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A CHARACTERIZATION OF ODD ORDER EXTENSIONS OF THE FINITE SIMPLE GROUPS PSp(4,q), G_2(q), D_4^2(q)

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Let p denote an odd prime integer and let $q = p^f$ where f is a positive integer. Let $\mathscr{H}$ denote the projective symplectic group $PSp(4, q)$, the Dickson group $G_2(q)$, or the Steinberg "triality twisted" group $D_4^2(q)$ over a field F_q of q elements. Then $\mathscr{H}$ is simple and the Sylow 2-subgroups of $\mathscr{H}$ have centers of order 2 so that involutions which centralize a Sylow 2-subgroup of $\mathscr{H}$ form a single conjugacy class of $\mathscr{H}$.

Let σ denote an automorphism of F_q. Then σ induces, in the natural way, an automorphism of $\mathscr{H}$ (cf. [2]) which fixes an involution in the center of a Sylow 2-subgroup of $\mathscr{H}$. In fact, $\langle\sigma\rangle$, the cyclic subgroup of $\mathrm{Aut}(F_q)$ generated by σ, acts faithfully on $\mathscr{H}$ and we may form the natural semi-direct product $\langle\sigma\rangle\mathscr{H}$. If σ is an odd ordered automorphism of F_q, then $\langle\sigma\rangle\mathscr{H}$ is an odd order extension of $\mathscr{H}$ with trivial 2-core. In fact, any odd order extension of $\mathscr{H}$ with trivial 2-core is of this form (cf. [2]).

Let j be an involution in the center of a Sylow 2-subgroup of $\mathscr{H}$ such that j is fixed by σ. Then the centralizer $C(j)$ of j in $\langle\sigma\rangle\mathscr{H}$ is a semi-direct product $\langle\sigma\rangle C_{\mathscr{H}}(j)$ with trivial 2-core.

For each of the 3 possibilities for $\mathscr{H}$, $C_{\mathscr{H}}(j)$ has a subgroup $\mathscr{Y}$ of index 2 containing subgroups L_1, L_2 such that $L_1 \cong SL(2, q_1)$, $L_2 \cong SL(2, q_2)$ (where q_1, q_2 are prime powers), $[L_1, L_2] = \{1\}$, $L_1 \cap L_2 = \langle j\rangle$ and $\mathscr{Y} = L_1L_2$.

It has been shown in [4], [5], and [9] that if a finite group G contains an involution j such that $C_G(j)$ has a subgroup $\mathscr{Y}$ of index 2 of the above type, then $G = C_G(j)\,O(G)$ ($O(G)$ denotes the 2-core of G; i.e., the largest normal subgroup of odd order in G) or $G \cong PSp(4, q)$ or $G \cong G_2(q)$ or $G \cong D_4^2(q)$ for some odd prime power q. However, for example, in classifying finite groups by the structure of their Sylow 2-subgroups, one may arrive at a situation

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in which the centralizer $C_G(j)$ of an involution j in a group G has trivial 2-core and has a normal subgroup $\mathscr{K}$ of odd index such that $\mathscr{K}$ has a subgroup $\mathscr{Y}$ of index 2 of the above type. This is, of course, the case with the groups $\langle\sigma\rangle\mathscr{H}$ above where $\mathscr{H}$ is $PSp(4, q)$, $G_2(q)$ or $D_4^2(q)$, q is an odd prime power and σ is an odd ordered automorphism of F_q. To handle this situation we prove the following more general result:

THEOREM. *If G is a finite group with an involution j such that*

a) $O(C_G(j)) = \{1\}$ *and*

b) *$C_G(j)$ contains a normal subgroup $\mathscr{Y}$ of index 2δ with δ odd or a normal subgroup $\mathscr{K}$ of index δ with δ odd such that $\mathscr{K}$ contains a subgroup $\mathscr{Y}$ of index 2 where in either case $\mathscr{Y}$ contains subgroups L_1, L_2 such that $L_1 \cong SL(2, q_1)$, $L_2 \cong SL(2, q_2)$ (where q_1, q_2 are prime powers), $[L_1, L_2] = \{1\}$, $L_1 \cap L_2 = \langle j\rangle$ and $\mathscr{Y} = L_1L_2$, then j is in the center of some Sylow 2-subgroup of G, g_1 and g_2 are both odd and one of the following holds:*

(i) $G = C_G(j)O(G)$.

(ii) *$q_1 = q_2$, L_1 and L_2 are not normal in $C_G(j)$ and $G \cong \langle\sigma\rangle PSp(4, q)$ where σ is an automorphism of order δ of a field of $q = q_1 = q_2$ elements.*

(iii) *$q_1 = q_2$, $L_1 \lhd C_G(j)$, $L_2 \lhd C_G(j)$ and $G \cong \langle\sigma\rangle G_2(q)$ where σ is an automorphism of order δ of a field of $q = q_1 = q_2$ elements.*

(iv) *one of the numbers q_1, q_2 is the cube of the other, $L_1 \lhd C_G(j)$, $L_2 \lhd C_G(j)$ and $G \cong \langle\sigma\rangle D_4^2(q)$ where σ is an automorphism of order δ of a field of $q = \min\{q_1, q_2\}$ elements.*

Thus, for the rest of the paper we assume that the theorem is false. Hence we assume that G is a finite group with an involution j such that $C_G(j)$ satisfies the hypotheses of the theorem and that G does not satisfy the conclusion of the theorem and we shall arrive at a contradiction. By induction, we may assume that all groups of order less than $|G|$ satisfy the theorem and that δ is minimal among all groups of order $|G|$ contradicting the theorem.

If $\delta = 1$, then the theorem follows from [4], [5], and [9]. Thus we have $\delta > 1$.

Note that $j \in Z(L_1) \cap Z(L_2)$ so both q_1 and q_2 are odd prime powers.

Our notation is fairly standard. If X is a finite group, then $O(X)$ denotes the 2-core of X; i.e., the largest odd order normal subgroup of G.

If $x^y = y^{-1}xy = z$, we write $y : x \to z$. If $y : x \to z$ and $y : z \to x$, then we write $y : x \longleftrightarrow z$. If $y : x \to x^{-1}$, then we say that y inverts x. If p is a prime, then an S_p-subgroup of a group X is a Sylow p-subgroup of X.

Let $q_1 = p_1^{n_1}$, $q_2 = p_2^{n_2}$ where p_1, p_2 are odd prime integers and n_1, n_2 are positive integers. Then:

$$q_i - \varepsilon_i = 2^{\alpha_i}u_i, \; q_i + \varepsilon_i = 2v_i$$

where $\varepsilon_i = \pm 1$, $\alpha_i \geq 2$ and u_i, v_i are odd for $i = 1, 2$.

Also let F_1, F_2 denote fields of q_1, q_2 elements respectively and view $SL(2, q_i)$ as the group of 2×2 matrices with coefficients in F_i of determinant 1 for $i = 1, 2$. As is well known, $\mathrm{Aut}(F_i)$ acts faithfully in the natural way on $GL(2, q_i)$ and $SL(2, q_i)$ as follows:

if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, q_i)$ or $SL(2, q_i)$ where

$a, b, c, d \in F_i$ and if $\sigma \in \mathrm{Aut}(F_i)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} \text{ for } i = 1, 2.$$

Finally fix isomorphisms

$$\phi_i : SL(2, q_i) \to L_i \quad \text{for} \quad i = 1, 2.$$

Clearly $\phi_i\left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\} = j$ for $i = 1, 2$.

