2}\right)$ is a circle of radius r_k in some Euclidean plane N_k, $N_k \perp N_m$ for $k \neq m$, and $N_k \perp \boldsymbol{R}^{n_l-t}$. Let ξ_{l+k-1} be the inward normal to $S^1\left(\frac{1}{r_k^2}\right)$ in N_k and let $\xi_{l+t}, \cdots, \xi_p$ be normal to $\boldsymbol{R}^{n_l-t}$ and N_k and constant. Let $M^n = M^{n_1}\times \cdots \times M^{n_l}$ and let ϕ be the product immersion. We may consider ξ_α as normal to M^n immersed in $\boldsymbol{R}^{n+p}$. Let A_α be the corresponding second fundamental forms. Then the normal connection is trivial, $D\eta = 0$, $f =$ constant, all sectional curvatures of $M^n \geq 0$, $D\xi_\alpha = 0$ on M^n, and the A_α's have the form of (b) in Lemma 4.

Let ψ be as in Lemma 4. We will show that M^n is locally a product of spheres and possibly a Euclidean space, in the manner of the example above.

Let $\xi_1, \cdots, \xi_p$ be chosen as in Lemma 4. We may assume $\lambda_\alpha \neq 0$ for $1 \leq \alpha \leq l-1$, $\lambda_\alpha = 0$ for $\alpha \geq l$ (if all $\lambda_\alpha = 0$ then the immersion is totally geodesic). Define distributions $T_1, T_2, \cdots, T_l$ by

$$T_\alpha(y) = \{X \in T_y(M^n) \mid A_\alpha X = \lambda_\alpha X\} \quad \text{for} \quad \alpha \leq l-1$$

$$T_l(y) = \{X \in T_y(M^n) \mid A_\alpha X = 0, \ 1 \leq \alpha \leq p\}$$

Let $n_\alpha = \dim T_\alpha$ (n_l may be 0). Assume M^n is connected, simply connected, and complete. Then each T_α is globally defined (for $\xi \in T_x^\perp$ parallel translation of ξ with respect to the normal connection is independent of path if $R^N = 0$ everywhere and M^n is simply connected). Each T_α has constant dimension and is differentiable (the eigenspaces of the A_α have constant dimension and thus we may find differentiable orthonormal eigenvector fields). The T_α's are orthogonal to each other and

$$T_x(M^n) = T_1(x) + \cdots + T_l(x) \quad \text{(orthogonal direct sum)} \tag{27}$$

LEMMA 6. *Each T_α is involutive, totally geodesic ($X, Y \in T_\alpha$ implies that $\nabla_X Y \in T_\alpha$), and parallel ($Y \in T_\alpha$, X tangent to M^n implies that $\nabla_X Y \in T_\alpha$).*

Proof. $0 = (\nabla_X A_\alpha)Y = \nabla_X(A_\alpha Y) - A_\alpha(\nabla_X Y)$ for X, Y tangent to M^n since $\nabla A_\alpha = 0$. If Y is an eigenvector field of A_α belonging to the eigenvalue λ_α (a constant) we have

$$\lambda_\alpha \nabla_X Y - A_\alpha(\nabla_X Y) = 0.$$

Thus $\nabla_X Y$ is an eigenvector field of A_α with eigenvalue λ_α. Thus each T_α is totally geodesic and therefore, because of (27), each T_α is parallel.

Let $x \in M^n$ and let M^{n_α} be the maximal integral submanifold of T_α through x. From Lemma 6 we conclude that

$$M^n = M^{n_1} \times \cdots \times M^{n_l} \quad \text{(Riemannian product)}$$

If $n_\alpha = 1$, then $M^{n_\alpha} = \boldsymbol{R}$ (we are assuming M^n is simply connected, complete). If $n_\alpha \geqq 2$, then the curvature tensor of M^{n_α} is the restriction of the curvature tensor of M^n since M^{n_α} is totally geodesic in M^n. Therefore the sectional curvature of M^{n_α} is constant and equals λ_α^2. Also, $M^{n_\iota} = \boldsymbol{R}^{n_\iota}$. Thus M^n is a product of spheres and possibly a Euclidean space. Clearly, the corresponding local result is true if we do not assume completeness since we only used completeness to obtain M^{n_α} as the entire sphere or Euclidean space.

The second fundamental forms and the normal connection forms of our isometric immersion ψ with respect to $\xi_1, \cdots, \xi_p$, chosen as in Lemma 4, are the same as those of our example ϕ. Thus by the classical rigidity theorem (see [1], volume 2, page 45, for the case $p = 1$) $\psi = \tau \circ \phi$ where τ is an isometry of $\boldsymbol{R}^{n+p}$. If M^n is complete and connected but not simply connected, let $\bar{M}^n$ be its simply connected Riemannian covering manifold and let π be the covering map. Define $\bar{\psi}$ by $\bar{\psi} = \psi \circ \pi$. Then $\bar{\psi}$ satisfies the assumptions of Lemma 4 and by the above there exists an isometry τ of $\boldsymbol{R}^{n+p}$ such that $\bar{\psi} = \tau \circ \phi$. If ϕ is 1-1, so is $\bar{\psi}$. If $\bar{\psi}$ is 1-1, then π and ψ are 1-1. Also, ϕ is 1-1 except possibly when $\phi(M^n)$ contains an S^1 as one of its products.

If $\tilde{M}^{n+p} = S^{n+p}$ and the hypothesis of Lemma 3 are satisfied then consider M^n as immersed in $\boldsymbol{R}^{n+p+1}$. Lemma 5 implies $\psi(M^n)$ is of the form $\phi(M^n)$ and hence a product of spheres, assuming M^n is complete.

We summarize our results as follows.

