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FUNCTION-THEORETIC METRICS AND BOUNDARY BEHAVIOUR OF FUNCTIONS MEROMORPHIC OR HOLOMORPHIC IN THE UNIT DISK

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§1. **Introduction.** The metrics to which the title of the present paper refers are expressed in the form of elements of arc length as follows:

(i) $|dw|$ in the finite w-plane $W_1 : |w| < \infty$.

(ii) $\dfrac{|dw|}{1+|w|^2}$ in the Riemann w-sphere $W_2 : |w| \leqq \infty$.

(iii) $\dfrac{|dw|}{1-|w|^2}$ in the open unit disk $W_3 : |w| < 1$.

Let $D : |z| < 1$ be the open unit disk and let $\Gamma : |z| = 1$ be the unit circle in the z-plane. We fix a constant ρ, $1/2 < \rho < 1$, once and for all and we denote by $\mathscr{D}(\zeta)$ the open disk $\{z;\ |z - \rho\zeta| < 1 - \rho\}$ for $\zeta \in \Gamma$. By a segment X at $\zeta \in \Gamma$ we mean an open rectilinear segment connecting ζ and a point of D. Let $w = f(z)$ be a function from D into $W_j (j = 1, 2, 3)$, being meromorphic or holomorphic in D, and set for $z = re^{i\theta} \in D$,

$$\delta_1(r, \theta) = |f'(re^{i\theta})|;$$

$$\delta_2(r, \theta) = \frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2};$$

$$\delta_3(r, \theta) = \frac{|f'(re^{i\theta})|}{1 - |f(re^{i\theta})|^2};$$

corresponding respectively to $j = 1$, 2 and 3. The word "capacity" always means "logarithmic capacity". Then our result is stated in the following

THEOREM. *Let M be a subset of Γ which is a Borel set in the plane and set*

$$\sigma = \bigcup_{\zeta \in M} \mathscr{D}(\zeta).$$

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Let $w = f(z)$ be a meromorphic or holomorphic function from D into W_j such that

$$\iint_{\sigma}\{\delta_j(r, \theta)\}^2 r dr d\theta < \infty \quad (j = 1, 2, 3). \tag{1}$$

Then there exists a subset E_j of M, being of capacity zero)*, *such that for any $\zeta \in M - E_j$ and for any segment X at ζ we have*

$$\int_X \delta_j(r, \theta)|dz| < \infty \qquad (z = re^{i\theta} \in X) \tag{2}$$

according as $j = 1, 2, 3$.

The condition (2) for $j = 1, 2, 3$ implies the existence of a limiting value $f(\zeta) \in W_j$ of $f(z)$ as $X \ni z \to \zeta$ according as $j = 1, 2, 3$. Then by the theorem of Lindelöf-Iversen-Gross [1, p. 5] combined with our condition (1), the function f has the angular limit $f(\zeta)$ at ζ, in other words, ζ is a Fatou point [1, p. 59] of f. It should therefore be noted that our theorem in the case $j = 1, 2$ gives "localization" of Beurling-Tsuji's theorem ([3, Theorems 3 and 4], [4, p. 344]).

An application of the theorem for $j = 3$ is the following. Let $G \subset W_3$ be a Jordan domain whose non-Euclidean area is finite and let $w = \Phi(z)$ be a one-to-one conformal map from D onto G in the w-plane. Furthermore, let $\Phi(\zeta)$ be the Carathéodory extension of Φ to Γ. Then we have $|\Phi(\zeta)| < 1$ except perhaps for a set of $\zeta \in \Gamma$ of capacity zero. Therefore, the boundary of G touches the circle $|w| = 1$ at a "thin" set in this sense.

§2. **Three lemmas.** Let $0 < \alpha < \pi/2$ and let $\Delta = \{re^{i\theta}; 0 < r \leqq 1, |\theta| \leqq \alpha\}$. We let $\Delta^* \supset \Delta$ be an open disc whose boundary contains the origin and we use the same notation $\delta_j(r, \theta)$ as in §1 for a function f defined in $\Delta^* (j = 1, 2, 3)$. We begin with two lemmas [4, p. 342, Theorem VIII. 47 and p. 343, Theorem VIII. 48] expressed in one.

LEMMA $j (j = 1, 2)$. *Let $w = f(z)$ be a function from Δ^* into W_j, being meromorphic or holomorphic in Δ^*. Assume that f does not take three distinct points of W_2 in Δ^* and set*

$$\Lambda_j(\theta) = \int_0^1 \delta_j(r, \theta) dr$$

for $|\theta| \leqq \alpha$. Assume furthermore that both $\Lambda_j(-\alpha)$ and $\Lambda_j(\alpha)$ are finite. Then $\Lambda_j(\theta)$ is bounded for $|\theta| \leqq \alpha$.

