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Nagoya Math. J.
Vol. 45 (1971), 39–78

REPRESENTATIONS OF CHEVALLEY GROUPS IN CHARACTERISTIC p

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Introduction

If G_K is a Chevalley group over a field K of prime characteristic p, the irreducible representations of G_K over K form a natural object of study. The basic results have been obtained by Steinberg [15], who showed that, if K is perfect, then each irreducible rational representation of G_K over K is a tensor product of representations obtained from certain basic representations by composing them with field automorphisms. These basic representations were obtained by "integrating" the irreducible restricted representations of a restricted Lie algebra associated with the group, which had been studied earlier by Curtis [7]. The present author had obtained the main results previously for the groups $SL(n, K)$, $Sp(2n, K)$ by different means, involving reduction (mod p) from the characteristic 0 case [16]. In this paper we extend this method to the other types of groups, in the hope that some additional insight may be gained.

We restrict ourselves to the case when K is finite, since the essential aspects of the situation already appear then. By a consideration of Brauer characters, we obtain the classification of the irreducible G_K-modules by highest weight in Theorem (3E). In Theorem (4H), we obtain a simple necessary and sufficient condition under which an irreducible module in the characteristic 0 case remains irreducible (mod p). This criterion is in terms of the discriminant of a certain integral quadratic form, and can be computed in any given case, although not as easily as Springer's sufficient condition [13].

We are able to give an explicit description of the irreducible G_K-modules (Theorem (5F)), and another proof of the tensor product theorem

Received July 1, 1970.
* Research partially supported by National Science Foundation Grant GP–11342 at the University of Notre Dame.

(Theorem (6C)), on the assumption that a certain property of the irreducible modules in the characteristic 0 case holds. It seems likely that this property is always valid, since it has been verified in the cases A_n, B_n, C_n, D_n, E_6, F_4, G_2 (Theorem (7E)).

Basic tools for this work are the notions of an admissible lattice on a module V for a classical semi-simple Lie algebra [9], and a non-degenerate bilinear form on V having a certain "contravariance" property. These ideas are developed in Sections 1 and 2.

In order to make the exposition as elementary as possible, we have avoided the language of group schemes, which might otherwise have been used to shorten the paper somewhat.

I would like to thank Professor Richard Brauer for the encouragement he gave me during the time I worked on my thesis at Harvard University, and for the gentle prodding since then which finally stimulated me to return to this subject after the lapse of so many years.

1. Lie Algebras

We shall use some standard facts about semi-simple Lie algebras and their modules. Proofs may be found in [8], [12].

Let $\mathfrak{g}$ be a (finite-dimensional) semi-simple Lie algebra over a field L of characteristic 0, which has a splitting Cartan subalgebra $\mathfrak{h}$ (i.e., each element of $\mathfrak{h}$ is mapped by the adjoint representation of $\mathfrak{g}$ on a linear transformation whose eigenvalues lie in L). The theory of such split semi-simple Lie algebras is identical with the classical Cartan-Killing theory of semi-simple Lie algebras over the complex field. If r is a member of the root system $\sum$ of $\mathfrak{h}$ in $\mathfrak{g}$, and $\mathfrak{g}_r$, $\mathfrak{g}_{-r}$ are the root subspaces of $\mathfrak{g}$ corresponding to r, $-r$, then the subspace $[\mathfrak{g}_{-r}, \mathfrak{g}_r]$ of $\mathfrak{h}$ contains a unique element H_r such that $r(H_r) = 2$.

If $\mathfrak{h}_0^*$ is the rational vector space generated by $\sum$ and $\prod = \{\alpha_1, \cdots \alpha_n\}$ is the fundamental system of positive roots corresponding to some ordering of $\mathfrak{h}_0^*$, then $\prod$ is a basis of $\mathfrak{h}_0^*$ and we may assume that the ordering on $\mathfrak{h}_0^*$ is the lexicographical one, in which $\sum k_i\alpha_i$ is positive if the first non-zero coefficient k_i is positive.

If V is a (finite-dimensional) $\mathfrak{g}$-module, then V is the direct sum of its weight spaces. The multiplicity of a weight is defined to be the dimension of the corresponding weight space. We denote by $P(V)$ the group generated by the weights of V, and by P the group generated by all weights of $\mathfrak{g}$ in all modules. Then P is contained in $\mathfrak{h}_0^*$, and inherits the lexicographcal ordering. If μ is a positive element of P, there is only a finite number of positive elements of P lower than μ. In particular, we may carry out induction on the positive elements of P.

