A. Kato
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SINGULARITIES OF PROJECTIVE EMBEDDING (POINTS OF ORDER n ON AN ELLIPTIC CURVE)

AKIKUNI KATO

In the Plücker formula for a curve embedded in a higher dimensional projective space, one encounters the notion of stationary point (cf, [*B*], [*W*]). W. F. Pohl gave new view point about it in terms of vector bundles and he defined "the singularities of embedding" (cf. [*P*]). At first, we shall give dual formulation of Pohl's one by means of the sheaf of principal parts of order n $\mathscr{P}^n_X$, and next we shall prove the following: If an elliptic curve is embedded in $(n-1)$-dimensional projective space $\underline{P}_{n-1}$ as a curve of degree n, singularities of projective embedding of order $n-1$ are exactly the points of order n with suitable choice of a neutral element on the curve which is an abelian variety of dimension one. The proof is given by making use of the relation between $\mathscr{P}^n_X$ and Schwarzenberger's secant bundle which we shall also give.

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§ 1. Singularities of embedding.

Let $f: X \to \underline{A}^m$ be an embedding (i.e. a closed immersion) of an affine S-scheme X into m-dimensional affine space $\underline{A}^m$ over $S = \mathrm{Spec}\,(A)$. We shall define singularities of a closed immersion f. Let $\mathscr{P}^n_X$, $\mathscr{P}^n_{\underline{A}}$ be the sheaf of principal parts of order n over X, $\underline{A}^m$ respectively. If $\underline{A}^m = \mathrm{Spec}\,(R)$, where $R = A[T_1, \cdots, T_m]$, T_i being indeterminates, then $\mathscr{P}^n_{\underline{A}}$ is the associated sheaf of R-module $P^n_R = R \otimes_A R/I^{n+1}$, where I being the kernel of multiplication $R \otimes_A R \to R$. Let U_i $(1 \leq i \leq m)$ be indeterminates and K be an ideal of $R[U_1, \cdots, U_m]$ generated by U_i $(1 \leq i \leq m)$. Then R-module P^n_R is isomorphic to $R[U_1, \cdots, U_m]/K^{n+1}$ (cf. EGA IV (16.4.10)). Since $P^1_R = R[U_1, \cdots, U_m]/K^2$ $RdT_1 \oplus \cdots \oplus RdT_m$, where dT_i being the class of U_i mod K^2, the correspondence $dT_i \longmapsto U_i$ mod K^{n+1} defines a homomorphism of (left) R-modules P^1_R

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$\to P^n_R$ and this defines a homomorphism of sheaves:

$$\omega_n : \mathscr{P}^1_{\underline{A}} \to \mathscr{P}^n_{\underline{A}}.$$

On the other hand, S-morphism $f : X \to \underline{A}^m$ induces a canonical homomorphism of $\mathscr{O}_X$-Algebra $P^n(f) : f^*(\mathscr{P}^n_{\underline{A}}) \to \mathscr{P}^n_X$.

Definition (1.1). For a closed immersion $f : X \to \underline{A}^m$, a point x of X is called *an n-regular point or a regular point of order n of f*, if the homomorphism of left $\mathscr{O}_X$-Modules $P^n(f)f^*(\omega_n) : f^*(\mathscr{P}^1_A) \to \mathscr{P}^n_X$ is surjective at x, and if not surjective at x, it is called an n-singular point (or a singular point of order n) of f.

Now suppose that projective embedding $f : X \to \underline{P}^m$ be given. Then there is a canonical surjective homomorphism of $\mathscr{O}_X$-Modules $\varphi : \mathscr{O}^{m+1}_X \to \mathscr{O}_X(1)$. Let $s : \mathscr{O}_X \to \mathscr{P}^n_X$ be the structure homomorphism of left $\mathscr{O}_X$-Algebra $\mathscr{P}^n_X$. This s defines a homomorphism $s^{m+1} : \mathscr{O}^{m+1}_X \to (\mathscr{P}^n_X)^{m+1} = \mathscr{P}^n_X \otimes_{\mathscr{O}_X} \mathscr{O}^{m+1}_X = \mathscr{P}^n_X(\mathscr{O}^{m+1}_X)$.

Definition (1.2). For a closed immersion $f : X \to \underline{P}^m$, a point x of X is called an *n-regular point of f*, if the homomorphism of left $\mathscr{O}_X$-Modules $E^n(f) = \mathscr{P}^n(\varphi) \circ s^{m+1} : \mathscr{O}^{m+1}_X \to \mathscr{P}^n_X(\mathscr{O}_X)(1))$ is surjective at x, and otherwise, it is called an *n-singular point of f*. We denote by $\mathscr{W}^n_X$ the sheaf of image of homomorphism $E^n(f)$.