The paper is organized as follows. In § 1, we study $C_G(j)$ to obtain various properties of G and to factorize $C_G(j)$ into a semi-direct product $C_G(j) = \mathscr{K}A$ where $|A| = \delta$ and A acts like "field automorphisms" on $\mathscr{K}$. In § 2, we examine $C_G(A)$ to show, among other facts, that $p_1 = p_2$. Then, in § 3 and 4, we construct a semi-direct product subgroup $\tilde{G}A$ of G such that $C_G(j) \leq \tilde{G}A$ and such that $\tilde{G}A$ is strongly embedded in G. Using [1], it is then easy to obtain a contradiction to prove the theorem.

§ 1. In this section we examine the structure of $C_G(j)$ and prove, among other facts, that $\alpha_1 = \alpha_2$, $O(G) = \{1\}$ and that G has only one conjugacy class of involutions.

LEMMA 1.1. $\mathscr{Y} = L_1L_2 \lhd C_G(j)$.

Proof. If $\mathscr{K} \lhd C_G(j)$ and $|\mathscr{K} : \mathscr{Y}| = 2$, then $O^2(\mathscr{K}) = \mathscr{Y}$ since L_1 and L_2 are generated by their elements of odd order and thus $\mathscr{Y} \lhd C_G(j)$.

The proof of [4, (2*B*)] yields:

LEMMA 1.2. *If* $H \leq G$ *and* T *is an* S_2*-subgroup of* $C_H(j)$ *such that* j *is characteristic in* T, *then* T *is an* S_2*-subgroup of* H. *In particular, an* S_2*-subgroup of* $C_G(j)$ *is an* S_2*-subgroup of* G *so that* j *is in the center of an* S_2*-subgroup of* G.

Clearly:

LEMMA 1.3. $Z(\mathscr{Y}) = \langle j \rangle$ *and all involutions of* $\mathscr{Y} - \langle j \rangle = L_1L_2 - \langle j \rangle$ *are conjugate in* $\mathscr{Y}$.

Since $|C_G(j)/\mathscr{Y}| = 2\delta$, there exists a unique subgroup $\mathscr{L}$ of $C_G(j)$ such that $|C_G(j) : \mathscr{L}| = 2$ and $\mathscr{L} > \mathscr{Y}$.

LEMMA 1.4. $\{L_1, L_2\}$ *is invariant in* $C_G(j)$ *and* $\mathscr{L} \leq N_G(L_1) = N_G(L_2)$.

Proof. The first part follows easily from the Krull-Schmidt theorem applied to the group $\mathscr{Y}/\langle j \rangle \cong PSL(2, q_1) \times PSL(2, q_2)$. Since $|\mathscr{L} : \mathscr{Y}| = \delta$ and $|C_G(j) : \mathscr{L}| = 2$, the lemma follows.

LEMMA 1.5. $C_{\mathscr{L}}(\mathscr{Y}) = \langle j \rangle$.

Proof. Since $|\mathscr{L} : \mathscr{Y}| = \delta$ and $C_{\mathscr{L}}(\mathscr{Y}) \cap \mathscr{Y} = \langle j \rangle$, $|C_{\mathscr{L}}(\mathscr{Y})| = 2d$ where $d | \delta$. But $C_{\mathscr{L}}(\mathscr{Y}) \lhd C_G(j)$ and $0(C_G(j)) = \{1\}$ so that $d = 1$.

LEMMA 1.6. *There exists a subgroup* A *of* $\mathscr{L}$ *of order* δ *and homomorphisms* $\beta_i : A \to Aut(F_i)$ *for* $i = 1, 2$ *such that: if* $a \in A$ *and* $k_i \in SL(2, q_i)$, *then*

$$\phi_i(k_i)^a = \phi_i(k_i^{\beta_i(a)}) \quad \text{for} \quad i = 1, 2. \tag{1.1}$$

Moreover, $\mathscr{L} = \mathscr{Y}A$, $\mathscr{Y} \cap A = \{1\}$, $\mathrm{Ker}(\beta_1) \cap \mathrm{Ker}(\beta_2) = \{1\}$ *and* A *is abelian on at most* 2 *genegrators.*

Proof. Clearly $C_{\mathscr{Y}}(L_1) = L_2 \lhd C_{\mathscr{L}}(L_1) \lhd \mathscr{L}$ and $|\mathscr{L} : L_1C_{\mathscr{L}}(L_1)|$ divides δ. It follows from the structure of $\mathrm{Aut}(L_1)$ that there exists a subgroup A_1 of $\mathscr{L}$ and a homomorphism $\beta_1 : A \to \mathrm{Aut}(F_1)$ such that $A_1 \geq C_{\mathscr{L}}(L_1)$, $\mathscr{L} = L_1A_1$, $L_1 \cap A_1 \leq C_{\mathscr{L}}(L_1)$ and such that $\phi_1(k_1)^a = \phi_1(k_1^{\beta_1(a)})$ for all $a \in A_1$ and $k_1 \in SL(2, q_1)$. Hence $L_1 \cap A_1 = C_{\mathscr{L}}(L_1) \cap L_1 = \langle j \rangle$, $\mathscr{Y} \cap A_1 = L_2 \lhd A_1$ and $|A_1/L_2|$ divides δ. Again it follows that there exists a subgroup A_2 of A_1 and a homomorphism $\beta_2 : A_2 \to \mathrm{Aut}(F_2)$ such that $A_2 \geq C_{A_1}(L_2)$, $A_1 = L_2A_2$, $L_2 \cap A_2 \leq C_{A_1}(L_2)$ and such that $\phi_2(k_2)^a = \phi_2(k_2^{\beta_2(a)})$ for all $a \in A_2$ and $k_2 \in SL(2, q_2)$. Hence $A_2 \cap \mathscr{Y} = A_2 \cap A_1 \cap \mathscr{Y} = A_2 \cap L_2 = C_{A_1}(L_2) \cap L_2 = \langle j \rangle$ so that $\langle j \rangle$ is an S_2-subgroup of A_2. Hence A_2 has a normal 2-complement A. Then $\mathscr{L} = \mathscr{Y}A$, $\mathscr{Y} \cap A = \{1\}$ and the restrictions of β_1, β_2 to A give the desired homomorphisms. Also $\mathrm{Ker}(\beta_1)\cap\mathrm{Ker}(\beta_2)=\{1\}$ follows from Lemma 1.5. Now it follows that conjugation induces a monomorphism of A into $O^{2'}((\mathrm{Aut}(L_1)/\mathrm{Inn}(L_1))\times(\mathrm{Aut}(L_2)/\mathrm{Inn}(L_2)))$ so that A is abelian on at most 2 generators as required.

Let $|\mathrm{Im}(\beta_i)|=\delta_i$; then $\delta_i|\delta$ and $\delta_i|n_i$ so that $n_i=\delta_i f_i$ where f_i is a positive integer for $i=1,2$. Hence if n_i is a 2-power, then $\delta_i=1$, A is cyclic, A centralizes L_i and is faithful on L_j where $\{i,j\}=\{1,2\}$. So that if both n_1, n_2 are 2-powers, then $\delta=1$ which is not the case. Thus we have:

(1.2) n_1 and n_2 are not both 2-powers.