THEOREM 1. *Let ψ be an isometric immersion of an n-dimensional, connected, complete Riemannian manifold M^n of non-negative sectional curvatures into $\boldsymbol{R}^{n+p}$ or S^{n+p}. Suppose that the mean curvature normal is parallel with respect to the normal connection and that the curvature tensor of the normal connection is zero. If either M^n is compact or has constant scalar curvature, then*

$$\psi(M^n) = M^{n_1} \times \cdots \times M^{n_\iota}$$

where each M^{n_i} is an n_i-dimensional sphere of some radius contained in some Euclidean space N^{n_i+1} of dimension $n_i + 1$, $N^{n_i+1} \perp N^{n_j+1}$ for $i \neq j$; except possibly one of the M^{n_i} is a Euclidean space $= N^{n_i}$ (this can only occur if $\tilde{M}^{n+p} = \boldsymbol{R}^{n+p}$). Furthermore, the immersion is an imbedding except possibly when some $M^{n_i} = S^1\left(\frac{1}{r^2}\right)$, a circle of radius r in some Euclidean plane. The corresponding local result is true with the assumption of constant scalar curvature.

We also have:

THEOREM 1′. *The assumption on the normal connection in Theorem* 1 *is not necessary in the following cases:*

(a) $p=2$ *and* $\eta\neq 0$,

(b) M^n *has constant sectional curvature* $\tilde{c}$, $p=2$, $\eta=0$, *and* $\tilde{M}^{n+2}(\tilde{c})=S^{n+2}(\tilde{c})$.

Proof. The proof will follow from Lemmas 7 and 8 below.

LEMMA 7. *Let* $\phi: M^n\to\tilde{M}^{n+2}(\tilde{c})$. *If* $D\eta=0$ *and* $\eta\neq 0$, *then the normal connection is trivial.*

Proof. Let ξ_1 and ξ_2 be orthonormal normal vector fields defined in a neighborhood U of x such that $\xi_1=\frac{\eta}{\|\eta\|}$. Now $D\eta=0$ implies $D\xi_1=0$ and hence $s_{12}=0$ in U. This implies the normal connection is trivial, as remarked in Section 1.

Note that if M^n is compact and $\tilde{M}^{n+p}=\boldsymbol{R}^{n+p}$, then $\eta\neq 0$.

LEMMA 8. *Let* M^n *have constant sectional curvature* $\tilde{c}$ *and isometrically immersed as a minimal submanifold of* $\tilde{M}^{n+2}(\tilde{c})$, *then the immersion is totally geodesic.*

Proof. The relative nullity is $\geqq n-2$ (see [1]). Thus, if

$$A_1=\begin{pmatrix}\lambda & & & & & \\ & -\lambda & & & & \\ & & 0 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 0\end{pmatrix},\qquad A_2=\left(\begin{array}{cc|c} a & b & \\ b & -a & 0 \\ \hline 0 & & 0\end{array}\right)$$

when represented with respect to the eigenvectors $E_1,\cdots,E_n$ of A_1 we have

$$K(E_1\wedge E_2)=\tilde{c}-\lambda^2-a^2-b^2=\tilde{c}.$$

Thus $\lambda=a=b=0$ and the immersion is totally geodesic.

Our results clearly imply the following Corollary to Theorem 1.

COROLLARY. *Let* $\phi: M^n\to\tilde{M}^{n+p}(\tilde{c})$ *be as in Theorem* 1. *Further assume that the sectional curvatures of* M^n *are strictly greater than zero. Then* M^n *has constant sectional curvature and is isometric to a sphere, and* $\phi(M^n)$ *is the usual sphere in some* $\boldsymbol{R}^{n+1}$.

3. Isometric immersions of space forms into space forms.

Let $\psi : M^n(c) \to \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of a Riemannian manifold $M^n(c)$ of constant sectional curvature c into a Riemannian manifold $\tilde{M}^{n+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$.

THEOREM 2. *Let $p=2$, $n \geqq 3$.*

(a) *If $c \neq \tilde{c}$, then the curvature tensor of the normal connection is zero.*

(b) *If $c=\tilde{c}$, then for each $x \in M^n$ the curvature tensor of the normal connection is zero at x or the relatively nullity (see [1]) at x is $n-2$.*

THEOREM 3. *If $p=3$, $n \geqq 4$, $D\eta=0$, $\eta \neq 0$, then we have (a) and (b) of Theorem 2.*

To prove Theorems 2 and 3 we will show that the second fundamental forms A_α commute. The proof is quite algebraic.

LEMMA 9. *Let B be a symmetric linear transformation defined on an inner product space V of dimension n. Let $E_1, \cdots, E_n$ be an orthonormal basis of V and $[B_{ij}]$ the matrix representing V with respect to this basis. If $BE_i \wedge BE_j = \sigma_{ij} E_i \wedge E_j$ then*

$$B_{ki}B_{lj} - B_{li}B_{kj} = 0 \text{ for } (k, l) \neq (i, j),\ k<l,\ i<j. \tag{28}$$

Proof.

$$\begin{aligned}\sigma_{ij}E_i \wedge E_j = BE_i \wedge BE_j &= \sum_{k,l} B_{ki}B_{lj}E_k \wedge E_l \\ &= \sum_{k<l}(B_{ki}B_{lj} - B_{li}B_{kj})E_k \wedge E_l.\end{aligned}$$

But $\{E_k \wedge E_l : k<l\}$ are linearly independent in the space of skew symmetric endomorphisms of V, from which the lemma follows.

LEMMA 10. *Let B be as in Lemma 9. Then for even n,*

$$\operatorname{Det} B = (-1)^\sigma \prod_{k=1}^{\frac{n}{2}} (B_{\sigma(2k-1)\sigma(2k-1)}B_{\sigma(2k)\sigma(2k)} - B^2_{\sigma(2k-1)\sigma(2k)}) \tag{29a}$$

where σ is any permutation of $1, \cdots, n$, and $(-1)^\sigma$ denotes the sign of σ.