The elements of P are characterized as those linear functions on $\mathfrak{h}$ which take integer values on all the $H_r (r \in \sum)$, or, equivalently, on $H_{\alpha_1}, \cdots, H_{\alpha_n}$. The *fundamental weights* are the elements $\lambda_1, \cdots, \lambda_n$ of P such that, for all i, j,

$$\lambda_i(H_{\alpha_j}) = \delta_{ij}.$$

We denote by P^+ the set of all μ in P such that $\mu(H_{\alpha_j}) \geqq 0$, all j. These μ are called dominant integral functions, and are just the non-negative integral linear combinations of $\lambda_1, \cdots, \lambda_n$. The λ_i are positive in the ordering of P.

The weights of a $\mathfrak{g}$-module are permuted among themselves by the action of the Weyl group W of $\mathfrak{g}$. Each orbit of W in P contains exactly one element of P^+, and this is the largest member of the orbit. The highest weight of an irreducible $\mathfrak{g}$-module occurs with multiplicity 1, and this gives a 1-1 correspondence between P^+ and the set of isomorphism classes of irreducible $\mathfrak{g}$-modules. Since every $\mathfrak{g}$-module is completely reducible, the isomorphism class of a $\mathfrak{g}$-module is determined by its weights and their multiplicities.

In [6], Chevalley proved that a set of root vectors X_r can be chosen in $\mathfrak{g}$ so that

$$[X_{-r}, X_r] = H_r,$$

and, whenever r, s, $r+s \in \sum$ and m is the greatest integer such that $s-mr$ is a root,

$$[X_r, X_s] = n_{r,s} X_{r+s}, \tag{1}$$

where $n_{r,s} = \pm(m+1)$. Further,

$$n_{-r,-s} = -n_{r,s}. \tag{2}$$

(1A) LEMMA. *There exists an automorphism θ of $\mathfrak{g}$ of order 2, such that*

$$\theta(H) = -H, \ \theta(X_r) = -X_{-r},$$

for all $H \in \mathfrak{h}$, $r \in \sum$.

Proof. By [8, p. 127], $\mathfrak{g}$ has an automorphism θ of order 2 such that $\theta(H_{\alpha_i}) = -H_{\alpha_i}$, $\theta(X_{\alpha_i}) = -X_{-\alpha_i}$, for $i = 1, \cdots, n$. If $r, s, r+s \in \sum$ and $\theta(X_r) = -X_{-r}$, $\theta(X_s) = -X_{-s}$, then by (1), (2), and the fact that $n_{r,s} \neq 0$, we have

$$\theta(X_{r+s}) = -X_{-r-s}.$$

An obvious induction shows that $\theta(X_r) = -X_{-r}$ for all positive roots r, and hence for all roots r, since θ has order 2. Since the H_{α_i} generate $\mathfrak{h}, \theta(H) = -H$ for all H in $\mathfrak{h}$.

(1B) Corollary. *If V is any $\mathfrak{g}$-module, there exists a non-degenerate bilinear form $(\ ,\ )$ on V such that*

$$(3) \qquad (vX_r, w) = (v, wX_{-r})$$

for all $v, w \in V, r \in \sum$. Weight vectors in V belonging to different weights are orthogonal with respect to this form. If V is irreducible, the form is symmetric and is unique to within multiplication by a scalar.

Proof. A new action of $\mathfrak{g}$ on V defined by setting

$$v \circ X = v\theta(X) \qquad (v \in V, X \in \mathfrak{g})$$

makes V into a new $\mathfrak{g}$-module V^*, whose weights are clearly the negatives of those of V. Hence V^* is isomorphic with the contragredient module to V, and there exists a non-degenerate bilinear form $(\ ,\ )$ on V such that

$$(vX, w) + (v, w \circ X) = 0$$

for $v, w \in V, X \in \mathfrak{g}$. Since $\theta(X_r) = -X_{-r}$, (3) is satisfied. If v and w are weight vectors belonging to distinct weights λ, μ, then we see by taking $X = H \in \mathfrak{h}$ that

$$\lambda(H)(v, w) - \mu(H)(v, w) = 0,$$

so that $(v, w) = 0$.

If V is irreducible, then V is absolutely irreducible. Since the form $(\ ,\ )$ amounts to a $\mathfrak{g}$-homomorphism of V into the contragredient of V^*, it is unique to within multiplication by a scalar, by Schur's Lemma. If v, w are interchanged and r is replaced by $-r$ in (3), we see that the "reversed" form $\langle\ ,\ \rangle$ given by

$$\langle v, w\rangle = (w, v)$$

satisfies (3). Hence, there is a scalar c such that

$$(w, v) = c(v, w)$$

for all $v, w \in V$. Clearly $c^2 = 1$, so that (,) is symmetric or skew-symmetric. Since a vector v_0 of highest weight in V is orthogonal to vectors of lower weights, and the highest weight occurs with multiplicity 1, (v_0, v_0) is non-zero, by non-degeneracy of (,). Hence the form is symmetric.