Let $\sigma_i\in\mathrm{Aut}(F_i)$ be such that $\sigma_i: x\to x^{p_i^{f_i}}$ for all $x\in F_i$; then $\mathrm{Im}(\beta_i)=\langle\sigma_i\rangle$ for $i=1,2$. Let F_i^* denote the fixed subfield of σ_i, and let $|F_i^*|=q_i^*$; then $q_i^*=p_i^{f_i}$ for $i=1,2$. Let γ_i be a primitive root of F_i for $i=1,2$. If $\varepsilon_i=1$, then $-\gamma_i^{u_i}\in F_i^*$ and $-\gamma_i^{u_i}$ is a non-square in F_i and we can choose $\lambda_i,\ \mu_i\in F_i$ such that $\lambda_i+\mu_i\sqrt{-\gamma_i^{u_i}}$ is a generator for the group of elements in the field $F_i(\sqrt{-\gamma_i^{u_i}})$ of F_i − norm 1. In this case, set:

$$\rho_i=\begin{bmatrix}\gamma_i & 0\\ 0 & \gamma_i^{-1}\end{bmatrix},\quad \kappa_i=\begin{bmatrix}\lambda_i & \mu_i\\ -\gamma_i^{u_i}\mu_i & \lambda_i\end{bmatrix},\quad b_i=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

If $\varepsilon_i=-1$, choose $\lambda_i,\ \mu_i\in F_i$ such that $\lambda_i+\mu_i\sqrt{-1}$ is a generator for the group of elements in the field $F_i(\sqrt{-1})$ of F_i − norm 1 and choose η_i, ζ_i in F_i^* such that $\eta_i^2+\zeta_i^2=1$. In this case, set:

$$\rho_i=\begin{bmatrix}\lambda_i & \mu_i\\ -\mu_i & \lambda_i\end{bmatrix},\quad \kappa_i=\begin{bmatrix}\gamma_i & 0\\ 0 & \gamma_i^{-1}\end{bmatrix},\quad b_i=\begin{bmatrix}\eta_i & \zeta_i\\ \zeta_i & \eta_i\end{bmatrix}.$$

Then we always have:

(1.3) $$b_i^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\quad \rho_i^{b_i}=\rho_i^{-1},\ 0(\rho_i)=q_i-\varepsilon_i,\ 0(\kappa_i)=q_i+\varepsilon_i.$$

Let:

(1.4) $$a_i=\rho_i^{u_i},\ t_i=a_i^{2^{\alpha_i-2}}\ \text{and}\ Q_i=\langle a_i,b_i\rangle.$$

Then:

(1.5) $\kappa_i^{t_i}=\kappa_i^{-1}$ and Q_i is an S_2-subgroup of $SL(2,q_i)$.

Moreover,

(1.6) $b_i^{\sigma_i} = b_i,\ \rho_i^{\sigma_i} = \rho_i^{\varepsilon_i p^{f_i}},\ a_i^{\sigma_i} = a_i$ and $\kappa_i^{\sigma_i} = \kappa_i^{-\varepsilon_i p^{f_i}}$.

In order to simplify the notation, we shall identify elements of $SL(2, q_i)$ with their ϕ_i-images in L_i and we shall suppress the homomorphism β_i in the action of the elements of A on L_i for $i = 1, 2$. Thus we shall utilize Lemma 1.6 with this in mind.

Set:

(1.7) $x = t_1t_2$ and $y = b_1b_2$.

Then x and y are involutions of $\mathscr{Y} - \langle j\rangle$.

A slight modification of the argument of [4, (2*D*)] yields:

LEMMA 1.7. *If* $L_i \ntriangleleft C_G(j)$ *for* $i = 1$ *or* $i = 2$, *then there exists an element* $n \in C_G(j) - \mathscr{L}$ *such that* $n^2 \in \langle j\rangle$ *and* $L_1^n = L_2$.

Note that if there is an involution $n \in C_G(j) - \mathscr{Y}$ such that $L_1^n = L_2$ then [7] implies that G satisfies conclusion (ii) of our theorem. However, we also have:

LEMMA 1.8. *If* $C_G(j)$ *contains an element* n *such that* $n^2 = j$ *and* $L_1^n = L_2$, *then* $G = C_G(j)O(G)$.

Proof. Since $[A, Q_1] = 1$, by conjugating n by an element of L_2, we may assume that $Q_2 = Q_1^n$. Now $C_{\mathscr{L}}(Q_1Q_2) = \langle j\rangle \times A$ so that n normalizes A. A slight modification of the proof of [4, (2*E*)] yields that $C_G(j) - \mathscr{Y}$ contains no involutions and then the remainder of the proof of [4, (2*E*)] applies directly to yield the lemma.

Thus we may henceforth assume:

$$L_i \lhd C_G(j) \quad \text{for} \quad i = 1 \quad \text{and} \quad i = 2. \tag{1.8}$$

LEMMA 1.9. $C_G(j)$ *contains a unique normal subgroup* $\mathscr{K}$ *of index* δ *containing* $\mathscr{Y}$ *such that* $C_G(j) = \mathscr{K}A$ *and* $\mathscr{K} \cap A = \{1\}$.

Proof. Conjugation induces a homomorphism $\theta : C_G(j) \to \mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2)$. By Lemma 1.5, $\mathrm{Ker}(\theta) \cap \mathscr{Y} = \langle j\rangle$. Thus an S_2-subgroup of $\mathrm{Ker}(\theta)$ has order 2 or 4. However, $\langle j\rangle \leq Z(\mathrm{Ker}(\theta))$ so that $\mathrm{Ker}(\theta)$ has a normal 2-complement which must be $\{1\}$ since $\mathrm{O}(C_G(j)) = \{1\}$. Thus $|\mathrm{Ker}(\theta)| = 2$ or 4. If $|\mathrm{Ker}(\theta)| = 4$, then $\mathscr{Y}\,\mathrm{Ker}(\theta)$ is a normal subgroup of $C_G(j)$ of index δ. If $\mathrm{Ker}(\theta) = \langle j\rangle$, consider the natural homomorphism

$\beta : \mathrm{Aut}(L_1) \times (\mathrm{Aut}(L_2) \to (\mathrm{Aut}(L_1)/\mathrm{Inn}(L_1)) \times (\mathrm{Aut}(L_2)/\mathrm{Inn}(L_2))$.

Then $\beta\cdot\theta$ has kernel $\mathscr{Y}$ so that $C_G(j)/\mathscr{Y}$ is abelian; hence, there always exists a normal subgroup $\mathscr{K}$ of $C_G(j)$ of index δ such that $\mathscr{Y}\leq\mathscr{K}$ and the rest readily follows.

Observe that $\mathscr{K}$ satisfies the hypotheses on the structure of the centralizer of an involution in [4] and that if $H\leq G$, then all S_2-subgroups of $C_H(j)$ lie in $\mathscr{K}\cap H$.

We now can prove:

LEMMA 1.10.

(i) *G has only one conjugacy class of involutions.*

(ii) *There exists an involution $n\in\mathscr{K}-\mathscr{Y}$ such that n acts by conjugation on L_i as:*

$$\begin{bmatrix} 0 & 1 \\ -\gamma_i^{u_i} & 0 \end{bmatrix} \text{ if } \varepsilon_i = 1 \text{ and as } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } \varepsilon_i = -1 \tag{1.9}$$

for $i = 1, 2$.