For odd n,

$$\operatorname{Det} B = (-1)^\sigma \left[\prod_{k=1}^{\frac{n-1}{2}} (B_{\sigma(2k-1)\sigma(2k-1)}B_{\sigma(2k)}B_{\sigma(2k)} - B^2_{\sigma(2k-1)\sigma(2k)})\right] B_{\sigma(n)\sigma(n)} \tag{29b}$$

Proof.

$$BE_{\sigma(1)} \wedge BE_{\sigma(2)} \wedge \cdots \wedge BE_{\sigma(n)}$$
$$= (-1)^{\sigma}(\text{Det } B)E_i \wedge E_2 \wedge \cdots \wedge E_n.$$

But by Lemma 9

$$BE_i \wedge BE_j = (B_{ii}B_{jj} - B_{ij}^2)\,(E_i \wedge E_j).$$

LEMMA 11. *Let $n \geq 3$. Let B be as in Lemmas 9 and 10. If B^2 is diagonal when expressed with respect to $E_1, \cdots, E_n$ and the rank of B is n, then B is diagonal when expressed with respect to $E_1, \cdots, E_n$.*

Proof. Let $\mu_1, \cdots, \mu_n$ be the eigenvalues of B and $(B^2)_{ij} = \mu_i^2\delta_{ij}$ where $\delta_{ij}=0$ for $i \neq j$, and $\delta_{ii}=1$. Suppose n is even. Then, since $\text{Det } B = \prod_{i=1}^{n} \mu_i$ and by Lemma 10,

$$0 \neq \prod_{i=1}^{n} \mu_i^2 = \prod_{k=1}^{\frac{n}{2}} (B_{\sigma(2k-1)\sigma(2k-1)}B_{\sigma(2k)\sigma(2k)} - B^2{}_{\sigma(2k-1)\sigma(2k)})^2. \tag{30}$$

And

$$\sum_k B_{ik}^2 = \mu_i^2 \tag{31}$$

$$\prod_1^n \mu_i^2 = \prod_{j=1}^{n} (\sum_k B_{jk}^2) \tag{32}$$

$$\begin{aligned} \mu_r^2\mu_s^2 &= (\sum_k B_{rk}^2)(\sum_l B_{sl}^2) \geq (B_{rr}^2 + B_{rs}^2)(B_{ss}^2 + B_{rs}^2) \\ &= B_{rr}^2B_{ss}^2 + (B_{rr}^2 + B_{ss}^2)B_{rs}^2 + B_{rs}^4 \\ &\geq B_{rr}^2B_{ss}^2 - 2B_{rr}B_{ss}B_{rs}^2 + B_{rs}^4 = (B_{rr}B_{ss} - B_{rs}^2)^2 \text{ for } r \neq s \end{aligned} \tag{33}$$

Comparing (30) and (32) we see that all the inequalities in (33) are equalities and hence

$$\mu_r^2\mu_s^2 = B_{rr}B_{ss} - B_{rs}^2 \text{ for } r \neq s \tag{34}$$

$$B_{rk}B_{sl} = 0 \text{ for } k \neq r \text{ or } s,\ l \neq r \text{ or } s. \tag{35}$$

Thus if $n \geq 3$ we conclude that $B_{ij} = 0$ for $i \neq j$. For odd n a similar argument holds.

Proof of Theorem 2, Choose orthonormal normal vectors ξ_1 and ξ_2 at $x \in M^n$ such that $tr\ A_2 = 0$ (If $\eta = 0$, any ξ_1 and ξ_2 will do; if $\eta \neq 0$, let

$\xi_1 = \frac{\eta}{\|\eta\|}$). Let $A_1 = A$ and $A_2 = B$. Diagonalize A with respect to its eigenvectors $E_1, \cdots, E_n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ respectively and express B with respect to $E_1, \cdots, E_n$. Then the Gauss equation and the Ricci tensor imply that

$$BE_i \wedge BE_j = (B_{ii}B_{jj} - B_{ij}^2)E_i \wedge E_j$$

for $i < j$ and B^2 is diagonal. If rank $B = n$ then we may conclude that B is diagonal and hence $AB = BA$. If rank $B < n$, then one of its eigenvalues is zero, say $\mu_1 = 0$. But $\mu_1^2 = \sum_k B_{1k}^2$. Thus $B_{1k} = 0$ for all k. Since

$$K(E_1 \wedge E_j) = \tilde{c} + \lambda_1\lambda_j = c$$

we get $\lambda_1\lambda_j = c - \tilde{c}$ for $j > 1$. If $c \neq \tilde{c}$ then $\lambda_j = \frac{c - \tilde{c}}{\lambda_1}$ for $j \geq 2$. Hence $AB = BA$, proving (a). If $c = \tilde{c}$ we can obtain (b) by noting that the relative nullity $\geq n - 2$, and therefore both A and B have rank ≤ 2. Recall that $\mu_i^2 = \sum_k B_{ik}^2$. Thus if rank $B = 1$, then B is diagonal. If rank $A = 1$ and rank $B = 2$, say $\mu_1 = \mu = -\mu_2 \neq 0$, then

$$\tilde{c} = K(E_1 \wedge E_2) = \tilde{c} - \mu^2.$$

Thus $\mu = 0$, contradicting rank $B = 2$. If rank $A =$ rank $B = 2$ we may suppose $\mu_1 = \mu = -\mu_2 \neq 0$. Then

$$K(E_1 \wedge E_2) = \tilde{c} + \lambda_1\lambda_2 - \mu^2 = \tilde{c}.$$

We conclude that $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. Thus A and B have the same null space. We may also prove (b) without appealing to the above fact on the relative nullity by a somewhat longer algebraic argument.