We shall call a form satisfying (3) *contravariant*. If $\{v_1, v_2, \cdots\}$ is a basis of V, we shall call the dual basis $\{w_1, w_2, \cdots\}$, defined by

$$(v_i, w_j) = \delta_{ij},$$

a basis *contragredient* to $\{v_1, v_2, \cdots\}$. If the v_i are weight vectors corresponding to weights μ_i, the w_i are also weight vectors corresponding to the same μ_i.

2. Chevalley Groups and Derived Modules

From now on we assume that K is a finite field of characteristic p. We choose a field L of characteristic 0 with a (discrete) valuation ring R having K as residue field. Thus,

$$K = R/\pi R,$$

where π is a prime element of R.

Let $\mathfrak{g}$ be a split semi-simple Lie algebra over L and suppose root vectors X_r to be chosen as in Section 1. We denote by $\mathfrak{U}_R$ the R-subalgebra of the universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{g}$ generated by the elements $X_r{}^m/m!$, $r \in \sum$, $m \geqq 0$. Following Ree and Kostant [11], [9], we define an *admissible lattice* on a $\mathfrak{g}$-module V to be a finitely generated R-submodule V_R of V, which generates V as a vector space over L, and which is invariant under $\mathfrak{U}_R$. Such a lattice is a free R-module, and an R-basis is a basis of V over L. In fact, an admissible lattice has an R-basis consisting of weight vectors; such a basis is called a *regular basis* of V. If V is irreducible and v_0 is a vector of highest weight in V then $v_0\mathfrak{U}_R$ is the smallest admissible lattice on V containing v_0.

(2A) LEMMA. *Let v_0 be a vector of highest weight in an irreducible $\mathfrak{g}$-module V and let (,) be the contravariant form on V, normalized so that $(v_0, v_0) = 1$. Then* *the form is integral on the admissible lattice* $V_R = v_0\mathfrak{U}_R$, *i.e.*, $(V_R, V_R) \subseteq R$. *Every* $\mathfrak{g}$*-module possesses a non-degenerate contravariant form and an admissible lattice on which the form is integral.*

Proof. The dual R-module to V_R with respect to the form $(\ ,\ )$,

$$V_R^* = \{v \in V \mid (v, V_R) \subseteq R\}$$

is easily checked to be an admissible lattice on V. Now V_R is the direct sum of Rv_0 and submodules consisting of vectors of lower weights. By (1B) and the condition that $(v_0, v_0) = 1$, we see that V_R^* contains v_0. Since V_R is the smallest admissible lattice containing v_0, $V_R^* \supseteq V_R$, so that $(V_R, V_R) \subseteq R$. The last statement of the lemma follows by complete reducibility.

If V is irreducible, v_0 is a vector of highest weight, and $(\ ,\ )$ is the contravariant form on V normalized so that $(v_0, v_0) = 1$, we shall call the discriminant of the integral form $(\ ,\ )$ on $V_R = v_0\mathfrak{U}_R$ the *discriminant* of V. (This can always be taken to be an ordinary integer.) We call V *p-unimodular* if its discriminant is a unit of R.

(2B) LEMMA. *If V is a p-unimodular irreducible* $\mathfrak{g}$*-module, there is only one admissible lattice on V, to within multiplication by a scalar.*

Proof. Let v_0 be a vector of highest weight in V. If V_R is an admissible lattice on V, then by multiplication by a scalar we may assume that V_R is the direct sum of Rv_0 with submodules of vectors of lower weights. If $(\ ,\ )$ is the contravariant form on V such that $(v_0, v_0) = 1$, and V_R^* is the corresponding dual lattice to V_R, we see as in (2A) that $v_0\mathfrak{U}_R \subseteq V_R^*$. Hence, V_R is contained in the dual of $v_0\mathfrak{U}_R$, which is $v_0\mathfrak{U}_R$ itself, since the form is unimodular on $v_0\mathfrak{U}_R$. Hence, $V_R = v_0\mathfrak{U}_R$.

We remark that the proof shows that every admissible lattice on an irreducible $\mathfrak{g}$-module V with vector v_0 of highest weight may be transformed by multiplication by a scalar into one which contains $v_0\mathfrak{U}_R$ and is contained in the dual of $v_0\mathfrak{U}_R$ with respect to the contravariant form, normalized so that $(v_0, v_0) = 1$. It follows easily that, to within multiplication by scalars, there is only a finite number of admissible lattices on V.