Let ξ_i $(0 \leq i \leq m)$ be the global sections which are images of canonical basis of free $\mathscr{O}_X$-Module $\mathscr{O}^{m+1}_X$ by φ. Their images $d^n\xi_i$ $(0 \leq i \leq m)$ in $\mathscr{P}^n_X(\mathscr{O}_X(1))$ generate (left) $\mathscr{O}_X$-Module $\mathscr{W}^n_X$. For a case $n = 1$, it is easy to check that $\mathscr{W}^1_X = \mathscr{P}^1_X(\mathscr{O}_X(1))$. Namely, every point is 1-regular point of f.

Proposition (1.3). *If $f : X \to \underline{P}^m$ is a closed immersion and $\underline{P}^m$ is obtained by patching affine spaces $\underline{A}_j$ $(0 \leq j \leq m)$ together, then for a point $x \in X$ such that $f(x) \in \underline{A}_j$, f is n-regular (or n-singular) at x if and only if $f|f^{-1}(\underline{A}_j)$ is n-regular (or n-singular) at x.*

Proof. Since, $\mathscr{P}^n_{\underline{P}}(\mathscr{O}_{\underline{P}}(1))|\underline{A}_j \simeq \mathscr{P}^n_{\underline{A}}$, the homomorphism ω_n defines homomorphisms $\mathscr{P}^1_{\underline{P}}(\mathscr{O}_{\underline{P}}(1))|\underline{A}_j \to \mathscr{P}^n_{\underline{P}}, (\mathscr{O}_{\underline{P}}(1))|\underline{A}_j$, $0 \leq j \leq m$. From these we get homomorphism $\mathscr{P}^1_{\underline{P}}(\mathscr{O}_{\underline{P}}(1)) \to \mathscr{P}^n_{\underline{P}}(\mathscr{O}_{\underline{P}}(1))$ (which maps $d^1\xi_i$ into $d^n\xi_i$) and the diagram

$$\mathcal{O}_{\underline{P}}^{m+1} \begin{array}{l} \nearrow \mathcal{P}_{\underline{P}}^{1}(\mathcal{O}_{\underline{P}}(1)) \\ \qquad\qquad \downarrow \\ \searrow \mathcal{P}_{\underline{P}}^{n}(\mathcal{O}_{\underline{P}}(1)) \end{array}$$

is commutative. Since $E^1(f)$ is surjective, we get the proposition.

PROPOSITION (1.4). *The set of n-singular points of a closed immersion f is a closed subset of X.*

Proof. Since structure morphism $X \to S$ is of finite type, $\mathcal{P}_X^n$, $\mathcal{P}_X^n(\mathcal{O}_X(1))$ are of finite type and cokernels of homomorphisms $f^*(\mathcal{P}_{\underline{A}}^1) \to \mathcal{P}_X^n$, $\mathcal{O}_X^{m+1} \to \mathcal{P}_X^n(\mathcal{O}_X(1))$ are also of finite type and this implies their support, i.e., the set of n-singular points of f is a closed subset of X.

PROPOSITION (1.5). *If f is n-regular at x, then f is k-regular at x for $1 \leq k \leq n$.*

Proof. This follows inductively from the following commutative diagrams:

$$f^*(\mathcal{P}_{\underline{A}}^1) \begin{array}{l} \nearrow \mathcal{P}_X^n \\ \qquad \downarrow \\ \searrow \mathcal{P}_X^{n-1} \end{array}, \quad \mathcal{O}_X^{m+1} \begin{array}{l} \nearrow \mathcal{P}_X^n(\mathcal{O}_X(1)) \\ \qquad\qquad \downarrow \\ \searrow \mathcal{P}_X^{n-1}(\mathcal{O}_X(1)) \end{array}$$

where vertical arrows are canonical surjective homomorphisms.

PROPOSITION (1.6). *Let f be an affine or projective embedding of X and g: $Y \to X$ be a closed immersion. If f is n-regular at $g(y)$, $y \in Y$, then $g \circ f$ is n-regular at y.*

Proof. Since homomorphis $i^*f^*(\mathcal{P}_{\underline{A}}^1) \to i^*(\mathcal{P}_X^n)$ is surjective at x and canonical homomorphism $i^*(\mathcal{P}_X^n)$ is surjective, their combined homomorphism $i^*f^*(\mathcal{P}_{\underline{A}}^1) \to \mathcal{P}_Y^n$ is surjective at y.