Proof of Theorem 3. Choose orthonormal normal vector fields ξ_1, ξ_2, and ξ_3 defined in a neighborhood U of x such that $\xi_1 = \frac{\eta}{\|\eta\|}$. Since $D\eta = 0$ implies that $D\xi_1 = 0$ we have $s_{1\alpha} = s_{\alpha 1} = 0$ in U. The Ricci equation then implies that A_1 and A_α commute. Let $A_1 = A$, $A_2 = B$, and $A_3 = C$. If we simultaneously diagonalize A and B, then the Gauss equation implies that

$$CE_i \wedge CE_j = (C_{ii}C_{jj} - C_{ij}^2)E_i \wedge E_j \text{ for } i < j$$

where $E_1, \cdots, E_n$ are the common eigenvectors of A and B corresponding to eigenvalues $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$, respectively. Let $\sigma_1, \cdots, \sigma_n$ be the eigenvalues of C; thus $\sigma_1^2, \cdots, \sigma_n^2$ are the eigenvalues of C^2 with the eigenvectors $E_1, \cdots, E_n$ above by the equation for the Ricci tensor. If one of B or C has rank ≥ 3 we may suppose it is C and apply Lemma 11 to $\bar{C} = C$ restricted to the image of C, say the span $\{E_{k+1}, \cdots, E_n\}$. Noting that

$$C = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{C} \end{array}\right)$$

when represented with respect to $E_1, \cdots, E_n$ we obtain the desired result. If one of B or C has rank ≤ 1, then we may suppose it is C. Then C^2 diagonal implies C is diagonal. Thus we are left to consider the case when both B and C have rank 2. Suppose B and C have rank 2. Let σ_1, σ_2 be the non-zero eigenvalues of C. Let

$$C = \left(\begin{array}{cc|c} a & b & \\ b & -a & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) \quad (\text{recall } tr\ C = 0)$$

Since $AC = CA$ we have $\lambda_1 b = \lambda_2 b$. If $b = 0$ we are done. If $b \neq 0$ then $\lambda_1 = \lambda_2$. Let $\lambda_1 = \lambda_2 = \lambda$. Then $K(E_1 \wedge E_j) = K(E_2 \wedge E_j)$ for $j \geq 3$ implies that $\mu_1\mu_j = \mu_2\mu_j$. Since rank $B = 2$, we see that $\mu_1 = 0$ if and only if $\mu_2 = 0$. If $\mu_1 = \mu_2 = 0$ then $BC = CB$. Thus we are reduced to considering the case that B and C have rank 2, the same null space, and $\lambda_1 = \lambda_2 = \lambda$. For $c \neq \tilde{c}$ we will show that this does not occur. From

$$K(E_1 \wedge E_j) - \tilde{c} = \lambda\lambda_j = c - \tilde{c},$$

we get $\lambda_j = \dfrac{c - \tilde{c}}{\lambda}$ for $j \geq 3$. Also

$$K(E_j \wedge E_k) - \tilde{c} = \lambda_j\lambda_k = c - \tilde{c} = \frac{(c - \tilde{c})^2}{\lambda^2}$$

for $j > k \geq 3$. Here we use $n \geq 4$. Thus $\lambda^2 = c - \tilde{c}$. But

$$K(E_1 \wedge E_2) - \tilde{c} = \lambda^2 - \mu^2 - \sigma^2 = c - \tilde{c}$$

where $\mu_1 = \mu = -\mu_2$ and $\sigma_1 = \sigma = -\sigma_2$. Thus $\mu = \sigma = 0$ contradicting rank B = rank $C = 2$. If $c = \tilde{c}$ then $\lambda\lambda_j = 0$ for $j > 2$. Hence $\lambda_j = 0$ for $j > 2$. Also $\lambda^2 - \mu^2 - \sigma^2 = 0$. If $\lambda = 0$ then $\mu = \sigma = 0$. Hence $BC = CB$ or the relative nullity is $n - 2$.

Theorems 1 and 2 imply:

THEOREM 4. *For* $n \geq 3$ *the real projective space* $P^n\left(\frac{1}{r^2}\right)$ *of curvature* $\frac{1}{r^2}$, $r \neq 1$, *cannot be isometrically immersed as a minimal submanifold of* $S^{n+2}(1)$.

Lemma 8 implies Theorem 4 is also true for $n = 2$, $r = 1$. Theorems 1 and 3 also imply:

PROPOSITION 1. *Let* $n \geq 4$. *Let* $M^n(c)$ *be compact and have constant sectional curvature* $c > 0$ *and isometrically immersed in* $\boldsymbol{R}^{n+3}$ *by* ψ *such that* $D\eta = 0$. *Then* $M^n(c) = S^n(c)$, ψ *is an imbedding and* $\psi(M^n)$ *is the usual n-dimensional sphere in some* $\boldsymbol{R}^{n+1}$.

PROPOSITION 2. *Let* $n \geq 4$. *Let* $M^n(c)$ *be compact and have constant sectional curvature* $c \geq 0$ *and isometrically immersed in* $S^{n+3}(\tilde{c})$, $\tilde{c} \neq c$, *such that* $D\eta = 0$, $\eta \neq 0$. *If* $c > 0$, *then* ψ *is an imbedding and* $\psi(M^n) = S^{n+3} \cap \boldsymbol{R}^{n+1}$ *for some Euclidean space* $\boldsymbol{R}^{n+1}$. *If* $c = 0$, *then* $\psi(M^n)$ *is a product of circles, said circles lying in perpendicular Euclidean planes.*

In the next section we characterize the isometric immersions of $M^n(1)$ into $\boldsymbol{R}^{n+2}$.