If V_R is an admissible lattice on V, then

$$\bar{V} = V_R/\pi V_R \simeq V_R \otimes_R K$$

is a vector space over K. If $v \in V_R$, we denote the residue class of v in $\bar{V}$

by $\bar{v}$. If $\{v_1, \cdots, v_m\}$ is a basis of V_R over R, then $\{\bar{v}_1, \cdots, \bar{v}_m\}$ is a basis of $\bar{V}$ over K. Thus, $\dim_K \bar{V} = \dim_L V$. Clearly, $\mathfrak{U}_R$ acts on $\bar{V}$ as well as on V. For $X \in \mathfrak{U}_R$, denote the corresponding linear transformations of V, $\bar{V}$ by $\rho(X)$, $\bar{\rho}(X)$, respectively. If $r \in \sum$, $m \geqq 0$, denote the element $X_r^m/m!$ of $\mathfrak{U}_R$ by $X_{r,m}$. Since V is a direct sum of finitely many weight spaces, it follows that $\rho(X_r)$ is nilpotent on V. Hence, for $t \in R$, $\bar{t} \in K$, the invertible linear transformations

$$x_r(t\ ;\ V) = \sum_{m=0}^{\infty} t^m \rho(X_{r,m}),$$

$$\bar{x}_r(\bar{t}\ ;\ V) = \sum_{m=0}^{\infty} \bar{t}^m \bar{\rho}(X_{r,m}),$$

of V, $\bar{V}$ are defined, the sums being finite. The sets

$$\{x_r(t\ ;\ V) \mid r \in \textstyle\sum,\ t \in R\},$$
$$\{\bar{x}_r(\bar{t}\ ;\ V) \mid r \in \textstyle\sum,\ \bar{t} \in K\},$$

generate groups $G_R(V)$, $G_K(V)$ of linear transformations on V, $\bar{V}$. (The notation differs slightly from that of [11], where, for example, $G_K(V)$ is denoted $G'_K(V)$). To within isomorphism, these groups depend only on the weight group $P(V)$ and not on V itself. Clearly, there is an epimorphism

$$G_R(V) \to G_K(V)$$

such that $x_r(t\ ;\ V)$ is mapped on $\bar{x}_r(\bar{t}\ ;\ V)$, where $\bar{t}$ is the residue class in K of the element t of R.

If χ is a character of the full weight group P in K, i.e., a homomorphism of P into the multiplicative group of K, then there is a linear transformation $h(\chi, V)$ of $\bar{V}$ such that if v is any vector of weight μ in V_R and $\bar{v}$ is the corresponding vector of $\bar{V}$, then

$$\bar{v}h(\chi, V) = \chi(\mu)\bar{v}.$$

These $h(\chi, V)$ form an Abelian subgroup $\mathfrak{H}_K(V)$ of $G_K(V)$. Since the Weyl group permutes the weights of $\mathfrak{g}$, it acts in a natural way on the characters χ and hence on the group $\mathfrak{H}_K(V)$.

If V_1 is a faithful $\mathfrak{g}$-module such that $P(V_1) \supseteq P(V)$, then there are epimorphisms

$$\phi : G_R(V_1) \to G_R(V),\ \ \bar{\phi} : G_K(V_1) \to G_K(V),$$

such that

$$\phi(x_r(t\ ;\ V_1)) = x_r(t\ ;\ V),$$
$$\bar{\phi}(\bar{x}_r(\bar{t}\ ;\ V_1)) = \bar{x}_r(\bar{t}\ ;\ V),$$
$$\bar{\phi}(h(\chi, V_1)) = h(\chi, V),$$

for $r \in \sum$, $t \in R$, $\bar{t} \in K$, and χ a character of P in K. In particular, this holds if $P(V_1) = P$. In this case we denote $G_R(V_1)$, $G_K(V_1)$ simply as G_R, G_K, so that we have epimorphisms

$$G_R \to G_R(V),\ \ G_K \to G_K(V),$$

for every $\mathfrak{g}$-module V. We shall also write $x_r(t)$, $\bar{x}_r(\bar{t})$, $h(\chi)$, $\mathfrak{H}_K$, for $x_r(t\ ;V_1)$, $\bar{x}_r(\bar{t}\ ;\ V_1)$, $h(\chi, V_1)$, $\mathfrak{H}_K(V_1)$. The group G_K is the "simply-connected" Chevalley group of type $\mathfrak{g}$ over K. More precisely, G_K is the group of rational points over K of a simply connected semi-simple algebraic group defined over the prime field. Thus, in particular, if $\mathfrak{g}$ is of type A_n, C_n, B_n, D_n, then G_K is isomorphic respectively to $SL(n+1, K)$, $Sp(2n, K)$, Spin $(2n+1, K)$, Spin $(2n, K)$, the last two groups being defined relative to forms of maximal Witt index.