PROPOSITION (1.7). *If X is an affine scheme or a projective scheme, then for a given integer, $n > 0$, there is an affine or projective embedding respectively which is everywhere n-regular.*

Proof. By proposition (1.6), we may assume that $X = \underline{A}^m$ or $X = \underline{P}^m$. From the canonical homomorphism $\mathcal{O}_{\underline{P}}^{m+1} \to \mathcal{O}_{\underline{P}}(1)$, we get a surjective homomorphism $\mathcal{O}_{\underline{P}}^{N+1} = (\mathcal{O}_{\underline{P}}^{m+1})^{\otimes n} \to \mathcal{O}_{\underline{P}}(n)$ and this defines a closed immersion f: $\underline{P}^m \to \underline{P}^N$ $((x_0, x_1, \cdots, x_m) \longmapsto (y_0, y_1, \cdots, y_N)$, $y_0 = x_0{}^n, \cdots, y_\iota = x_0^{i_0} \cdots x_m^{i_m}$,

$\cdots, y_N = x_m{}^n, i_0 + i_1 + \cdots + i_m = n)$. We show that f is n-regular at every point $(x_0, x_1, \cdots, x_m)$. We may assume that $x_0 \neq 0$. If we restrict to an affine open subset $\underline{A}^N = (\underline{P}^N)_{y_0}$ of $\underline{P}^N$, it is enough to show that the closed immersion $\underline{A}^m \to \underline{A}^N$ $(\xi_1, \cdots, \xi_m) \longmapsto (\eta_1, \cdots, \eta_N)$, $\eta_i = \eta_1{}^{i_1} \cdots \eta_m{}^{i_m}$, $i_1 + \cdots i_m \leqslant n$) is everywhere n-regular (and this proves the case $X = \underline{A}^m$). Put $\underline{A}^N = \mathrm{Spec}(B)$, $\underline{A}^m = \mathrm{Spec}(C)$, where $B = A[Y_1, \cdots, Y_N]$, $C = A[X_1, \cdots, X_m]$. The closed immersion $\underline{A}^m \to \underline{A}^N$ corresponds to a surjective homomorphism $\varphi : B \to C$ $(Y_i \longmapsto X_1^{i_1} \cdots X_m^{i_m})$. Then P_C^n can be identified to $A[X_1, \cdots, X_m, U_1, \cdots, U_m]/K^{n+1}$, and the C-module W^n which defines $\mathscr{W}_X^n$, is generated by 1 and $(X_1 + U_1)^{i_1} \cdots (X_m + U_m)^{i_m} \equiv U_1^{i_1} \cdots U_m{}^{i_m} +$ (terms of lower degrees of $U_1, \cdots, U_m$), $\mathrm{mod}\, K^{n+1}$. This shows $W^n = P_C^n$.

PROPOSITION (1.8). *For a closed immersion $X \subsetneq \underline{A}^m$ or $X \subsetneq P^m$, of r-dimensional variety X, if a positive integer n satisfies inequality $m < r + \binom{r+1}{2} + \cdots + \binom{r+n-1}{n}$, then the closed immersion is everywhere n-singular.*

Proof. If the closed immersion is n-regular at $x \in X$, we may assume that x is a simple point of X, because the set of n-singular points is closed. Then there is an affine neighborhood of $U = \mathrm{Spec}(B)$ of x such that B is a formally smooth A-algebra. Over U, there is an isomorphism

$$S^{\cdot}{}_{\mathscr{O}_X}(\Omega_X^1) \rightrightarrows \mathscr{G}r.\ (\mathscr{P}_X)$$

(cf. [EGA] IV (16.10.1), (16.10.2)).

Since Ω_X^1 is a locally free of rank r over U, by the exact sequence on U:

$$0 \to S^n_{\mathscr{O}_X}(\Omega_X^1) \to \mathscr{P}_X^n \to \mathscr{P}_X^{n-1} \to 0,$$

we see that $\mathscr{P}_X^n$ is locally free of rank $r + \binom{r+1}{2} + \cdots + \binom{r+n-1}{n} + 1$ on U. Since $\mathscr{W}_X^n$ is generated by $m+1$ sections on U, it can not be $\mathscr{W}_X^n \neq \mathscr{P}_X^n$, if n satisfies the inequality.

§ 2. Stationary points.

Let X be an r-dimensional algebraic variety over an algebraically closed field k. We assume that X is embedded in $\underline{A}^m$ or $\underline{P}^m$. Let x be a simple point of X. If $t_1, \cdots, t_r$ are uniformizing parameters at x, then $\mathscr{O}_{X,x}$ is contained in the formal power series ring $k[[t_1, \cdots, t_r]]$. Since the property that an embedding is n-regular at x is invariant under linear transformation of ambient space $\underline{A}^m$ or $\underline{P}^m$, we may assume that (inhomogeneous) coordinate $x_1, \cdots, x_m$ of x and their power series $x_i = \varphi_i(t)$ $(1 \leqslant i \leqslant m)$ are as follows:

$$(2.1)\qquad \varphi_1(t) = H_{i,j}(t) + H_{i,j+1}(t) + \cdots,\ (l_{j-1} < i \leqslant l_j)$$

where $H_{ik}(t)$ is a homogeneous polynomial of $t_1, \cdots, t_r$ of degree k and $H_{l_{j-1}+1,j}, H_{l_{j-1}+2,j}, \cdots, H_{l_j+j}$ are linearly independent over k, $l_0 = 0$ $l_1 = r \leqslant l_2 \leqslant \cdots \leqslant l_j \leqslant \cdots \leqslant m$.