(iii) $\alpha_1 = \alpha_2$

(iv) $\mathscr{K} = \mathscr{Y}\langle n\rangle$

Proof. The proof of [4, (2*F*)] yields (i), (iii), and the fact that there exists an involution $n\in\mathscr{K}-\mathscr{Y}$ which acts on $L_i = SL(2, q_i)$ as an involution in $PGL(2, q_i) - SL(2, q_i)$ for $i = 1, 2$. Then, by conjugating n by an element of $\mathscr{Y} = L_1L_2$, we arrive at (ii).

Henceforth, let $\alpha = \alpha_1 = \alpha_2$.

We now have:

$$\rho_i^n = \rho_i^{-1},\ a_i^n = a_i^{-1},\ \kappa_i^n = \kappa_i \quad \text{for} \quad i = 1, 2. \tag{1.10}$$

When $\varepsilon_i=1$, we also have $b_i^n=b_ia_i$. However, if $\varepsilon_i=-1$, then $b_i^{-1}b_i^n\in\langle\rho_i\rangle$ and $b_i^{-1}b_i^n$ is fixed by σ_i. Then, as in [4, p. 146], there is an integer m such that ρ_i^m is fixed by σ_i and such that $(b_i\rho_i^m)^n = (b_i\rho_i^m)a_i$. Replacing b_i by $b_i\rho_i^m$ in this case, we have:

$$b_i^n = b_ia_i \quad \text{for} \quad i = 1, 2. \tag{1.11}$$

COROLLARY 1.10.1. $O(G) = \{1\}$.

Proof. Clearly $\langle n, j\rangle$ is a 4-subgroup of G whose three involutions are conjugate in G. Since $C_G(t)\cap O(G)\leq O(C_G(t)) = \{1\}$ for any involution t of $\langle n, j\rangle$ and $O(G) = \langle C_G(t)\cap O(G)\,|\,t\in\langle n, j\rangle^{\#}\rangle$, the corollary follows.

LEMMA 1.11.

(i) *n normalizes Q_i for $i=1$ and 2 so that $Q_1Q_2\langle n\rangle$ is an S_2-subgroup of G.*

(ii) $[n, A]=\{1\}$.

(iii) $C_G(\mathscr{Y})=\langle j\rangle$.

Proof. Clearly (i) holds. Since n normalizes $C_{\mathscr{L}}(Q_1Q_2)=\langle j\rangle\times A$, n normalizes A so that $[n, A]\leq A\cap\mathscr{K}=\{1\}$ and (ii) holds. Finally, if θ denotes the homomorphism defined in the proof of Lemma 1.9, then $\langle j\rangle\leq \operatorname{Ker}(\theta)=C_G(\mathscr{Y})\leq C_{\mathscr{X}}(\mathscr{Y})=\langle j\rangle$ and (iii) follows.

LEMMA 1.12. $O^{2'}(C_G(j))=\mathscr{K}$.

Proof. If $q_i>3$, then L_i is generated by 2-elements and if $q_i=3$, then $L_i\langle n\rangle\cong GL(2,3)$ is also generated by 2-elements. Hence $\mathscr{K}=L_1L_2\langle n\rangle\leq O^{2'}(C_G(j))\leq\mathscr{K}$ and we are done.

COROLLARY 1.12.1. *If an involution t of $C_G(j)$ inverts an odd order subgroup Q of $C_G(j)$, then $Q\leq\mathscr{K}$.*

For future reference, we have:

(1.12) $C_{\mathscr{X}}(x, j)=\langle\rho_1, \rho_2, y, n\rangle$ and $C_G(x, j)=C_{\mathscr{X}}(x, j)A$.

(1.13) $C_{\mathscr{X}}(n, j)=\langle\kappa_1, \kappa_2, x, n\rangle$ and $C_G(n, j)=C_{\mathscr{X}}(n, j)A$.

Since n inverts ρ_1, ρ_2 and x inverts κ_1, κ_2 we have:

(1.14) $O^{2'}(C_G(x, j))=C_{\mathscr{X}}(x, j)$.

(1.15) $O^{2'}(C_G(n, j))=C_{\mathscr{X}}(n, j)$.

§2. In this section, we obtain information about $C_G(A)$ and show, among other facts, that $p_1=p_2$

The proof of [4, (3A)] yields:

LEMMA 2.1. *If D is a 4-subgroup of G, then D is conjugate in G to $\langle x, j\rangle$ or $\langle n, j\rangle$ and $N_G(D)/C_G(D)\cong S_3$, the symmetric group on 3 symbols.*

From this lemma, we can demonstrate:

LEMMA 2.2. *If $E=\langle n, j\rangle$, then $E\leq C_G(A)$ and $(N_G(E)\cap C_G(A))/(C_G(E)\cap C_G(A))\cong S_3$, the symmetric group on 3-symbols.*

Proof. Let $V_i = \langle \kappa_i^2 \rangle$ for $i = 1, 2$ and $V = V_1 \times V_2$; then $|V_i| = v_i$ for $i = 1, 2$ and $|V| = v_1 v_2$ is odd. By (1.15), $O^{2'}(C_G(E)) = (V \times E)\langle x \rangle$ and the Frattini argument yields $N_G(E) = C_G(E)\ (N_G(E) \cap N_G(E \times \langle x \rangle))$. Hence there exists a 3-element $\zeta \in N_G(E) \cap N_G(E \times \langle x \rangle)$ such that $\zeta : j \rightarrow n \rightarrow nj \rightarrow j$. Since ζ normalizes $C_G(E \times \langle x \rangle) = E \times \langle x \rangle \times A$, it follows that $\zeta \in N_G(A)$. On the other hand, t_1 fixes j, $t_1 : n \longleftrightarrow nj$ and $[t_1, A] = \{1\}$. Consequently t_1^ζ fixes $n, t_1^\zeta : nj \longleftrightarrow j$ and $[t_1^\zeta, A] = \{1\}$. Thus $\langle t_1, t_1^\zeta \rangle \leq N_G(E) \cap C_G(A)$ and the lemma follows.

LEMMA 2.3.

(i) *$C_{\mathscr{L}}(A) = L_1^* L_2^*$ where $L_i^* = C_{L_i}(A) \cong SL(2, q_i^*)$ for $i = 1, 2$ and $[L_1^*, L_2^*] = \{1\}$ and $L_1^* \cap L_2^* = \langle j \rangle$.*

(ii) *$C_G(A, j) = C_{\mathscr{K}}(A) \times A$ where $C_{\mathscr{K}}(A) = C_{\mathscr{L}}(A)\langle n \rangle$.*

Proof. If $l_i \in L_i$ for $i = 1, 2$ and $(l_1 l_2)^a = l_1 l_2$ for $a \in A$, then $l_1^{-1} l_1^a \in L_1 \cap L_2 = \langle j \rangle$. Since a is of odd order, $l_1^a = l_1$, then $l_2^a = l_2$ and the lemma follows easily.

LEMMA 2.4.

(i) $p_1 = p_2$.

(ii) *$C_G(A)/A$ is isomorphic to $G_2(q^*)$ or $D_4^2(q^*)$ where $q^* = \min\{q_1^*, q_2^*\}$.*

Proof. Let $M = C_G(A)$; then $C_{M/A}(jA) = C_M(j)A/A = C_G(A, j)/A \cong C_{\mathscr{K}}(A)$. But $L_i^* = L_i \cap C_G(A) \lhd C_G(A, j)$ for $i = 1, 2$ and j is conjugate to n in M. Hence $M/A \neq C_{M/A}(jA) O(M/A)$ and [4], [5] yield the result.