4. Codimension two isometric immersions of spheres into Euclidean space.

Consider the following example. Let ϕ be an isometric immersion of $\boldsymbol{R}^{n+1}$ into $\boldsymbol{R}^{n+2}$ and let ψ be the restriction of ϕ to $S^n(1)$. Then ψ is an isometric immersion of S^n into $\boldsymbol{R}^{n+2}$. Let M^{n+1} be the image of $\boldsymbol{R}^{n+1}$ under ϕ; M^{n+1} is locally smooth and flat. Let ξ be the inward pointing normal on $S^n \subset \boldsymbol{R}^{n+1}$ and let $\xi_1 = \phi_* \xi$. Let ξ_2 be normal to M^{n+1}. Let A_1 and A_2 be the second fundamental forms associated with ξ_1 and ξ_2 and $s_{\alpha\beta}$ the normal connection forms; let $s_{12} = s$. Then an easy calculation shows that $A_1 = I$ and A_2 has at most one non-zero eigenvalue μ. If $E_1, \cdots, E_n$ are orthonormal eigenvectors of A_2 with $A_2 E_1 = \mu E_1$, then $s(E_i) = 0$ for $i \geq 2$.

In the rest of this section let $n \geq 4$ and let $\psi : M^n(1) \to \boldsymbol{R}^{n+2}$ be an isometric immersion of an n-dimensional Riemannian manifold $M^n(1)$ of constant sectional curvature 1 into $(n+2)$-dimensional Euclidean space. From Theorem 2 we conclude that the normal connection is trivial.

LEMMA 12.

(a) *For each* $x \in M^n$ *there exist orthonormal normal vectors* ξ_1 *and* ξ_2 *at* x *such that* $A_1 = I$ *and* A_2 *has at most one non-zero eigenvalue* μ. *If* $E_1, \cdots, E_n$ *are* *the common orthonormal eigenvectors of* A_1 *and* A_2 *with* $A_2E_1 = \mu E_1$*, then*

$$A_1 = I \quad \textit{and} \quad A_2 = \begin{pmatrix} \mu & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 0 \end{pmatrix}$$

when represented with respect to $E_1, \cdots, E_n$. μ *is clearly uniquely determined up to sign.*

(b) *If* $\mu(x) \neq 0$ *or* $\mu = 0$ *in a neighborhood of* x*, then* ξ_1 *and* ξ_2 *may be chosen continuously in a neighborhood of* x *such that* A_1 *and* A_2 *are as in* (*a*). *Furthermore, since the eigenvalues of* A_2 *are continuous and have constant multiplicities we may find continuous orthonormal eigenvector fields in this case.*

Proof. Let ξ_1 and ξ_2 be any differentiable orthonormal normal vector fields defined in a neighborhood U of x. Let A_1 and A_2 be the associated second fundamental forms. Then the eigenvalues of A_1 and A_2 are continuous. Let $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$ be the eigenvalues of A_1 and A_2 respectively with corresponding eigenvectors $E_1, \cdots, E_n$. We do not know yet that $E_1, \cdots, E_n$ can be chosen continuously; as remarked, when the eigenvalues of A_2 have constant multiplicity, this will follow. Since

$$1 = K(E_i \wedge E_j) = \lambda_i\lambda_j + \mu_i\mu_j \quad \text{for} \quad i \neq j,$$

we may assume $\lambda_1 \neq 0$. Letting

$$\xi_1' = (\lambda_1\xi_1 + \mu_1\xi_2)/(\lambda_1^2 + \mu_1^2)^{\frac{1}{2}}$$

and

$$\xi_2' = (\mu_1\xi_1 - \lambda_1\xi_2)/(\lambda_1^2 + \mu_1^2)^{\frac{1}{2}}$$

we see that we may assume that we have continuous ξ_1 and ξ_2 with $\mu_1 = 0$ in U. Since

$$1 = K(E_1 \wedge E_j) = \lambda_1\lambda_j + \mu_1\mu_j = \lambda_1\lambda_j$$

for $j \geq 2$, we have $\lambda_j = \frac{1}{\lambda_1}$ for $j \geq 2$. Let $\lambda = \lambda_1$. We now distinguish three possibilities:

(i) all $\mu_i(x) \neq 0$ for $i \geq 2$ (and therefore by continuity of the μ_i, this is so in a neighborhood of x),

(ii) at least one $\mu_i(x) = 0$ for $i \geq 2$ but not all $\mu_i(x) = 0$ for $i \geq 2$, and

(iii) all $\mu_i(x) = 0$.

Case (i): We have

$$1 = \frac{1}{\lambda^2} + \mu_i \mu_j = \frac{1}{\lambda^2} + \mu_i \mu_k$$

for $k \geq j > i \geq 2$. Thus all μ_i are equal for $i \geq 2$. Let σ be their common value. Let $\xi_1' = \frac{1}{\lambda} \xi_1 + \sigma \xi_2$ and $\xi_2' = \sigma \xi_1 - \frac{1}{\lambda} \xi_2$. Then ξ_1' and ξ_2' have the properties in (a) and (b) with $\mu = \sigma\lambda \neq 0$.

Case (ii): We may suppose $\mu_2(x) \neq 0$, and therefore, by continuity of μ_2, $\mu_2 \neq 0$ in a neighborhood of x; and we may suppose $\mu_3(x) = 0$. Then

$$1 = K(E_j \wedge E_3) = \frac{1}{\lambda^2} + \mu_j \mu_3 = \frac{1}{\lambda^2}$$

at x for $j > 3$. Hence $\lambda(x) = \pm 1$; we may suppose $\lambda(x) = 1$. Since

$$1 = \lambda_i \lambda_j + \mu_i \mu_j = 1 + \mu_i \mu_j$$

at x for $i \neq j$, at most one μ_i is non-zero at x. We now claim that $\lambda = 1$ and $\mu_i = 0$ for $i \neq 2$ in neighborhood of x. By continuity of the eigenvalues there exists an $\varepsilon > 0$ and a neighborhood V of x such that $|\mu_i(y)| < \varepsilon$ for $i \geq 3$ and $|\mu_2(y)| > \varepsilon$ for all $y \in V$. But the argument in case (i) and the above applied to such y imply that either all $\mu_i(y)$ are equal for $i \geq 2$ or at most one of them is non-zero. Clearly we must have the latter case and $\mu = \mu_2 \neq 0$. Reorder the eigenvalues to obtain the desired result.