In particular, if X is a curve, (2.1) can be also written by the following form (cf. [W]):

$$(2.2)\qquad \varphi_i(t) = t^{\delta_i} + \cdots,\ 1 \leqslant i \leqslant n$$

where $\delta_1 < \delta_2 < \cdots < \delta_n$.

Let s^n, d^n, be structure homomorphism of left or right $\mathcal{O}_{X,x}$-algebra $\mathcal{P}^n_{X,x}$ respectively. Put $d = d^n - s^n$. Then d satisfies following equality:

$$d(f \cdot g) = fdg + gdf + (df)(dg),\ f,\ g \in \mathcal{O}_{X,x}.$$

By the above equality, it is easily verified following lemma:

LEMMA (**2.3**). *If $\varphi(t) = \sum_{\nu=k}^{\infty} H_\nu(t_1, \cdots, t_r)$, where $H_\nu(t)$ is a homogeneous polynomial of $t_1, \cdots, t_r$ of degree ν, then $d\varphi(t) = \sum_{l=1}^{n} F_l(t_1, \cdots, t_r; dt_1, \cdots, dt_r)$, where $F_l(t; dt)$ is a homogeneous polynomial of $dt_1, \cdots, dt_r$ of degree l with coefficients in $\mathcal{O}_{X,x}$ such that coefficients of F_l are formal power series of order $k - l$ for $l < k$ and $F_l(0, \cdots, 0; dt_1, \cdots, dt_r) = H_l(dt_1, \cdots, dt_r)$ for $k \leqslant l \leqslant n$.*

THEOREM (**2.4**). *A point x of X, whose coordinates satisfies* (2.1), *is an n-regular point of embedding if and only if $l_j - l_{j-1} = \binom{j+r-1}{r-1}$, for all j, $1 \leqslant j \leqslant n$.*

Proof. A basis of free (left) $\mathcal{O}_{X,x}$-module $\mathcal{P}^n_{X,x}$ is given by $(dt_1)^{i_1} \cdots (dt_r)^{i_r}$ $(0 \leqslant i_1 + \cdots + i_r \leqslant n)$. Clearly, it holds that $l_j - l_{j-1} \leqslant \binom{j+r-1}{r-1} =$ number of monomials of degree j of r-variables = number of $(dt_1)^{i_1} \cdots (dt_r)^{i_r}$, $(i_1 + \cdots + i_r = j)$. We denote by $\omega_0 = 1, \omega_1, \cdots, \omega_N$ the above basis with lexicographically order. Put $dx_i = d\varphi_i(t) = \sum_{j=1}^{n} f_{ij}(t)\omega_j$. Then, x is an n-regular point $\Longleftrightarrow \mathcal{P}^n_{X,x}$ is generated by 1, $dx_1, \cdots, dx_m \Longleftrightarrow$ rank $(f_{ij}) = N$.

By lemma (2.3), matrix (f_{ij}) is following form:

$$(f_{ij}) = \begin{pmatrix} A_1 & & \\ & A_2 & * \\ \# & & A_m \\ \cdots & \cdots & \cdots \\ & \# & \end{pmatrix},$$ where A_j is a matrix with $l_j - l_{i-1}$ rows, $\binom{j+r-1}{r-1}$ columns and components at $\#$ are elements of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,x}$,

Hence, if rank $(f_{ij}) = N$, it must be $l_j - l_{j-1} = \binom{j+r-1}{r-1}$. Conversely, if

$l_j - l_{j-1} = \binom{j+r-1}{r-1}$, then det A_j is invertible, since A_j mod $\mathfrak{m}$ is a coefficient matrix of $H_{l_{j-1}+1,j}, \cdots, H_{l_j,j}$, This implies that det $\begin{pmatrix} A_1 & & \\ & A_2 & * \\ \sharp & & A_m \end{pmatrix}$ is invertible in $\mathcal{O}_{X,x}$.

Remark (2.5). A point x of a curve X, whose coordinates satisfies (2.2), is called a stationary point of rank n, if $\delta_n - \delta_{n-1} > 1$ (cf. [W] p. 45).

§ 3. Secant bundle.