We shall be concerned with modules for the group algebra of G_K over K which are finite-dimensional over K, and call these simply G_K-modules. Similarly a finite-dimensional module for the group algebra of G_R over L will be called a G_R-module. Since $G_K(V)$ is a group of linear transformations on $\bar{V}$, the homomorphism $G_K \to G_K(V)$ makes $\bar{V}$ into a G_K-module, which we call the module *derived* from the $\mathfrak{g}$-module V by means of the admissible lattice V_R. Similarly, V itself becomes a G_R-module.

If $x \in G_R$, we denote its image in G_K under the natural epimorphism $G_R \to G_K$ by $\bar{x}$. From our definitions it is clear that

$$\overline{vx} = \bar{v}\,\bar{x} \qquad (v \in V_R,\ x \in G_R).$$

The derived module $\bar{V}$ is in general not independent of the choice of admissible lattice V_R.

(2C) LEMMA. *If V_R, V_R' are two admissible lattices on a $\mathfrak{g}$-module V, then the corresponding derived G_K-modules $\bar{V}$, $\bar{V}'$ have the same irreducible constituents. If V is irreducible and p-unimodular, then $\bar{V}$ is independent of the choice of admissible lattice V_R, to within isomorphism.*

Proof. The first statement is essentially the same as a result of Brauer [2, p. 954]. An alternative argument is as follows. Since $V_R \cap V'_R$ is another admissible lattice on V, we may suppose that $V_R \supseteq V'_R$. We consider V_R and V'_R as $\mathfrak{U}_R$-modules. We have $\mathfrak{U}_R$-isomorphisms

$$V_R/V'_R \simeq \pi V_R/\pi V'_R,$$
$$(\pi V_R + V'_R)/V'_R \simeq \pi V_R/(\pi V_R \cap V'_R).$$

Since V_R/V'_R is finite, it follows that $V_R/(\pi V_R + V'_R)$ and $(\pi V_R \cap V'_R)/\pi V'_R$ have the same composition factors. Since

$$(\pi V_R + V'_R)/\pi V_R \simeq V'_R/(\pi V_R \cap V'_R),$$

it follows that $\bar{V} = V_R/\pi V_R$ and $\bar{V}' = V'_R/\pi V'_R$ have the same composition factors, as $\mathfrak{U}_R$-modules. This implies that they have the same composition factors as G_K-modules.

The second assertion of the lemma follows immediately from (2B).

The lemma implies in particular that whether or not $\bar{V}$ is irreducible does not depend on the choice of admissible lattice V_R. It is possible for $\bar{V}$ to be a reducible G_K-module even though V is an irreducible $\mathfrak{g}$-module.

(2D) LEMMA. *If V is an irreducible $\mathfrak{g}$-module which is not $\mathfrak{p}$-unimodular, then $\bar{V}$ is a reducible G_K-module.*

Proof. Let v_0 be a vector of highest weight in V, and $(\ ,\ )$ the contravariant form on V such that $(v_0, v_0) = 1$. By (2A), the form is integral on the admissible lattice $V_R = v_0\mathfrak{U}_R$.

We define a symmetric bilinear form on the derived module $\bar{V}$, which we shall also denote by $(\ ,\ )$, by taking $(\bar{v}, \bar{w})$ to be the residue class in K of the element (v, w) of R, where $v, w \in V_R$. By induction on m, using (3), we have

$$(\bar{v}\bar{\rho}(X_{r,m}), \bar{w}) = (\bar{v}, \bar{w}\bar{\rho}(X_{-r,m})),$$

where $r \in \sum$, $m \geqq 0$. It follows that

$$(\bar{v}\bar{x}_r(\bar{t}), \bar{w}) = (\bar{v}, \bar{w}\bar{x}_{-r}(\bar{t})),$$

where $\bar{t} \in K$. Thus, the radical of the form $(\ ,\ )$ on $\bar{V}$,

$$\bar{V}_0 = \{\bar{v} \in \bar{V} \mid (\bar{v}, \bar{V}) = \{0\}\}$$

is a G_K-submodule of $\bar{V}$.

Since the discriminant of the form on V_R is not a unit of R, the form $(\ ,\ )$ on $\bar{V}$ is degenerate. It is not identically zero, since $(\bar{v}_0, \bar{v}_0) = \dot{1}$. Hence, $\{0\} \subset \bar{V}_0 \subset \bar{V}$, so that $\bar{V}$ is reducible.