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*) In other words, the outer logarithmic capacity of E_j is zero.

The following lemma needs a proof.

LEMMA 3. *Let $w = f(z)$ be a holomorphic function from Δ^* into W_3. Set*

$$\Lambda_3(\theta) = \int_0^1 \delta_3(r, \theta) dr$$

for $|\theta| \leqq \alpha$ and assume that both $\Lambda_3(-\alpha)$ and $\Lambda_3(\alpha)$ are finite. Then $\Lambda_3(\theta)$ is bounded for $|\theta| \leqq \alpha$.

Proof. As f is bounded in Δ^*, by the same argument as in the next paragraph to the theorem in § 1 the origin is a Fatou point of f at which f has the angular limit $f(0)$ with $|f(0)| < 1$. This implies that we have a positive constant B such that $(1 - |f(re^{i\theta})|^2)^{-1} < B$ on Δ. On the other hand, both $\Lambda_1(-\alpha)$ and $\Lambda_1(\alpha)$ are finite because of $\delta_3(r, \theta) \geqq \delta_1(r, \theta)$ for $|\theta| \leqq \alpha$. Lemma 3 follows from Lemma 1 combined with $\Lambda_3(\theta) \leqq B\Lambda_1(\theta)$ for $|\theta| \leqq \alpha$.

§ 3. **Proof of Theorem.** In the following $z = re^{i\theta}$ and $e^{i\omega}$ are *always* points of D and M respectively. To avoid unnecessary complexity we drop the suffix j of $\delta_j(r, \theta)$ if the argument is true for $j = 1, 2, 3$. We remark that $\delta_2(r, \theta)$ is not defined at the poles of f; but this is not essential in the following proof.

We set

$$h(r, \theta) = \begin{cases} \delta(r, \theta) & \text{for } z \in \sigma, \\ 0 & \text{for } z \in D - \sigma. \end{cases}$$

Let $\psi \equiv \psi(r, \theta) = \pi - \arg(re^{i\theta} - 1)$, where $0 < r < 1$, $|\theta| \leqq \pi$ and $\pi/2 < \arg(re^{i\theta} - 1) < 3\pi/2$. Then by $\tan \psi = r \sin \theta/(1 - r \cos \theta)$ we have

$$\begin{aligned} \frac{\partial \psi}{\partial \theta} &= -\frac{\partial}{\partial \theta} \arg(re^{i\theta} - 1) \\ &= \frac{\partial}{\partial \theta} \operatorname{Im} \log \{1/(re^{i\theta} - 1)\} \\ &= r(\cos \theta - r)/(1 - 2r \cos \theta + r^2). \end{aligned} \tag{3}$$

We next consider the function

$$H(\omega; r, \theta) = h(r, \theta + \omega) \frac{\partial \psi}{\partial \theta}. \tag{4}$$

Then $H(\omega; r, \theta)$, for a fixed ω, is Lebesgue measurable for $0 < r < 1$ and $|\theta| \leqq \pi$; and $H(\omega; r, \theta) \geqq 0$ in the disk

$$S = \{re^{i\theta};\ \cos\theta > r\}$$

and further $H(\omega; r, \theta) \leqq 0$ in $D - S$ by (3). Therefore we may consider two integrals:

$$J_1(\omega) = \iint_S H(\omega; r, \theta) dr d\theta \geqq 0$$

and

$$J_2(\omega) = -\iint_{D-S} H(\omega; r, \theta) dr d\theta \geqq 0$$

for $e^{i\omega} \in M$. We first assert that

(I) $J_2(\omega) < +\infty$ *for any* $e^{i\omega} \in M$, *so that* $H(\omega; r, \theta)$ *possesses a definite integral on* D [2, p. 20] *and that*

(5) $$J(\omega) \equiv \iint_D H(\omega; r, \theta) dr d\theta = J_1(\omega) - J_2(\omega).$$

We let, for the proof, C_r be the circle $|z| = r$, $0 < r < 1$. Then

$$-\frac{\partial\phi}{\partial\theta} = r(r - \cos\theta)/(1 - 2r\cos\theta + r^2) \leqq r/(r+1) < r$$

for $re^{i\theta} \in C_r - S$. This can be proved by considering $-\dfrac{\partial\phi}{\partial\theta}$ as a function of $\cos\theta$ (cf. [4, p. 346]). Therefore by (3) and (4) we have