Let us consider a commutative diagram of S-prescheme,

$$(P) \qquad \begin{array}{ccc} & & X \\ & \overset{f}{\nearrow} & \uparrow p \\ W & \longrightarrow & X \times Y \\ & \underset{g}{\searrow} & \downarrow q \\ & & Y \end{array} \qquad (p,\ q \text{ being projections})$$

which we denote simply by $P = (W, X, Y, f, g)$ (Schwarzenberger called it a product scheme, if f is a covering map [S]). For a quasi-coherent $\mathcal{O}_Y$-Module $\mathcal{F}$, there is an $\mathcal{O}_X$-Module $\sum_P(\mathcal{F})$ defined by the relation

$$\textstyle\sum_P(\mathcal{F}) = f_* g^*(\mathcal{F}).$$

By abuse of language, we shall call this $\mathcal{O}_X$-Module $\sum_P(\mathcal{F})$ secant sheaf which defines a secant bundle in particular case (cf. [S]).

Let $X^{(n)}$ be the n-th infinitesimal neighbourhood of X for the diagonal morphism (cf. [EGA] IV) (16.1.2)). If we consider a diagram

$$(I_n) \qquad \begin{array}{ccc} & & X \\ & \overset{p_1^{(n)}}{\nearrow} & \uparrow p_1 \\ X^{(n)} & \overset{h_n}{\longrightarrow} & X \times_S X \\ & \underset{p_2^{(n)}}{\searrow} & \downarrow p_2 \\ & & X \end{array}$$

where h_n is canonical morphism and p_1, p_2, projections, then for a quasi-coherent $\mathcal{O}_X$-Module, $\mathcal{F}$, we obtain a secant sheaf $\sum_{I_n}(\mathcal{F})$. In this case $\sum_{I_n}(\mathcal{F})$ is nothing else than $(p_1^{(n)})_*(p_2^{(n)})^*(\mathcal{F}) = \mathcal{P}_X^n(\mathcal{F})$. Another diagram with which we shall concern is that of cartesian product. Let X_n be an n-fold cartesian product of S-prescheme X. Identity morphim $1_{X_n} : X_n \to X_n$ and projection to t-th factor $X_n \to X$ define a closed immersion $h_t : X_n \to X_n \times_S X$. Let W_n be the union of subschemes $h_t(X_n)$ of $X_n \times_S X$ and i inclusion $W_n \to X_n \times_S X$. Then these give a diagram

$$(C_n)\qquad \begin{array}{ccccc} & & & & X_n \\ & \overset{f}{\nearrow} & & \nearrow p & \\ W_n & \xrightarrow[i]{} & X_n \times_S X & & \\ & \underset{g}{\searrow} & & \searrow q & \\ & & & & X \end{array}$$

and secant sheaf $\sum_{C_n}(\mathscr{F})$, if quasi-coherent $\mathscr{O}_X$-Module $\mathscr{F}$ is given. We denote $\sum_{C_n}(\mathscr{F})$ by $\sum^n(\mathscr{F})$. In this section we shall prove the following: If Δ is a diagonal morphism $\Delta : X \to X_n$, there is a canonical isomorphism $\Delta^*(\sum^{n+1}(\mathscr{F})) \rightleftarrows \mathscr{P}^n_X(\mathscr{F})$.

For two diagrams of S-preschemes $P = (W, X, Y, f, g)$, $P' = (W', X', Y', f', g')$, triple of morphisms of S-preschemes, $r_W : W' \to W$, $r_X : X' \to X$, $r_Y : Y' \to Y$ is defined to be a morphism of $P' = (W', X', Y', f', g')$ into $P = (W, X, Y, f, g)$, if $f \circ r_W = r_X \circ f'$ and $g \circ r_W = r_Y \circ g'$. For such a morphism $\underline{r} = (r_W, r_X, r_Y)$ and a quasi-coherent $\mathscr{O}_Y$-Module $\mathscr{F}$, there is a canonical homomorphism $\rho : g^*(\mathscr{F}) \to (r_W)_*(r_W)^* g^*(\mathscr{F}) = (r_W)_* g'^* r_Y{}^*(\mathscr{F})$ and this induces a homomorphism $f_*(\rho) : \sum_P(\mathscr{F}) \to (r_X)_* \sum_{P'}(r_Y^*(\mathscr{F}))$. The adjoint homomorphism of $f_*(\rho)$ is denoted by $\beta(\underline{r})$, $\beta(\underline{r}) : r_X^*(\sum_P(\mathscr{F})) \to \sum_{P'}(r_Y^*(\mathscr{F}))$.