On the other hand, an irreducible $\mathfrak{g}$-module is irreducible as a G_R-module by the following result.

(2E) Lemma. *If V is any $\mathfrak{g}$-module, the $\mathfrak{g}$-submodules of V are the same as the G_R-submodules of V.*

Proof. By definition of $G_R(V)$, every $\mathfrak{g}$-submodule of V is a G_R-submodule. If $r \in \sum$, $t \in R$, then

$$x_r(t\ ;\ V) = 1 + t\rho(X_r) + t^2\rho(X_r)^2/2! + \cdots + t^N\rho(X_r)^N/N!,$$

for some N. If we choose $N+1$ distinct values of $t_1, t_2, \cdots, t_{N+1}$ of t in R, then we can solve the resulting equations to obtain

$$\rho(X_r) = \sum_{i=1}^{N+1} a_i x_r(t_i\ ;\ V)$$

for suitable a_i in L. It now follows that every G_R-submodule of V is a $\mathfrak{g}$-submodule.

We shall need some information concerning the arrangement of the constituents of $\bar{V}$, $\bar{V}'$ in (2C) in a special situation.

(2F) Lemma. *Let V_R, V'_R be admissible lattices on a $\mathfrak{g}$-module V, giving derived G_K-modules $\bar{V}$, $\bar{V}'$. Suppose that $V_R \supseteq V'_R$, and that V'_R contains an element v which is not in πV_R. Then $\bar{V}'$ has a non-trivial quotient module which is isomorphic with a submodule of $\bar{V}$ containing the residue class $\bar{v}$ of v modulo πV_R.*

Proof. The inclusion map of V'_R in V_R induces a G_K-homomorphism $\bar{V}' \to \bar{V}$ whose image contains $\bar{v}$, since $v \in V'_R$. Since $v \notin \pi V_R$, $\bar{v}$ is non-zero.

The proof of the following lemma is straightforward and is omitted.

(2G) Lemma. *Let V_R be an admissible lattice on a $\mathfrak{g}$-module V. If W is a $\mathfrak{g}$-submodule of V, then $W_R = W \cap V_R$ is an admissible lattice on W and the derived G_K-module $\bar{W}$ may be identified with a submodule of $\bar{V}$. When embedded in the natural way, V_R/W_R is an admissible lattice on V/W and the corresponding derived G_K-module is isomorphic with $\bar{V}/\bar{W}$. If U_R is an admissible lattice on another $\mathfrak{g}$-module U, then $U_R \otimes_R V_R$ is an admissible lattice on $U \otimes_L V$ when embedded in the natural way, and the corresponding derived G_K-module is isomorphic with $\bar{U} \otimes_K \bar{V}$.*

If V^* is the dual space of V, regarded as the contragredient $\mathfrak{g}$-module, then the dual module V_R^* of V_R, naturally embedded in V^*, is an admissible lattice on V^*, and the corresponding derived G_K-module is isomorphic with the contragredient of $\bar{V}$.

The group analogue of (1A) is the following result.

(2H) LEMMA. *There exist automorphisms ϕ, $\bar{\phi}$ of G_R, G_K respectively, both of order 2, such that*

$$\phi(x_r(t)) = x_{-r}(t),$$
$$\bar{\phi}(\bar{x}_r(\bar{t})) = \bar{x}_{-r}(\bar{t}),$$

for all $r \in \sum$, $t \in R$, $\bar{t} \in K$.

Proof. We take a faithful $\mathfrak{g}$-module V such that $P(V) = P$, so that $G_R = G_R(V)$, $G_K = G_K(V)$. Using the automorphism θ of $\mathfrak{g}$ given in (1A), we make V into a new $\mathfrak{g}$-module V^* by setting

$$v \circ X = v\theta(X) \qquad (v \in V, X \in \mathfrak{g}).$$

We have seen that V^* is the $\mathfrak{g}$-module contragredient to V. If ρ, ρ^* are the representations of the universal enveloping algebra $\mathfrak{U}$ in V, V^* respectively, then we see that

$$\rho^*(X_{r,m}) = (-1)^m \rho(X_{-r,m}),$$

for all $r \in \sum$, $m \geqq 0$. Thus, an admissible lattice V_R on V is also an admissible lattice on V^*, and an analogous relation holds for the action of $\mathfrak{U}_R$ on the derived $\bar{V}$, $\bar{V}^*$. It follows that

$$x_r(t\ ;\ V^*) = x_{-r}(-t),$$
$$\bar{x}_r(\bar{t}\ ;\ V^*) = \bar{x}_{-r}(-\bar{t}),$$

for all $r \in \sum$, $t \in R$, $\bar{t} \in K$. Hence, $G_R(V^*) = G_R$, $G_K(V^*) = G_K$.