Case (iii): If all $\mu_i(x) = 0$ then $\lambda(x) = \pm 1$ and we may suppose $\lambda(x) = +1$. It remains to prove (b) when $\mu = 0$ in a neighborhood V of x. If ξ_1 and ξ_2 chosen as above with $\mu_1 = 0$ and $\lambda_1 = \lambda = \frac{1}{\lambda_i}$ for $i \geq 2$ in a neighborhood U of x, $U \subset V$, with $\lambda(x) = 1$ and $\mu_i(x) = 0$ for $i \geq 2$, then we claim $\lambda = 1$ and $\mu_i = 0$ in U. For if $\mu_i(y) \neq 0$ for some $y \in V$ and some i, then (i) and (ii) applied to y imply $\mu(y) \neq 0$, a contradiction. This completes the proof of Lemma 12.

LEMMA 13. *If $\mu(x) \neq 0$ or $\mu = 0$ in a neighborhood of x, then we may choose ξ_1 and ξ_2 differentiably in a neighborhood of x such that A_1 and A_2 are as in Lemma 12. Since the eigenvalues of A_2 have constant multiplicities μ is differentiable and we may find differentiable orthonormal eigenvector fields $E_1, \cdots, E_n$ of A_2.*

Proof. Let ξ_1 and ξ_2 be continuous orthonormal normal vector fields defined in a neighborhood of x such that

$$A_1 = I \quad \text{and} \quad A_2 = \begin{pmatrix} \mu & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

when represented with respect to continuous orthonormal eigenvector fields $E_1, \cdots, E_n$ of A_2. Let $\bar{\xi}_1$ and $\bar{\xi}_2$ be any differentiable orthonormal normal vector fields defined in a neighborhood of x such that $\bar{\xi}_1 = a\xi_1 + b\xi_2$ and $\bar{\xi}_2 = -b\xi_1 + a\xi_2$ with $a(x) = b(x) = \frac{1}{\sqrt{2}}$ and a, b continuous since $a = g(\xi_1, \bar{\xi}_1)$ and $b = g(\xi_2, \bar{\xi}_1)$. Then

$$\bar{A}_1 = \begin{pmatrix} a + b\mu & & & & \\ & a & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & a \end{pmatrix} \quad \text{and} \quad \bar{A}_2 = \begin{pmatrix} -b + a\mu & & & & \\ & -b & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & -b \end{pmatrix}$$

when represented with respect to $E_1, \cdots, E_n$. Thus by the assumptions on μ the eigenvalues of $\bar{A}_1$ and $\bar{A}_2$ have constant multiplicities in a neighborhood of x and are therefore differentiable in this neighborhood. Thus a and b are differentiable. But $\xi_1 = a\bar{\xi}_1 - b\bar{\xi}_2$ and $\xi_2 = b\bar{\xi}_1 + a\bar{\xi}_2$. Hence ξ_1 and ξ_2 are differentiable.

LEMMA 14. *If $\mu(x) \neq 0$ and ξ_1 and ξ_2 chosen differentiably in a neighborhood U of x such that $\mu \neq 0$ in U,*

$$A_1 = I \quad \textit{and} \quad A_2 = \begin{pmatrix} \mu & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

when represented with respect to orthonormal differentiable eigenvector fields $E_1, \cdots, E_n$ of A_2, then

(a) *The distribution $\mathscr{S}(y)$ defined by $\mathscr{S}(y) = span\{E_2(y), \cdots, E_n(y)\}$ is integrable,*

(b) *The normal connection 1-form s satisfies $s(E_i) = 0$ for $i \geq 2$.*

Proof. Codazzi's equation applied to E_i and E_j for $i > j \geqq 2$ implies that

$$-A_2(\nabla_{E_i}E_j - \nabla_{E_j}E_i) + s(E_i)E_j - s(E_j)E_i = 0.$$

Since $g(A_2X, E_k) = 0$ for $k > 1$ we conclude that $s(E_i) = s(E_j) = 0$ for $i > j \geqq 2$. Since $\nabla_{E_i}E_j - \nabla_{E_j}E_i = [E_i, E_j]$ we conclude that $g(E_1, [E_i, E_j]) = 0$.

LEMMA 15. *If $\mu = 0$ in a neighborhood of x and $\xi_1, \xi_2, A_1, A_2, E_1, \cdots, E_n$ as in Lemma* 13, *then $s(E_i) = 0$ for all i.*

Proof. Codazzi's equation implies that $s(E_i)E_j - s(E_j)E_i = 0$ for all i, j.

Note that the set of x such that $\mu(x) \neq 0$ or μ identically zero in a neighborhood of x is a dense open subset of M^n.

PROPOSITION 3. *If $\mu = 0$ in a neighborhood of x, then there exists a neighborhood U of x such that $\psi(U)$ is part of a sphere S^n in some $\boldsymbol{R}^{n+1}$.*

Proof. Choose differentiable orthonormal normal vector fields ξ_1 and ξ_2 defined in neighborhood U of x such that $A_1 = I$, $A_2 = 0$, and $s = 0$. From the classical rigidity theorem (see [1], volume 2, page 45 for the rigidity theorem in codimension 1) we conclude the desired result.