(6) $$-H(\omega; r, \theta) \leqq rh(r, \theta + \omega),\quad re^{i\theta} \in C_r - S.$$

We estimate $J_2(\omega)$ upwards by (6) and by Schwarz's inequality as follows:

$$\begin{aligned} J_2(\omega) &= -\int_0^1 dr \int_{C_r - S} H(\omega; r, \theta) d\theta \leqq \int_0^1 dr \int_{C_r - S} rh(r, \theta + \omega) d\theta \\ &= \iint_{D-S} h(r, \theta + \omega) r dr d\theta \leqq \iint_D h(r, \theta + \omega) r dr d\theta \\ &= \iint_D h(r, \theta) r dr d\theta \leqq \pi^{1/2} \left[\iint_D \{h(r, \theta)\}^2 r dr d\theta\right]^{1/2} \\ &= (\pi U)^{1/2} < +\infty, \end{aligned}$$

where

(7) $$U = \iint_D \{h(r, \theta)\}^2 r dr d\theta = \iint_\sigma \{\delta(r, \theta)\}^2 r dr d\theta < +\infty$$

by our assumption (1) in the theorem. This completes the proof of (I).

Let $\ell(\omega, \varphi)$ be the chord of the circle $|z-\rho e^{i\omega}|=1-\rho$, with one end-point $e^{i\omega}$, making the directed angle φ, $|\varphi|<\pi/2$, with the radius of D at $e^{i\omega}$. We shall use the notation $\ell(0, \varphi)$ though $\zeta=1$ may not be in M. The chord $\ell(\omega, \varphi)$ has the length

$$(8) \qquad \lambda(\varphi)=(2-2\rho)\cos\varphi,$$

being independent of ω. We then set for $-\pi/2<\varphi<\pi/2$,

$$(9) \qquad L(\omega, \varphi)=\int_{\ell(\omega, \varphi)}\delta(r, \theta)|dz| \quad (z=re^{i\theta}\in\ell(\omega, \varphi))$$

and we consider the function $\chi(\omega)$ on M defined by

$$(10) \qquad \chi(\omega)=\int_{-\pi/2}^{\pi/2}L(\omega, \varphi)\cos\varphi d\varphi.$$

(II) *The function $\chi(\omega)$ is Borel measurable on M.*

We shall prove this for $\delta_2(r, \theta)$*⁾. In other cases the proofs are simpler and hence are omitted.

Let γ_k $(k=1,2,\cdots)$ be the circle $|z|=r_k$, $2\rho-1\leqq r_k<1$, such that $r_k\nearrow 1$ and the set $\bigcup_{k=1}^{\infty}\gamma_k$ contains all the poles of f in the half-open ring $\{z;\ 2\rho-1\leqq|z|<1\}$. Let R_ν $(\nu=1,2,\cdots)$ be the open set, being o the form of a summation of ring domains whose boundaries are concentric circles with the centre $z=0$, such that

$$R_1\supset R_2\supset\cdots\supset\bigcap_{\nu=1}^{\infty}R_\nu=\bigcup_{k=1}^{\infty}\gamma_k.$$

Let $2\rho-1<\beta_1<\cdots<\beta_m<\cdots<1$, $\beta_m\nearrow 1$ and let D_m be the closed ring $\{z;2\rho-1\leqq|z|\leqq\beta_m\}$. We then set $D_{m\nu}=D_m-R_\nu$ for $m, \nu=1,2\cdots$. We note first that

$$(11) \qquad L(\omega, \varphi)=\int_{\ell(\omega, \varphi)}\delta_2(r, \theta)|dz|=\int_{\ell(0, \varphi)}\delta_2(r, \theta+\omega)|dz|$$

$$(z=re^{i\theta}\in\ell(0, \varphi) \text{ in the last expression})$$

and we then consider

$$L_{m\nu}(\omega, \varphi)\equiv\int_{\ell(0, \varphi)\cap D_{m\nu}}\delta_2(r, \theta+\omega)|dz|$$

$$(z=re^{i\theta}\in\ell(0, \varphi)\cap D_{m\nu}).$$

*⁾ δ_2 may be extended continuously to the poles of f and our proof will be rather simplified (*Added in proof*).