Since $P(V^*) = P(V)$, there are isomorphisms ϕ, $\bar{\phi}$ of $G_R = G_R(V)$ and $G_K = G_K(V)$ on $G_R(V^*)$ and $G_K(V^*)$ respectively, transforming $x_r(t)$ into $x_r(t\ ;\ V^*)$ and $\bar{x}_r(\bar{t})$ into $\bar{x}_r(\bar{t}\ ;\ V^*)$. These are the desired automorphisms.

If $g \in G_R$, $\bar{g} \in G_K$, we shall denote $\phi(g)$, $\phi(\bar{g})$ as g^*, $\bar{g}^*$. We remark that if $\{v_1, v_2, \cdots\}$ is a basis of an admissible lattice V_R on a $\mathfrak{g}$-module V and $\{w_1, w_2, \cdots\}$ is the contragredient basis relative to a contravariant form $(\ ,\ )$, then

$$v_i g = \sum_j g_{ij} v_j, \tag{4}$$
$$w_i g^* = \sum_j g^*_{ij} w_j,$$

where the matrices (g_{ij}), (g^*_{ij}) are contragredients of each other. If the form is integral and unimodular on V_R, so that the w_i also form a basis of V_R, then we have the corresponding relations also for the action of G_K on the derived module $\bar{V}$:

$$\bar{v}_i \bar{g} = \sum_j \bar{g}_{ij} \bar{v}_j, \tag{5}$$
$$\bar{w}_i \bar{g}^* = \sum_j \bar{g}^*_{ij} \bar{w}_j,$$

where the matrices $(\bar{g}_{ij})$, $(\bar{g}^*_{ij})$ are contragredients of each other.

Finally, we remark that the construction of G_K can be carried out with K replaced by any commutative R-algebra with identity. In particular, if Ω is an extension field of K and V_R is an admissible lattice on a $\mathfrak{g}$-module V, then $G_\Omega(V)$ is defined as a group of linear transformations on the vector space $V_R \otimes_R \Omega$ over Ω. There is an obvious embedding of $G_K(V)$ in $G_\Omega(V)$. If $P(V) = P$, then $G_\Omega(V)$ is simply denoted G_Ω, and G_K is embedded in G_Ω.

3. Brauer Characters

We require some results of Steinberg concerning the p-regular classes of G_K [15]. From now on, q will denote the number of elements of K.

Let Ω be the algebraic closure of K and take G_K as embedded in G_Ω. Then, two p-regular elements of G_K are conjugate in G_K if and only if they are conjugate in G_Ω, and each p-regular element of G_K is conjugate in G_Ω to an element of the Cartan subgroup $\mathfrak{H}_\Omega$. Two elements of $\mathfrak{H}_\Omega$ are conjugate in G_Ω if and only if they are conjugate under the action of the Weyl group W. If χ is a character of P in Ω, then the element $h(\chi)$ of $\mathfrak{H}_\Omega$ is conjugate to an element of G_K if and only if χ is conjugate to χ^q under W. Thus, if A is the set of all such χ, there is a 1-1 correspondence between the set of p-regular classes of G_K and the set of orbits in A under W. The number of p-regular classes in G_K is q^n, where n is the rank of $\mathfrak{g}$.

Let m be the p'-part of $|G_K|$. We choose a fixed isomorphism between the group of m-th roots of 1 in Ω and the group of m-th roots of 1 in the complex field C. If ε is an m-th root of 1 in Ω, we denote the corresponding complex m-th root of 1 by ε_0. Since the order of every p-regular element of G_K divides m, $\chi^m = 1$ for all $\chi \in A$. Thus, for $\mu \in P$, $\chi(\mu)_0$ is defined. Since χ is determined by its values on the fundamental weights λ_i, every complex valued function f on the finite set A can be expressed as a polynomial in the functions $\chi \longmapsto \chi(\lambda_i)_0$, and thus as a linear combination

$$f(\chi) = \sum_{\mu} c_\mu \chi(\mu)_0, \tag{6}$$

where μ ranges over certain elements of P. If f is constant on each orbit in A under W, then

$$f(\chi) = \sum_{\mu} c_\mu \chi(w(\mu))_0,$$

for all $w \in W$. Summing over W and dividing by $|W|$, we see that the expression (6) for f may be taken to be symmetric under the action of W. By the 1–1 correspondence between the p-regular classes of G_K and the orbits in A under W, these symmetric expressions give all the complex valued functions on the p-regular classes of G_K.