For a diagram $P = (W, X, Y, f, g)$ and a morphism $r_X : X' \to X$, it is obtained new diagram $P' = (W', X', Y, f', g \circ r_W)$ in which W' is the fibered product $X' \times_X W$ and f', r_W are projections. Then $\underline{r} = (r_W, r_X, l_Y)$ is a morphism of P' into P. Let $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_W$-Module. If f is an affine morphism, there is an isomorphism (EGA II (1.5.2)),

$$r_X^* \circ f_*(\mathscr{E}) \rightleftarrows f'_* r_W^*(\mathscr{E}),$$

in particular if $\mathscr{E} = g^*(\mathscr{F})$, where $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-Module, this isomorphism is

$$\beta(\underline{r}) : r_X^*(\textstyle\sum_P(\mathscr{F})) \rightleftarrows \sum_{P'}(\mathscr{F}). \tag{3.1}$$

The diagonal $\Delta : X \to X_{n+1}$ factors through $X \xrightarrow{j} W_{n+1} \xrightarrow{f} X_{n+1}$, where j is a closed immersion such that $j(X)$ is diagonal of $X_{n+1} \times_S X$. The composite morphism $r : X^{(n)} \to W_{n+1}$ of morphisms $p_1^{(n)} : X^{(n)} \to X$ and $j : X \to W_{n+1}$ is also a closed immersion, hence it is an affine morphism. Two morphisms $p_1^{(n)}$: $X^{(n)} \to X$ and $r : X \to W_{n+1}$ induce a morphism $\sigma : X^{(n)} \to X \times_{X_{n+1}} W_{n+1}$

PROPOSITION (3.2). *σ is an isomophism, $\sigma : X^{(n)} \rightleftarrows X \times_{X_{n+1}} W_{n+1}$.*

Proof. Since r is affine, σ is also affine, and we can assume that X, S

are affine schemes such that $X = \mathrm{Spec}\,(B)$, $S = \mathrm{Spec}\,(A)$. Then $X^{(n)} = \mathrm{Spec}\,(P^n_{B/A})$, where $P^n_{B/A} = B \otimes_A B / I^{n+1}_{B/A}$. Put $T^{n+1}(B) = B \otimes_A B \otimes_A \cdots \otimes_A B$ $(n+1$ times), then $X_{n+1} = \mathrm{Spec}\,(T^{n+1}(B))$, $X_{n+1} \times_S X = \mathrm{Spec}\,(T^{n+1}(B) \otimes_A B)$. Let J be the ideal of W_{n+1} in $T^{n+1}(B) \otimes_A B$ such that $W_{n+1} = \mathrm{Spec}\,(T^{n+1}(B) \otimes_A B/J)$. The diagonal morphism $\Delta : X \to X_{n+1}$ determines a homomorphism of rings $T^{n+1}(B) \to B$ which makes B a $T^{n+1}(B)$-module. Tensoring an exact sequence of $T^{n+1}(B)$-modules

$$0 \to J \to T^{n+1}(B) \otimes_A B \to T^{n+1}(B) \otimes_A B/J \to 0$$

with B, we get an exact sequence

$$(*) \qquad B \otimes_{T^{n+1}(B)} J \xrightarrow{\psi} B \otimes_{T^{n+1}(B)} (T^{n+1}(B) \otimes_A B) \to C \to 0$$

where $C = B \otimes_{T^{n+1}(B)} (T^{n+1}(B) \otimes_A B/J)$ and $\mathrm{Spec}\,(C) = X \times_{X_{n+1}} W_{n+1}$. On the other hand, we have another exact sequence

$$(**) \qquad 0 \to I^{n+1}_{B/A} \to B \otimes_A B \to P^n_{A/B} \to 0$$

Since there is a canonical isomorphism between middle terms of exact sequences (*) and (**), in order to prove $P^n_{B/A} \simeq C$, it reduces to show that the image of ψ is canonically isomorphic to $I^{n+1}_{B/A}$. Let J_i be an ideal of $T^{n+1}(B) \otimes_A B$ generated by elements $\varphi_i(a) \otimes 1 - \varphi_i(1) \otimes a$, $a \in B$, where $\varphi_i(a) = 1 \otimes 1 \otimes \cdots \otimes \overset{i}{\breve{a}} \otimes \cdots \otimes 1$. Then J_i is a kernel of multiplication $T^{n+1}(B) \otimes_A B \to T^{n+1}(B)$ of last component with i-th component, and it holds that $J = J_1 \cap J_2 \cap \cdots \cap J_{n+1}$ and $\psi(J_i) = I$, hence it suffices to prove that $J = J_1 \cdot J_2 \cdots J_{n+1}$. Let $\sum_\nu a_1^{(\nu)} \otimes a_2^{(\nu)} \cdots \otimes a_{n+1}^{(\nu)} \otimes b^{(\nu)}$ be an arbitrary element of $J = J_1 \cap J_2 \cap \cdots \cap J_{n+1}$. Then, $\sum_\nu a_1^{(\nu)} \otimes \cdots \otimes \overset{i}{\breve{1}} \otimes \cdots \otimes a_{n+1}^{(\nu)} a_i^{(\nu)} b^{(\nu)} = 0$, for every i, repeatedly, we have $\sum_\nu a_1^{(\nu)} \otimes \cdots \otimes \overset{i_1}{\breve{1}} \otimes \cdots \overset{i_k}{\breve{1}} \otimes \cdots \otimes a_{n+1}^{(\nu)} \otimes a_{i_1}^{(\nu)} \cdots a_{i_k}^{(\nu)} b^{(\nu)} = 0$.