We shall show that for any $e^{i\omega_0}\in M$ we have $L_{m\nu}(\omega,\varphi)\to L_{m\nu}(\omega_0,\varphi)$ as $\omega\to\omega_0$ uniformly for $-\pi/2<\varphi<\pi/2$, so that

$$\chi_{m\nu}(\omega)\equiv\int_{-\pi/2}^{\pi/2}L_{m\nu}(\omega,\varphi)\cos\varphi d\varphi$$

is continuous on M. Indeed,

$$\begin{aligned}&|L_{m\nu}(\omega,\varphi)-L_{m\nu}(\omega_0,\varphi)|\\ &\leqq\int_{\ell(0,\varphi)\cap D_{m\nu}}|\delta_2(r,\theta+\omega)-\delta_2(r,\theta+\omega_0)||dz|\\ &\leqq\{\max{}_{re^{i\theta}\in D_{m\nu}}|\delta_2(r,\theta+\omega)-\delta_2(r,\theta+\omega_0)|\}\times\\ &\times\{\sup{}_{|\varphi|<\pi/2}\int_{\ell(0,\varphi)\cap D_{m\nu}}|dz|\},\end{aligned}$$

so that our assertion follows from the uniform continuity of the function $\delta_2(r,\theta)$ on the compact set $D_{m\nu}$. Set

$$L_m(\omega,\varphi)=\int_{\ell(0,\varphi)\cap D_m}\delta_2(r,\theta+\omega)|dz|$$

and further set

$$\chi_m(\omega)=\int_{-\pi/2}^{\pi/2}L_m(\omega,\varphi)\cos\varphi d\varphi.$$

Then $\chi_{m\nu}(\omega)\nearrow\chi_m(\omega)$ as $\nu\nearrow\infty$ and $\chi_m(\omega)\nearrow\chi(\omega)$ as $m\nearrow\infty$. This proves our proposition (II).

(III) *The inequality* $J_1(\omega)\geqq(2\rho-1)\chi(\omega)$ *holds for any* $e^{i\omega}\in M$.

We remember that $\mathscr{D}(1)$ is the disk $|z-\rho|<1-\rho$ and we let

$$J_1^*(\omega)=\iint_{\mathscr{D}(1)}H(\omega;r,\theta)drd\theta.$$

Then $J_1(\omega)\geqq J_1^*(\omega)$ since $S\supset\mathscr{D}(1)$ and $H(\omega;r,\theta)\geqq0$ in S. To estimate $J_1^*(\omega)$ downwards, we set for $re^{i\theta}\in\mathscr{D}(1)$,

$$\begin{aligned}&t=|re^{i\theta}-1|\text{ and }\psi=\pi-\arg(re^{i\theta}-1)\text{ for}\\ &\pi/2<\arg(re^{i\theta}-1)<3\pi/2.\end{aligned}$$

Then $1>r=(1-2t\cos\psi+t^2)^{1/2}$, and on the chord $\ell(0,\psi)$, for a fixed ψ, $|\psi|<\pi/2$, we have

$$\begin{aligned}dr&=(t-\cos\psi)(1-2t\cos\psi+t^2)^{-1/2}dt\\ &\geqq(\cos\psi-t)(-dt)\quad(\text{for }dt\leqq0).\end{aligned}$$

We note that r decreases as t increases on $\angle(0, \psi)$ and $\cos\psi \geqq t$ since $re^{i\theta} \in \mathscr{D}(1) \subset S$. Furthermore, on the circle $C_r : |z| = r,\ 0 < r < 1$, we have

$$H(\omega;\, r,\, \theta)d\theta = h(r,\, \theta + \omega)d\psi$$

by (4). We therefore obtain

$$\begin{aligned} J_1^*(\omega) &= \int_{2\rho-1}^{1} dr \int_{C_r \cap \mathscr{D}(1)} H(\omega;\, r,\, \theta)d\theta \\ &= \int_{2\rho-1}^{1} dr \int_{C_r \cap \mathscr{D}(1)} h(r,\, \theta + \omega)d\psi \\ &= \iint_{\mathscr{D}(1)} h(r,\, \theta + \omega)dr d\psi \\ &= \int_{-\pi/2}^{\pi/2} d\psi \int_{\angle(0,\,\psi)} h(r,\, \theta + \omega)dr \\ &\geqq \int_{-\pi/2}^{\pi/2} d\psi \int_0^{\lambda(\psi)} \delta(r,\, \theta + \omega)(\cos\psi - t)dt \end{aligned}$$

(where $\lambda(\psi)$ is defined in (8); we note that $h(r,\, \theta + \omega) = \delta(r,\, \theta + \omega)$ for $re^{i\theta} \in \mathscr{D}(1)$ since $\sigma \supset \mathscr{D}(e^{i\omega})$)

$$\geqq (2\rho - 1)\int_{-\pi/2}^{\pi/2} d\psi \int_0^{\lambda(\psi)} \delta(r,\, \theta + \omega)\cos\psi dt$$