Suppose $\mu(x_0) \neq 0$. Choose ξ_1 and ξ_2 as in Lemma 13. Let $y_1, \cdots, y_n$ be local coordinates defined in a neighborhood U of x_0 with $y_i = 0$ for all i at x_0 and such that $\partial/\partial y_2, \cdots, \partial/\partial y_n$ span the distribution $\mathscr{S}(y)$ for $y \in U$. Let $P(y)$ be the hyperplane in $\boldsymbol{R}^{n+2}$ spanned by $T_y(M^n) + \operatorname{span}\{\xi_1(y)\}$ and passing through y. Thus we have an n-parameter family of $(n+1)$-dimensional hyperplanes given by

$$g(\vec{X}, \xi_2(y_1, \cdots, y_n)) + p(y_1, \cdots, y_n) = 0,$$

where $\vec{X}$ is the position vector, and, putting $\vec{x} = \overrightarrow{Oy}$, $p(y_1, \cdots, y_n)$ is given by

$$g(\vec{x}(y_1, \cdots, y_n), \xi_2(y_1, \cdots, y_n)) + p(y_1, \cdots, y_n) = 0. \tag{36}$$

Since $\tilde{\nabla}_{E_k}\xi_2 = 0$ ($\tilde{\nabla}$ is covariant differentiation in $\boldsymbol{R}^{n+2}$) for $k \geqq 2$, ξ_2 depends only on y_1. Differentiating (36) we have

$$g\Big(\frac{\partial \vec{x}}{\partial y_k}, \xi_2\Big) + g\Big(\vec{x}, \frac{\partial \xi_2}{\partial y_k}\Big) + \frac{\partial p}{\partial y_k} = 0. \tag{37}$$

The first term on the left hand side of (37) is zero since $\frac{\partial\vec{x}}{\partial y_k}$ is tangent to M^n. For $k \geqq 2$ the second term is zero. Thus $\frac{\partial p}{\partial y_k} = 0$ for $k \geqq 2$ and we really have a one-parameter family of hyperplanes. For $k = 1$ (37) is

$$g\left(\vec{x},\ \frac{\partial \xi_2}{\partial y_1}\right) + \frac{\partial p}{\partial y_1} = 0. \tag{38}$$

Since $\mu(x) \neq 0$ we also have near x:

$$g(\tilde{\nabla}_{E_1}\xi_2,\ E_1) \neq 0. \tag{39}$$

Since $g(E_1,\ \partial/\partial y_1) \neq 0$ and by (39) we have near x:

$$\tilde{\nabla}_{\partial/\partial y_1}\xi_2 \neq 0 \text{ and } g(\tilde{\nabla}_{\partial/\partial y_1}\xi_2,\ \partial/\partial y_1) \neq 0. \tag{40}$$

We claim that the envelope (see below) of this one parameter family of hyperplanes is a smooth flat manifold near x.

LEMMA 16. *Let $\gamma(y)$ be a smooth curve in $\boldsymbol{R}^{n+2}$ and $P(y)$ a one-parameter family of hyperplanes with normals $\xi(y)$ such that $P(y)$ passes through $\gamma(y)$ and contains the tangent vector $\partial/\partial y$ to γ at $\gamma(y)$. Suppose $g\left(\frac{\partial\xi}{\partial y},\ \frac{\partial\gamma}{\partial y}\right) \neq 0$ at $y = 0$. Then the envelope of $P(y)$ (see below) is a smooth flat $(n+1)$-dimensional Riemannian manifold near $\gamma(0)$.*

Proof. We may choose Euclidean coordinates $x_1, \cdots, x_{n+2}$ such that $x_i = 0$ for all i at $\gamma(0)$, $\partial/\partial x_1 = \xi(0)$, and $\partial/\partial x_2$ is in the direction of $\frac{\partial\xi}{\partial y}(0)$. The family of hyperplanes $P(y)$ is given by

$$g(\vec{X},\ \xi(y)) + p(y) = 0 \tag{41}$$

where $\vec{X}$ is the position vector, and, putting $\vec{x} = \overrightarrow{Oy}$, $p(y)$ is given by

$$g(\vec{x}(y),\ \xi(y)) + p(y) = 0. \tag{42}$$

Differentiating (42) with respect to y we obtain

$$g\left(\vec{x},\ \frac{\partial\xi}{\partial y}\right) + \frac{\partial p}{\partial y} = 0 \tag{43}$$

since $\frac{\partial\vec{x}}{\partial y}$ is tangent to γ. Since $\vec{x} = 0$ at $\gamma(0)$ we have

$$p(0) = \frac{\partial p}{\partial y}(0) = 0.$$

We also consider the $(n+1)$-dimensional planes defined by

(44) $$g\left(\vec{X}, \frac{\partial \xi}{\partial y}(y)\right) + \frac{\partial p}{\partial y}(y) = 0.$$

The characteristics of the family of hyperplanes P is defined to be the family of n-dimensional planes defined by (41) and (44). We define the envelope to be the set of characteristic planes.

If one writes out (41) and (44) in terms of the coordinates $x_1, \cdots, x_{n+2}$, and y, the assumptions that $\xi(0) = \partial/\partial x_1$ and that $\frac{\partial \xi}{\partial y}(0)$ is in the direction of $\partial/\partial x_2$ imply that we may solve for x_1 and x_2 as functions of $x_3, \cdots, x_{n+2}$, and y:

$$x_1 = F(x_3, \cdots, x_{n+2}, y)$$

$$x_2 = G(x_3, \cdots, x_{n+2}, y)$$

If we calculate $\frac{\partial G}{\partial y}(0, \cdots, 0, 0)$ we find that

(45) $$\frac{\partial G}{\partial y}(0, \cdots, 0, 0) = \frac{-\partial^2 p}{\partial y^2}(0) \Big/ g\left(\frac{\partial \xi}{\partial y}(0), \frac{\partial}{\partial x_2}\right)$$