$$\begin{aligned}
\text{Since} \quad & \prod_{i=1}^{n+1} (\varphi_i(a_i^{(\nu)}) \otimes 1 - \varphi_i(1) \otimes a_i^{(\nu)}) \\
&= (a_1^{(\nu)} \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes a_1^{(\nu)}) \\
&\qquad \cdots (1 \otimes \cdots \otimes a_{n+1}^{(\nu)} \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes a_{n+1}^{(\nu)}) \\
&= a_1^{(\nu)} \otimes a_2^{(\nu)} \otimes \cdots \otimes a_{n+1}^{(\nu)} \otimes 1 - 1 \otimes a_2^{(\nu)} \otimes \cdots \otimes a_{n+1}^{(\nu)} \otimes a_1^{(\nu)} + \cdots
\end{aligned}$$

we see that $\sum_\nu a_1^{(\nu)} \otimes a_2^{(\nu)} \otimes \cdots \otimes a_{n+1}^{(\nu)} \otimes b^{(\nu)}$

$$= \sum_\nu (1 \otimes \cdots \otimes 1 \otimes b^{(\nu)}) \prod_{t=1}^{n+1} (\varphi_i(a_i^{(\nu)}) \otimes 1 - \varphi_i(1) \otimes a_i^{(\nu)})$$

is an element of $J_1 \cdot J_2 \cdots J_{n+1}$. Since it is clear that $J_1 \cdot J_2 \cdots J_{n+1} \subset J$, $J = J = J_1 \cdot J_2 \cdots J_{n+1}$.

THEOREM (3.3). *If $\Delta : X \to X_{n+1} = X \times \cdots \times X$ is a diagonal morphism and $\sum^{n+1}(\mathscr{F})$ is a secant sheaf on X_{n+1} associated with a quasi-coherent $\mathscr{O}_X$-Module $\mathscr{F}$, then there is a canonical isomorphism, $\Delta^*(\sum^{n+1}(\mathscr{F})) \rightleftarrows \mathscr{P}_X^n(\mathscr{F})$.*

Proof. By roposition (3.2), isomorphism (3.1) gives the isomorphism in question.

Remark. We can also consider a diagram for n-fold symmetric product $X_{(n)}$ of X and a secant sheaf $\sum^{(n)}(\mathscr{F})$ on $X_{(n)}$ cf. [S], p. 375). Then there is a canonical morphism $r_X : X_n \to X_{(n)}$ and $r_X^*(\sum^{(n)}(\mathscr{F})) = \sum^n(\mathscr{F})$, hence we have also a canonical isomorphism $\Delta^* r_X^*(\sum^{(n+1)}(\mathscr{F})) \simeq \mathscr{P}_X^n(\mathscr{F})$.

§ 4. Points of order n on elliptic curve.

Suppose that an elliptic curve X is embedded in $(n-1)$-dimensional projective space $\underline{P}^{n-1}$ over an algebraically closed field k, as a curve of degree n and not contained in a proper linear subspace of $\underline{P}^{n-1}$. Then by Riemann-Roch theorem, $H^1(X, \mathscr{O}_X(1)) = 0$. Consider an exact sequence:

$$0 \to J(W) \to \mathscr{O}_{X_n \times X} \to i_* \mathscr{O}_{W_n} \to 0, \tag{4.1}$$

where $J(W)$ is an Ideal of $\mathscr{O}_{X_n \times X}$ corresponding the subscheme W_n. Tensor by $q^*(\mathscr{O}_X(1))$ and apply p^*. The result is a cohomology exact sequence which begins

$$\begin{aligned} 0 \to p_*(J(W) \otimes q^*(\mathscr{O}_X(1))) \to p_* q^*(\mathscr{O}_X(1)) \overset{\alpha}{\to} \\ p_*(i_*(\mathscr{O}_{W_n}) \otimes q^*(\mathscr{O}_X(1)) \to R^1 p_*(q^*(\mathscr{O}_X(1) \otimes J(W)) \to 0 \end{aligned} \tag{4.2}$$