(because of $\cos\psi - t \geqq (2\rho - 1)\cos\psi$ for $0 \leqq t \leqq \lambda(\psi)$)

$$= (2\rho - 1)\int_{-\pi/2}^{\pi/2} L(\omega,\, \psi)\cos\psi d\psi$$

(cf. (11); the formula (11) is true for δ)

$$= (2\rho - 1)\chi(\omega).$$

(IV) *The set* $E = \{e^{i\omega} \in M;\ \chi(\omega) = +\infty\}$ *is of capacity zero.*

By (II) the set E is a Borel set in the plane, so that E is capacitable by the celebrated Choquet theorem. Therefore we have only to prove that E is of inner capacity zero. Assume on the contrary that E contains a closed set F of positive capacity and let

$$u(z) = \int_F \log(1/|z - e^{i\omega}|)d\mu(\omega) \leqq V < +\infty$$

be the conductor potential [4, p. 55] of F, where V is a constant and μ is a Borel measure on F of total mass $\mu(F) = 1$. Then we have [4, p. 345]

(12) $$\iint_D \left(\frac{\partial u}{\partial r}\right)^2 r dr d\theta \leqq \pi V/2$$

and

(13) $$r\frac{\partial u}{\partial r} = -\int_F \frac{\partial}{\partial \theta} \arg (re^{i\theta} - e^{i\omega}) d\mu(\omega).$$

We next consider the function

(14) $$\begin{aligned} Q(\omega; r, \theta) &\equiv H(\omega; r, \theta - \omega) \\ &= -h(r, \theta)\frac{\partial}{\partial \theta} \arg (re^{i\theta} - e^{i\omega}) \\ &= h(r, \theta) r\{\cos(\theta - \omega) - r\}/\{1 - 2r\cos(\theta - \omega) + r^2\} \end{aligned}$$

for $re^{i\theta} \in D$ and $e^{i\omega} \in F$ (cf. (3), (4)). Then Q is a Borel measurable function on the product space $D \times F$ and by (13) and (14) we have

$$h(r, \theta) r\frac{\partial u}{\partial r} = \int_F Q(\omega; r, \theta) d\mu(\omega).$$

On the other hand, both $h(r, \theta)$ and $\frac{\partial u}{\partial r}$ are square summable on D with respect to the measure $r dr d\theta$ by (7) and (12). Therefore, we have by Schwarz's inequality,

$$\begin{aligned} J &\equiv \iint_D dr d\theta \int_F Q(\omega; r, \theta) d\mu(\omega) \\ &= \iint_D h(r, \theta) r\frac{\partial u}{\partial r} dr d\theta \neq \pm\infty. \end{aligned}$$

By Fubini's theorem [2, p. 87] applied to the positive and the negative parts of Q respectively we have

(15) $$J = \int_F d\mu(\omega) \iint_D Q(\omega; r, \theta) dr d\theta \neq \pm\infty.$$

Now, by (3), (4), (5) and (14) we have

$$\begin{aligned} J(\omega) &= \iint_D h(r, \theta + \omega)\frac{\partial}{\partial \theta}\{-\arg (re^{i\theta} - 1)\} dr d\theta \\ &= \iint_D h(r, \theta)\frac{\partial}{\partial \theta}\{-\arg (re^{i\theta} - e^{i\omega})\} dr d\theta \\ &= \iint_D Q(\omega; r, \theta) dr d\theta, \end{aligned}$$

so that by (15),

$$J = \int_F J(\omega) d\mu(\omega) \neq \pm \infty.$$

However, by (5), (III) and the very definition of E we have $J(\omega) = +\infty$ for $e^{i\omega} \in F \subset E$. This is a contradiction.

(V) *The set E is the exceptional set in the statement of the theorem.*

Let $e^{i\omega} \in M - E$. Then $\chi(\omega) < +\infty$, so that by the definition of $\chi(\omega)$ (cf. (10)), the quantity $L(\omega, \varphi)$ (cf. (9)) is finite for a.e., φ, $|\varphi| < \pi/2$. Consequently, there are two chords $\ell(\omega, \varphi_1)$ and $\ell(\omega, \varphi_2)$, $-\pi/2 < \varphi_1 < \varphi_2 < \pi/2$, at $e^{i\omega}$ such that $L(\omega, \varphi_k) < +\infty$, $k = 1, 2$. By Lemma j for $j = 1, 2, 3$ and by our assumption (1) we know that $L(\omega, \varphi) < +\infty$ for any φ, $\varphi_1 < \varphi < \varphi_2$. Repeating this process, we have the required property (2) at the point $e^{i\omega} \in M - E$.

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