§ 3. In this section, we lay the groundwork for the construction of a strongly embedded subgroup of G and we prove that an S_p-subgroup of $C_G(j)$ is not an S_p-subgroup of G.

Let $p = p_1 = p_2$. Since $\alpha_1 = \alpha_2 = \alpha$, we have $\varepsilon_1 = \varepsilon_2$, so let $\varepsilon = \varepsilon_1 = \varepsilon_2$.

We now introduce the following notation: let the images of

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

under the isomorhism $\phi_i : SL(2, q_i) \rightarrow L_i$ be denoted by $x_i(\alpha)$, $x_{-i}(\alpha)$, $h_i(\alpha), \omega_i$ respectively and let X_i, X_{-i}, H_i be the subgroups of L_i generated by elements of the form $x_i(\alpha)$, $x_{-i}(\alpha)$, $h_i(\alpha)$ respectivley for $i = 1, 2$. We have:

$$L_i = X_iH_i \cup X_iH_i\omega_iX_i \quad \text{for} \quad i = 1, 2.$$

Set:

$$d_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ if } \varepsilon = 1 \text{ and } d_i = 1 \text{ if } \varepsilon = 1 \text{ for } i = 1, 2 \text{ and let:}$$

(3.1) $$h_0 = nd_1d_2.$$

Then $\mathscr{H} = \langle L_1L_2, h_0\rangle$, $h_0 \in C_G(H_1H_2)$ and $h_0^2 \in H_1H_2$. Let:

(3.2) $$H = \langle H_1H_2,\ h_0\rangle.$$

Then $|H| = (q_1 - 1)(q_2 - 1)$ and, as on [4, p. 163], H is abelian and is the direct product of two cyclic subgroups of orders $q_1 - 1$ and $q_2 - 1$.

Let D denote the 4-subgroup contained in H and denote the involutions in D by:

$$j = j_0,\ j_1,\ j_2.$$

Clearly $D = \langle x, j\rangle$ if $\varepsilon = 1$ and $D = \langle n, j\rangle$ if $\varepsilon = -1$. The involution $\omega_1\omega_2$ inverts H and $|C_{\mathscr{H}}(D)| = 2(q_1 - 1)(q_2 - 1)$; hence:

(3.3) $$C_G(D) = C_{\mathscr{H}}(D)A \text{ where } C_{\mathscr{H}}(D) = H\langle\omega_1\omega_2\rangle.$$

(3.4) $$O^{2'}(C_G(D)) = C_{\mathscr{H}}(D).$$

The case $q_1 = q_2 = 3$ has been excluded so that $D \neq H$ and H is the unique subgroup of its isomorphism type in $0^{2'}(C_G(D))$ so that $N_G(D) = N_G(H)$.

By Lemma 2.1, there exists a 3-element $\eta \in N_G(H) = N_G(D)$ such that $\eta : j_0 \to j_1 \to j_2 \to j_0$. Since ω_1 fixes $j = j_0$ and $\omega_1 : j_1 \longleftrightarrow j_2$, $N_G(D) = \langle C_G(D), \omega_1, \omega_1^\eta\rangle$ by Lemma 2.1. Also ω_1 centralizes $C_G(D)/O^{2'}(C_G(D)) = O^{2'}(C_G(D))A/O^{2'}(C_G(D))$ so that ω_1^η also centralizes $C_G(D)/O^{2'}(C_G(D))$ and hence $\langle O^{2'}(C_G(D)),\ \omega_1, \omega_1^\eta\rangle \trianglelefteq N_G(D)$. Thus $O^{2'}(N_G(D)) = \langle O^{2'}(C_G(D)), \omega_1, \omega_1^\eta\rangle$ and $C_G(D)/O^{2'}(C_G(D)) \leq Z(N_G(D)/O^{2'}(C_G(D)))$. By Lemma 2.1, there exists a unique normal subgroup M of $N_G(D)$ such that $|N_G(D) : M| = 2$, $C_G(D) \leq M$, $\omega_1 \notin M$ and $|M : C_G(D)| = 3$. Then $M/0^{2'}(C_G(D))$ is abelian of order 3δ and ω_1 acts non-trivially on this abelian group. It follows that M contains a normal subgroup M^* containing $O^{2'}(C_G(D))$ such that ω_1 normalizes M^*, ω_1 inverts $M^*/O^{2'}(C_G(D))$, $|M^*/O^{2'}(C_G(D))| = 3$, $M = M^*C_G(D)$ and $M^* \cap C_G(D) = O^{2'}(C_G(D))$. Moreover, $M^*\langle\omega_1\rangle \leq O^{2'}(N_G(D))$, $M = M^*A$, $M^* \cap A = M^* \cap C_G(D) \cap A = C_{\mathscr{H}}(D) \cap A = \{1\}$ and $M^*\langle\omega_1\rangle \lhd N_G(D) = M^*\langle\omega_1\rangle A$. Thus $O^{2'}(N_G(D)) = M^*\langle\omega_1\rangle$ and $O^{2'}(N_G(D)) \cap A = M^* \cap A = \{1\}$ so that $N_G(D) =$

$O^{2'}(N_G(D))A$, $O^{2'}(N_G(D))\cap A=\{1\}$, $O^{2'}(N_G(D))\cap C_G(D)=O^{2'}(C_G(D))$ and $O^{2'}(N_G(D))/O^{2'}(N_G(D))\cap C_G(D))\cong S_3$. We may now assume that the 3-element η lies in $O^{2'}(N_G(D))$. Let T denote the S_2-subgroup of H; we have $[A, T\cap(H_1H_2)]=\{1\}$ and $[A, h_0]=\{1\}$ so that $[A, T]=\{1\}$ since $T=(T\cap(H_1H_2))h_0$. Consequently, $[A, T\langle\omega_1\omega_2\rangle]=\{1\}$ where $T\langle\omega_1\omega_2\rangle$ is an S_2-subgroup of $O^{2'}(C_G(D))$. The Frattini argument implies that we may assume that $\eta\in N_G(T\langle\omega_1\omega_2\rangle)$ also. Then [8, Kapital I, Satz 18.6] implies that η centralizes an element $\gamma\in T\langle\omega_1\omega_2\rangle-T$ which must be an involution. Note that $C_G(D)=H(\langle\omega_1\omega_2\rangle\times A)$ has a normal 2-complement $O(H)A$. However, $C_{O(H)A}(\gamma)=A$ which implies that $\eta\in N_G(A)$ and hence $[\eta, A]\leq O^{2'}(N_G(D))\cap A=\{1\}$. Since $C_A(H_1H_2)=\{1\}$ and $O^{2'}(N_G(D))/H$ is clearly dihedral of order 12, we have proved:

LEMMA 3.1. *Let D be the 4-subgroup of H. Then $D\neq H$, $N_G(H)=N_G(D)=O^{2'}(N_G(D))A$, $O^{2'}(N_G(D))\cap A=\{1\}$, $H\leq O^{2'}(N_G(D))$ and $W=O^{2'}(N_G(D))/H$ is dihedral of order 12. Moreover, $C_G(H)=H$, there exists a 3-element $\eta\in O^{2'}(N_G(D))\cap C_G(A)$ such that $O^{2'}(N_G(D))=\langle H,\omega_1,\omega_2,\eta\rangle$ and $O^{2'}(N_G(D))\cap C_G(j)=O^{2'}(C_G(D))=H\langle\omega_1\omega_2\rangle$.*