Differentiating (43) we obtain

$$g\left(\frac{\partial \vec{x}}{\partial y}, \frac{\partial \xi}{\partial y}\right) + g\left(\vec{x}, \frac{\partial^2 \xi}{\partial y^2}\right) + \frac{\partial^2 p}{\partial y^2} = 0$$

which evaluated at $y = 0$ is

(46) $$g\left(\frac{\partial \vec{x}}{\partial y}, \frac{\partial \xi}{\partial y}\right) + \frac{\partial^2 p}{\partial y^2} = 0.$$

Since the first term on the LHS of (46) is not zero, $\frac{\partial^2 p}{\partial y^2}(0) \neq 0$. Thus we may solve for y as a differentiable function of $x_2, \cdots, x_{n+2}$ near $y = 0$. Hence we obtain x_1 as a differentiable function of $x_2, \cdots, x_{n+2}$ on the envelope near $y = 0$. Thus near $y = 0$ the envelope is a smooth manifold with $P(y)$ as its tangent plane. It is clear that it is also flat.

Let us return to the immersion $\psi : M^n(1) \to \boldsymbol{R}^{n+2}$. Let $\gamma(y_1)$ be an integral curve of $\partial/\partial y_1$ through $(0, \cdots, 0)$. Using this for γ in the previous lemma we see that we have proved our claim. Call this envelope $\bar{M}^{n+1}$. It is clear that for $y \in M^n$, y near x, $y \in \bar{M}^{n+1}$.

Thus we have proved:

THEOREM 5. *Let $n \geq 4$. Let $\psi : M^n(1) \to \boldsymbol{R}^{n+2}$ be an isometric immersion of an n-dimensional Riemannian manifold M^n of constant sectional curvature* 1 *into*

$(n+2)$-dimensional Euclidean space. Then there exists a dense open set $V \subset M^n$ such that each point $x \in V$ has a neighborhood U and an isometric imbedding g of U into $S^n(1) \subset \boldsymbol{R}^{n+1}$ and an isometric immersion f of an open set W of $\boldsymbol{R}^{n+1}$ into $\boldsymbol{R}^{n+2}$ such that $\psi_{|U} = f \circ g$.

5. **Remarks.** Compact hypersurfaces of $\boldsymbol{R}^{n+1}$ of constant mean curvature $\neq 0$ satisfy a variational principle. Namely, a compact hypersurface M^n of $\boldsymbol{R}^{n+1}$ has constant mean curvature $\neq 0$ if and only if its n-dimensional area $\mathscr{A}$ is stationary with respect to $(n+1)$-dimensional volume preserving variations; where the above $(n+1)$-dimensional volume is the volume in $\boldsymbol{R}^{n+1}$ enclosed by M^n. More precisely: Let $\{\psi_t\}$ be a 1-parameter family of immersions of a compact M^n into $\boldsymbol{R}^{n+1}$, defined for $t \in (-\varepsilon, \varepsilon)$, with $\psi_0 = \psi$ and such that the map $\Psi : M^n \times (-\varepsilon, \varepsilon) \to \boldsymbol{R}^{n+1}$ defined by $\Psi(m, t) = \psi_t(m)$ is C^∞. Then Ψ is called a variation of ψ. Let $\mathscr{A}(t)$ be n-dimensional area of $\psi_t(M^n)$ and $V(t)$ $(n+1)$-dimensional volume enclosed by $\psi_t(M^n)$. We are assuming that $\psi_t(M^n)$ is a simple closed hypersurface of $\boldsymbol{R}^{n+1}$; i.e., $\psi_t(M^n)$ is a manifold—no self intersections. An $(n+1)$-dimensional volume preserving variation is one for which $V(t) = V(0)$ for all t. Now, $\psi : M^n \to \boldsymbol{R}^{n+1}$ has constant mean curvature if and only if $\frac{dA}{dt}(0) = 0$ for all $(n+1)$-dimensional volume preserving variations.

A fundamental question seems to be: Do n-dimensional submanifolds of $\boldsymbol{R}^{n+p}$ of constant mean curvature $\neq 0$ satisfy a variational principle?

If M^1 is a compact connected 1-dimensional submanifold of R^{p+1} such that $D\eta = 0$, then it is quite easy to show that M^1 is a circle that lies in some 2-dimensional Euclidean plane. Bryan Smyth has communicated to me that he has shown that if M^2 is a compact 2-dimensional submanifold of $\boldsymbol{R}^4$ such that $D\eta = 0$ and M^2 is topologically a sphere, then M^2 is isometric to S^2 and lies in some 3-dimensional Euclidean space. The above result of Bryan Smyth and our results Theorem 1′ and Proposition 1 suggest the following question: How necessary are our assumptions on the triviality of the normal connection and the sectional curvatures in Theorem 1? Can we replace one or both of them by some topological condition or some other condition?

Bryan Smyth has also pointed out to me that by considering the Laplacian of $\operatorname{tr} A_\eta^2$ one can show that a connected compact submanifold M^n of $\boldsymbol{R}^{n+p}$ of positive curvature and constant mean curvature is a minimal submanifold of some sphere S^{n+p-1}.

Cartan (Oeuvres Completes, partie III, vol. 1, p. 417) has shown that if an n-dimensional space form $M^n(c)$ is isometrically immersed in an $(n+p)$-dimensional space form $\tilde{M}^{n+p}(\tilde{c})$, $c < \tilde{c}$, then $p \geqq n-1$; and if $p = n-1$, then the normal curvature tensor is zero. John Moore has used this result in his Berkeley Thesis to show that in the case $p = n-1$, if in addition $D\eta = 0$, then M^n is flat, i.e. $c = 0$.

Do Theorems 2 and 3 have analogues for higher codimension? Do the algebraic lemmas used in the proof of Theorems 2 and 3 extend? Finally, is Theorem 5 true for $n = 3$?

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