Its last term is zero (apply principle of exchange (cf. [M] p. 785) and $H^1(X, \mathscr{O}_X(1)) = 0$). Apply $\sum^n$ to a canonical surjective homomorhpism $\mathscr{O}_X^n \to \mathscr{O}_X(1)$ and combine canonical homomorphism $\mathscr{O}_{X_n}^n \to \sum^n(\mathscr{O}_X)^n = \sum^n(\mathscr{O}_X^n)$, then resulting homomorphism is $\alpha : \mathscr{O}_{X_n}^n \to \sum^n(\mathscr{O}_X(1))$ by our assumption. Thus by theorem (3.3), $\Delta^*(\alpha)$ is the homomorphism $E^{n-1}(f) : \mathscr{O}_X^n \to \mathscr{P}_X^{n-1}(\mathscr{O}_X(1))$ in definition (1.2). By Nakayama's lemma, projective embedding $X \to P^{n-1}$ is $(n-1)$-singular at x if and only if α is not surjective at $(x, x, \cdots, x) \in X_n$, i.e. if and only if $(x, x, \cdots, x) \in \operatorname{Supp} R^1 p_*(q^*(\mathscr{O}_X(1) \otimes J(W))$. Now we calculate(*) $\operatorname{Supp} R^1 p_*(q^*(\mathscr{O}_X(1)) \otimes J(W))$. For a given geometric point $\xi =$

*) This calculation is suggested by H. Yamada.

Spec $(k) \xrightarrow{i} X_n$, $i(\xi) = y$, consider a diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X_n \times_k X \\ \downarrow r & i & p \downarrow \\ \xi & \longrightarrow & X_n \end{array}$$

where s is a morphism $x \longmapsto (y, x), r$, structure morphism, and p, projection. Apply principle of exchange:

$$R^1p_*(q^*(\mathcal{O}_X(1)) \otimes J(W)) \otimes k(y) \simeq R^1r_*(s^*(q^*(\mathcal{O}_X(1)) \otimes J(W)) \\ = H^1(X, s^*(q^*(\mathcal{O}_X(1)) \otimes J(W)).$$

Hence, $R^1p_*(q^*(\mathcal{O}_X(1)) \otimes J(W)) = 0$ if and only if $H^1(X, s^*(q^*(\mathcal{O}_X(1)) \otimes J(W)) = 0$. From the exact sequence (4.1), we get following diagram:

$$\begin{array}{ccccccccc} & & s^*(q^*(\mathcal{O}_X(1)) \otimes J(W)) & \longrightarrow & s^*q^*(\mathcal{O}_X(1)) & \longrightarrow & s^*(i_*\mathcal{O}_W \otimes q^*(\mathcal{O}_X(1))) & \longrightarrow & 0 \\ & & \downarrow \varphi & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_X(-[y]) \otimes \mathcal{O}_X(1) & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & \mathcal{O}_{X \cap s^{-1}(W)} \otimes \mathcal{O}_X(1) & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

where $[y]$ is the corresponding divisor on X to point $y \in X_n$. A surjective homomorphism φ induces a surjective homomorphism $H^1(X, s^*(q^*(\mathcal{O}_X(1)) \otimes J(W)) \rightarrow H^1(X, \mathcal{O}_X(-[y]) \otimes \mathcal{O}_X(1))$, since $\dim X = 1$, but it is also injective, since dimension of supports of kernel of φ is zero. By duality, $H^1(X, \mathcal{O}_X(-[y]) \otimes \mathcal{O}_X(1)) \neq 0$ if and only if $H^0(X, \mathcal{O}([y]) \otimes \mathcal{O}_X(-1)) \neq 0$, i.e. $[y]$ is contained in the linear system of hyperplanesections.

THEOREM (4.3). *If an elliptic curve X is embedded in $(n-1)$-dimensional projective space P^{n-1} over an algebraically clased field k as a curve of degree n $(n \geqslant 3)$ and not contained in a proper subspace, then the points of order n of abelian variety X with suitable choice of a neutral element are exactly the $(n-1)$-singularities of the embedding.*

Proof. There exists a point on X at which the projective embedding is $(n-1)$-singular, for otherwise, $E^{n-1}(f) : \mathcal{O}_X^n \rightarrow \mathcal{P}_X^{n-1}(\mathcal{O}_X(1))$ is a surjective homomorphism of locally free sheaves of same rank, since X is a curve, it must be an isomorphism, but this cannot be happen, since the following sequence is exact:

$$0 \to \Omega_X^{\otimes k} \otimes \mathcal{O}_X(1) \to \mathcal{P}_X^k(\mathcal{O}_X(1)) \to \mathcal{P}_X^{k-1}(\mathcal{O}_X(1)) \to 0$$

for $k = 1, \cdots, n-1$. We choose 0 as a neutral element. A point x of X is $(n-1)$-singular if and only if the divisors $[(x, \cdots, x)]$, $[(0, \cdots, 0)]$ are linearly equivalent, but this is equivalent to $nx = 0$ by Abel's theorem.

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Nagoya Institute for Technology
Department of Mathematics