Combining Corollary 1.12.1 and the proof of [4, (4*B*)], we have:

LEMMA 3.2. *If P is a p-subgroup of $C_G(j)$ which is inverted by j_1 or j_2, then $P\leq X_aX_b$ where $a\in\{1,-1\}$ and $b\in\{2,-2\}$.*

As in [4, (4*C*)], since H normalizes X_i and X_{-i} for $i=1$ and 2, we have:

LEMMA 3.3. *Let $\pi\in(X_aX_b)^{\#}$ with $a\in\{1,-1\}$ and $b\in\{2,-2\}$. Then:*

(i) *If $\pi\notin X_a\cup X_b$, then π has $\frac{1}{2}(q_1-1)(q_2-1)$ conjugates under H, all of which belong to $X_aX_b-X_a-X_b$.*

(ii) *If $\pi\in X_a^{\#}$ or $X_b^{\#}$, then the conjugates of π under H consist of $X_a^{\#}$ or $X_b^{\#}$, respectively.*

Let $X=X_1\times X_2$. Then:

$$N_{\mathscr{X}}(X)=XH\lhd N_G(X)\cap C_G(j)=N_{\mathscr{X}}(X)A. \tag{3.5}$$

$$C_G(X,j)=X\times\langle j\rangle \tag{3.6}$$

We can now demonstrate:

LEMMA 3.4. *An S_p-subgroup of $C_G(j)$ is not an S_p-subgroup of G.*

Proof. Assume that the lemma is false. If $p\nmid\delta=|A|$ then X is an S_p-subgroup of G which contradicts Lemma 2.4 (ii) since an S_p-subgroup of

$G_2(q^*)$ or of $D_4^2(q^*)$ is not elementary abelian; hence $p|\delta$. Let A_p denote the S_p-subgroup of A. Then $T = XA_p$ is an S_p-subgroup of G. We claim that X is the unique elementary abelian subgroup of T of order q_1q_2. For, let U be an elementary abelian subgroup of T such that $U \not\leq X$. Since A_p is abelian on at most 2-generators, $|U/U\cap X||p^2$. If $|U/U\cap X| = p$, let $u = za \in U - U\cap X$ where $z\in X$ and $a\in A_p^\#$. But $|C_X(u)| = |C_X(a)|$ and a acts as a non-trivial "field automorphism" of order p on X_1 or X_2. It follows that $|C_X(a)|p < q_1q_2$ and hence $|U| = p|U\cap X| \leq p|C_X(u)| = p|C_X(a)| < q_1q_2$. If $|U/U\cap X| = p^2$, then the "projection" of U into A_p is a subgroup Y of A_p which is elementary abelian of type (p, p). Since Y is not cyclic, $[Y, X_i] \neq \{1\}$ for $i = 1$ and 2 and hence $|C_X(Y)| = q_1^{\frac{1}{p}} q_2^{\frac{1}{p}}$. But $X\cap U \leq C_X(U) = C_X(Y)$ so that $|U| = p^2|U\cap X| \leq p^2 q_1^{\frac{1}{p}} q_2^{\frac{1}{p}} < q_1q_2$. Thus X is the unique elementary abelian subgroup of T of order q_1q_2 and X is weakly closed in T with respect to G.

Now Lemma 1.2 and (3.6) imply that $\langle j\rangle$ is an S_2-subgroup of $C_G(X)$. Hence $C_G(X)$ has a normal 2-complement and the Frattini argument gives:

(3.7) $N_G(X) = O(C_G(X))\,(N_G(X)\cap C_G(j)) = O(C_G(X))\,HA$ where $O(C_G(X)) \cap (HA) = O(C_G(X)) \cap C_G(X) \cap (HA) = O(C_G(X)) \cap \langle j\rangle = \{1\}$.

Thus $(0(C_G(X))H)\cap A = (O(C_G(X)H)\cap (HA)\cap A = H\cap A = \{1\}$, $O(C_G(X))H \lhd N_G(X)$ and $N_G(X)/O(C_G(X))H \cong A$. Since $p||A|$, $N_G(X)$ has a normal subgroup of index p.

Applying [6, Theorem 14.4.2] to the weakly closed subgroup X of T with respect to G, we conclude that G has a normal subgroup J of index p. Now Lemma 1.12 implies that $C_J(j) \geq \mathscr{K}$ so that $C_J(j) = \mathscr{K}(A\cap J) \trianglelefteq C_G(j)$ and hence $O(C_J(j)) = \{1\}$. Also Lemma 2.1 implies that there is a 2-element $w\in N_G(\langle n, j\rangle)$ such that $w : j \longleftrightarrow n$. Since $w \in J$, $J \neq C_J(j)0(J)$; then, since $|J| < |G|$, we conclude that J satisfies conclusions (iii) or (iv) of the theorem. Hence $J' \lhd G$, $|G/J'|$ is odd and either $q_1 = q_2 = q$ and $J' \cong G_2(q)$ or one of the numbers q_1, q_2 is the cube of the other and $J' \cong D_4^2(q)$ where $q = \min\{q_1, q_2\}$. However, $O(G) = \{1\}$ by Corollary 1.10.1 so that G satisfies conclusions (iii) or (iv) of the theorem. But this is false and the lemma follows.

§4. In this section we construct a strongly embedded subgroup and use it to obtain a final contradiction.

Let $\{a, b\} = \{1, 2\}$ and let U be a non-trivial subgroup of X_b. Then:

(4.1) $$C_{\mathscr{X}}(U) = X_b \times L_a \trianglelefteq C_G(U, j).$$

Thus Q_b is an S_2-subgroup of $C_G(U, j)$ and Lemma 1.2 and the proof of [4, (3C)] yields:

LEMMA 4.1. *If* $\{a, b\} = \{1, 2\}$ *and* U *is a non-trivial subgroup of* X_b, *then*:

(i) $C_G(U) = O(C_G(U))C_G(U, j)$ *and* $X_b \leq O(C_G(U))$.

(ii) $N_G(U) = O(C_G(U))(C_G(j) \cap N_G(U))$.

(iii) Q_a *is an* S_2-*subgroup of* $C_G(U)$.

For $U = X_b$, we have:

(4.2) $$N_{\mathscr{X}}(X_b) = X_b L_a H \triangleleft N_G(X_b) \cap C_G(j) = N_{\mathscr{X}}(X_b)A.$$

In this case, we also have:

LEMMA 4.2.

(i) $O(C_G(X_b)) \cap C_G(j) = X_b$.

(ii) $O(C_G(X_b))/X_b$ *is abelian and inverted by* j.

(iii) $O(C_G(X_b))$ *is nilpotent.*

Proof. Let $Y = O(C_G(X_b)) \cap C_G(j)$; clearly $X_b \leq Y$ and $[Y, L_a] \leq L_a \cap O(C_G(X_b)) = \{1\}$. Hence $Y \leq C_G(L_a)$ so that $Y \leq O(C_G(X_b)) \cap C_G(X_a) \cap C_G(L_a) \leq X_b$ which implies (i). Since (ii), (iii) follow immediately from (i), we are done.

LEMMA 4.3. *Let* $\{a, b\} = \{1, 2\}$ *and let* M *denote the* S_p-*subgroup of* $O(C_G(X_b))$. *Then* $M \neq X_b$.

Proof. Assume that $M = X_b$ and let Q be an S_p-subgroup of $C_G(j)$ such that $X = X_1 \times X_2 \leq Q$. By (3.6), $\langle j \rangle$ is an S_2-subgroup of $C_G(Q, j)$ so that $\langle j \rangle$ is an S_2-subgroup of $C_G(Q)$. Hence $C_G(Q) = L\langle j \rangle$ where $L = O(C_G(Q))$. Since $Z(Q) \leq L$, we may choose an S_p-subgroup Q^* of $C_L(j)$ such that $Q^* \geq Q$. Then Q^*Q is a p-subgroup of $C_G(j)$ and hence $Q^* = Z(Q)$. By Lemma 3.4, Q is not an S_p-subgroup of $N_G(Q)$. Since $N_G(Q) = L(N_G(Q) \cap C_G(j))$, we have:

$$\frac{|N_G(Q)|_p}{|Q|} = \frac{|L|_p}{|C_L(j)|_p} \geq p.$$

Now let Q_1 be an S_p-subgroup of L normalized by j. Since $Z(Q)$ is the S_p-sbugroup of $C_L(j)$, $Q_1 > Z(Q) = L \cap Q$. Hence there exists an element

$z \in Q_1^\#$ inverted by j. But $Q_1 \leq C_G(X_b) = O(C_G(X_b))C_G(X_b, j)$ so that $z \in O(C_G(X_b))$ and $z \notin O(C_G(X_b)) \cap C_G(j) = X_b$ which implies that $M \neq X_b$, proving the lemma.

Again let $\{a, b\} = \{1, 2\}$ and let M denote the S_p-subgroup of $O(C_G(X_b))$. Then D, the 4-subgroup contained in H, acts on M. Since $D^\# = \{j_0, j_1, j_2\}$, letting

(4.3) $$M_i = M \cap C_G(j_i) \quad \text{for} \quad i = 0, 1, 2,$$

we have:

(4.4) $$H \text{ normalizes } M_i \text{ for } i = 0, 1, 2.$$

(4.5) $$M_0 = X_b \text{ and } M = M_0 M_1 M_2.$$

(4.6) $$M_1^{\omega_a} = M_2 \text{ and } M_2^{\omega_a} = M_1.$$

Since j inverst M_1 and M_2, we have:

(4.7) $$M_i \leq O^{2'}(C_G(j_i)) \text{ and } M_i \text{ is elementary abelian for } i = 1, 2.$$

(4.8) $$|M_1| = |M_2| \leq q_1 q_2 \text{ and } |M| = q_b |M_1|^2.$$

Hence $1 < |M/X_b| \leq q_1^2 q_2^2$, M/X_b is an elementary abelian p-group which admits L_a and M/X_b has L_a-composition factors which are faithful irreducible L_a-modules over the field of p elements (since j inverts M/X_b).

Also if $\eta \in O^{2'}(N_G(D))$ is the 3-element of Lemma 3.1, then (4.7) and Lemma 3.2 yield:

(4.9) $$M_1 \leq (X_a X_b)^\eta \text{ and } M_2 \leq (X_c X_d)^{\eta^2} \text{ where } a, c \in \{1, -1\} \text{ and } b, d \in \{2, -2\}.$$

Arrange notation so that $q_1 \geq q_2$ and set $a = 1$ and $b = 2$ in the above. Since $1 < |M/X_2| \leq q_1^2 q_2^2 \leq q_1^4$, we have the following 3 possiblities by [4, (1E)]: (I) $|M/X_2| = q_1^4$, (II) $|M/X_2| = q_1^{8/3}$, and (III) $|M/X_2| = q_1^2$.

If (I), then $q_1 = q_2$, $|M_1| = |M_2| = q_1 q_2$, equality holds in (4.9), and $|M/X_2| = q_1^2 q_2^2$. If (II), then $|M_1| = |M_2| = q_1^{4/3} > q_1 \geq q_2$, and, as on [4, p. 165], we conclude that $q_1 = q_2^3$, equality holds in (4.9) and that $|M/X_2| = q_1^2 q_2^2$.

However, in order to treat possibility (III), we shall need a deeper analysis. To this end, let $P = MX_1$, let $\tilde{M}$ denote the S_p-subgroup of $O(C_G(X_1))$, let $\tilde{P} = \tilde{M}X_2$, let $\tilde{M}_0 = \tilde{M} \cap C_G(j) = X_1$, and let $\tilde{M}_i = \tilde{M} \cap C_G(j_i)$ for $i = 1, 2$.

Lemma 4.4. *If* $|M/X_2| = q_1^2$, *then* $|\tilde{M}/X_1| = q_1^2 q_2^2$.

Proof. Assume $|M/X_2| = q_1^2$, then $|M_1| = |M_2| = q_1 > 3$ and it follows from the discussion in [4, p. 166] that:

(i) $M_1 = X_a^\eta$ or $q_1 = q_2$ and $M_1 = X_b^\eta$ and

(ii) $M_2 = X_c^{\eta^2}$ or $q_1 = q_2$ and $M_2 = X_d^{\eta^2}$. Taking into account (4.6), we have one of the following 4 cases:

(1) $M = X_2 X_{-1}^\eta X_1^{\eta^2}$

(2) $M = X_2 X_1^\eta X_{-1}^{\eta^2}$

(3) $M = X_2 X_2^\eta X_2^{\eta^2}$ and $q_1 = q_2$

(4) $M = X_2 X_{-2}^\eta X_{-2}^{\eta^2}$ and $q_1 = q_2$.

If we have case (1), set $\omega = \omega_1\omega_2\eta \in N_G(H)$; then, as in [5, (6*C*)], $P \cap P^\omega = X_{-1}^\eta X_2^{\eta^2}$. The proof of [5, (6*C*)] yields that M is elementary abelian and $X_{-1}^\eta \leq Z(P)$. Hence $P^{\omega_1\eta} = X_{-1}X_2X_{-1}^{\eta^2}$ centralizes X_1; since j inverts X_{-1}^η, X_2^η and $X_{-1}^{\eta^2}$, Lemma 4.1 implies that $P^{\omega_1\eta} \leq \tilde{M}$. Thus $(X_{-1}X_2)^\eta \leq \tilde{M}$ and $|\tilde{M}_1|^2 = |M/X_1| \geq q_1^2 q_2^2$ so that $|\tilde{M}/X_1| = q_1^2 q_2^2$. A similar argument yields the result for case (2). If we have case (3), then η normalizes M and $X_2 \leq Z(M)$ so that M is elementary abelian. Then, as in the proof of [5, (6*C*)], $X_2^{\eta^i} \leq C_G(X_1)$ for $i = 1$ or 2. Since j inverts $X_2^{\eta^i}$, we have $X_2^{\eta^i} \leq \tilde{M}$. Thus, if the conclusion of the lemma is false, then we would have case (1) or (2) for $\tilde{M}$ which implies that $|M/X_2| = q_1^2 q_2^2$ and the lemma holds in case (3). Finally, assume case (4), then, as in the proof of [5, (6*C*)],

$$[X_{-2}^\eta, X_{-2}^{\eta^2}] = X_2, \; Z(M) = X_2 \text{ and } Z(P) = X_2.$$

Let A_p denote the S_p-subgroup of A (possibly $A_p = \{1\}$). Then $P^* = PA_p$ is an S_p-subgroup of $N_G(X_2)$ by Lemma 4.1. Let R be an S_p-subgroup of G containing P^*; clearly $Z(R) \leq R \cap C_G(X_2) \cap C_G(X_1) = (P(A_p \cap C_G(X_2)) \cap C_G(X_1)$. However, $A_p \cap C_G(X_2)$ acts faithfully on $P/M \cong X_1$ so $Z(R) \leq P \cap Z(P) = Z(P) = X_2$. Now Lemma 4.1 implies that Q_1 is an S_2-subgroup of $C_G(Z(R))$. However, if $|M/X_1| \neq q_1^2 q_2^2$, then we must have case (4) for $\tilde{M}$. Then there would exist an S_p-subgroup $\tilde{R}$ of G such that $Z(\tilde{R}) \leq X_1$ and Q_2 is an S_2-subgroup of $C_G(Z(\tilde{R}))$. It follows that Q_1 and Q_2 are conjugate in G which implies that Q_1 and Q_2 are conjugate to each other in $C_G(j)$ which is impossible so the lemma also holds in case (4).

We now have:

LEMMA 4.5. *There is a choice for* α, β *with* $\{\alpha, \beta\} = \{1, 2\}$ *such that* $|O(C_G(X_\beta))/X_\beta|_p = q_1^2 q_2^2$.

We now assume that α, β are chosen so as to satisfy this lemma.

The proof of [4, (4H)] now yields:

LEMMA 4.6. *Let* M *be the* S_p*-subgroup of* $O(C_G(X_\beta))$ *and let* $P = X_\alpha M$. *Then*:

(i) M/X_β *is elementary abelian of order* $q_1^2 q_2^2$.

(ii) *With a suitable choice of notation, we have*:

$$\text{(a)} \quad P = X_\alpha X_\beta (X_{-\alpha} X_\beta)^\eta (X_\alpha X_\beta)^{\eta^2}$$

(4.10) or

$$\text{(b)} \quad P = X_\alpha X_\beta (X_{-\alpha} X_{-\beta})^\eta (X_\alpha X_{-\beta})^{\eta^2}.$$

Let $\mathcal{N} = O^{2'}(N_G(H)) = O^{2'}(N_G(D)) = \langle H, \omega_1, \omega_2, \eta \rangle$ and let $W = \mathcal{N}/H$. As we have seen in Lemma 3.1, W is dihedral of order 12.

As in [4, §6] it follows that:

LEMMA 4.7. *With suitable notation, we may assume that* (4.10) (*b*) *holds.*

Since $H \leq N_G(P)$ and $H \cap P = \{1\}$, if we set

$$\mathfrak{B} = HP \tag{4.1}$$

then $\mathfrak{B}$ is a subgroup of G of order $(q_1 - 1)(q_2 - 1)q_1^3 q_2^3$. Set

$$\tilde{G} = \mathfrak{B}\mathcal{N}\mathfrak{B}. \tag{4.12}$$

Then [4, (6D) and (6E)] yield:

LEMMA 4.8. $\tilde{G}$ *is a subgroup of* G.

We can now show:

LEMMA 4.9. *Let* $\omega : W \to \mathcal{N}$ *be a transversal*; *then*:

(i) $\tilde{G}$ *is the disjoint union of the* 12 *double cosets* $\mathfrak{B}\omega(w)\mathfrak{B}$ *for* $w \in W$.

(ii) $A \leq N_G(\tilde{G})$ *and* $\tilde{G} \cap A = \{1\}$.

(iii) $C_{\tilde{G}}(j) = \mathfrak{B}$

(iv) *Either* $q_1 = q_2$ *and* $\tilde{G} \cong G_2(q)$ *where* $q = q_1 = q_2$ *or one of* q_1, q_2 *is the cube of the other and* $\tilde{G} \cong D_4^2(q)$ *where* $q = \min\{q_1, q_2\}$.

Proof. Clearly $\tilde{G}$ is the union of the double cosets $\mathfrak{B}\omega(w)\mathfrak{B}$ for $w\in W$ and $\mathfrak{B}\cap\mathscr{N} = H(P\cap\mathscr{N})$. Let $\pi\in P\cap\mathscr{N}$, then $[\pi, H]\leq H\cap P=\{1\}$ so $\pi\in C_G(H)\cap\mathscr{N}\cap P=H\cap P=\{1\}$. Now (i) follows. Since $A\leq N_G(P)\cap N_G(H)$, $A\leq N_G(\tilde{G})$ and $Y=\tilde{G}A$ is a subgroup of G. Clearly $N_Y(P) = N_{\tilde{G}}(P)A = \mathfrak{B}A$; let $a\in\tilde{G}\cap A$, then $a\in N_{\tilde{G}}(P)=\mathfrak{B}$ and $a\in PH\cap C_G(j)\cap A=(X_\alpha X_\beta H)\cap A\leq\mathscr{K}\cap A=\{1\}$, so (ii) holds. Now $L_\alpha = \langle X_\alpha, X_\alpha^{\omega_\alpha}\rangle\leq\tilde{G}$; similarly $L_\beta\leq\tilde{G}$, so that $\mathscr{K}=L_1L_2H\leq G$ and (iii) follows. But then (iv) follows from [4] and [5] and the fact that $\eta: j\rightarrow j_1$. Q.E.D.

Using the subgroup $\tilde{G}A$ we can arrive at a final contradiction. Since $\tilde{G}A$ satisfies conclusions (iii) or (iv) of the theorem, $G\neq\tilde{G}A$. Assume that G has a proper normal subgroup N; then $|N|$ is even by Corollary 1.10.1. But $Q_1Q_2\langle n\rangle\leq\tilde{G}$ and $Q_1Q_2\langle n\rangle$ is an S_2-subgroup of G, hence $N\cap\tilde{G}\neq\{1\}$. Since $\tilde{G}$ is simple by Lemma 4.10 (iv), $\tilde{G}\leq N$ and $C_N(j) = \mathscr{K}(A\cap N)\trianglelefteq C_G(j)$. Thus $O(C_N(j)) = \{1\}$ and the theorem holds for N by our choice of G. But then either $q_1=q_2$ and $N'\cong G_2(q)$ where $q=q_1=q_2$ or one of q_1, q_2 is the cube of the other and $N'\cong D_4^2(q)$ where $q=\min\{q_1,q_2\}$. Since $N'\lhd G$ and $O(G)=\{1\}$, the theorem holds for G, which is false. Thus G is simple and $\tilde{G}A$ cannot contain all elements of G of even order. However, $\tilde{G}A$ has only one conjugacy class of involutions and $C_G(j)\leq\tilde{G}A$. Thus, in the language of [1], $\tilde{G}A$ is strongly embedded in G. Then [1, Satz 4] implies that $G\cong PSL(2,q)$ or $Sz(q)$ or $PSU(3,q)$ where $q=2^m\geq 4$. Hence an S_2-subgroup of G has center of order $q\geq 4$ (cf. [1, § 4]). However, we know that $Q_1Q_2\langle n\rangle$ is an S_2-subgroup of G and $Z(Q_1Q_2\langle n\rangle)=\langle j\rangle$ which is a contradiction and the theorem follows.

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