For each element μ of the set P^+ of dominant integral functions in P, we now define a complex valued function s_μ on the p-regular classes of G_K by the formula

$$s_\mu(g) = \sum_{\nu} \chi(\nu)_0, \tag{7}$$

where g is conjugate in G_Ω to $h(\chi)$ $(\chi \in A)$, and the sum is taken over all ν in the orbit of μ under W. Since every orbit in P under W contains an element of P^+, the symmetric expression (6) shows that every complex valued function on the p-regular classes of G_K is a linear combination of the s_μ, for various $\mu \in P^+$. Since χ and χ^q are conjugate under W when $\chi \in A$, we see easily from (7) that

$$s_{q\mu} = s_\mu. \tag{8}$$

Since μ is the highest element of P occurring in (7), it follows easily that, if μ, $\rho \in P^+$, then

$$s_\mu s_\rho = s_{\mu+\rho} + \sum_{\nu} s_\nu, \tag{9}$$

where the sum is taken over certain elements ν of P^+ lower than $\mu + \rho$ (with possible repetitions).

(3A) LEMMA. *Let $\mu=\sum_i m_i\lambda_i\in P^+$. If $m_i\geqq q$ for some i, then $s_\mu=\sum_\nu \pm s_\nu$, where the sum is taken over certain elements ν of P^+ lower than μ.*

Proof. Since $\mu-q\lambda_i\in P^+$, we see from (8) and (9) that

$$s_\mu = s_{\mu-q\lambda_i}s_{\lambda_i} - \text{terms } s_\nu,\ \nu<\mu,$$
$$s_{\mu-q\lambda_i}s_{\lambda_i} = s_{\mu-(q-1)\lambda_i} + \text{terms } s_\rho,\ \rho<\mu-(q-1)\lambda_i.$$

Since λ_i is positive, $\mu-(q-1)\lambda_i$ is lower than μ, and we obtain the desired expression for s_μ.

We now write P_q for the set of all elements $\mu=\sum m_i\lambda_i$ of P^+ such that $0\leqq m_i\leqq q-1$ for all i. Since the number of p-regular classes of G_K is q^n, the number of elements of P_q, we deduce the following result from (3A) by induction in P^+.

(3B) COROLLARY. *The q^n functions $s_\mu, \mu\in P_q$, form a basis for the vector space of complex valued functions on the p-regular classes of G_K. In particular, if $\rho\in P^+$, then s_ρ is a linear combination of certain s_μ, where $\mu\in P_q$ and $\mu\leqq\rho$.*

Now let $U(\mu)$ be the irreducible $\mathfrak{g}$-module with highest weight μ, $\mu\in P^+$. The derived G_K-module $\bar{U}(\mu)$ obtained with a choice of admissible lattice in $U(\mu)$ corresponds to the representation $G_K\to G_K(U(\mu))$, which obviously extends to the representation $G_\Omega\to G_\Omega(U(\mu))$. An element $h(\chi)$ of $\mathfrak{H}_\Omega$ is represented by a linear transformation whose matrix with respect to a suitable basis is diagonal, with entries $\chi(\mu_i)$, where μ_i ranges over the weights of $U(\mu)$, each counted with its multiplicities. If g is a p-regular element of G_K and χ is an element of A such that g is conjugate to $h(\chi)$ in G_Ω, then the $\chi(\mu_i)$ are m-th roots of 1 in Ω, and the Brauer character ϕ_μ of $\bar{U}(\mu)$ is given by

$$\phi_\mu(g) = \sum_i \chi(\mu_i)_0.$$

(If L is taken as a subfield of the complex field C, so that G_C is defined as remarked at the end of Section 2, and χ_0 is the character of P in C given by $\chi_0(\nu)=\chi(\nu)_0 (\nu\in P)$, then $\phi_\mu(g)=\phi_\mu^0(h(\chi_0))$, where ϕ_μ^0 is the character of the irreducible representation of G_C with highest weight μ. Thus, ϕ_μ can be computed from Weyl's character formula [12]. We remark also that ϕ_μ is independent of the choice of admissible lattice on $U(\mu)$, and this gives an alternative method of proving (2C).)

Since the μ_i are permuted among themselves by the Weyl group W, and μ occurs only once, we can express ψ_μ in the form

$$\psi_\mu = s_\mu + \sum_\nu b_{\mu\nu} s_\nu, \tag{10}$$

where the coefficients $b_{\mu\nu}$ are positive integers and the sum ranges over certain elements ν of P^+ which are lower than μ, By (3B), we can express the s_ν in terms of various s_ρ, where $\rho \in P_q$, $\rho < \mu$. Thus, we may assume that all the ν appearing in (10) are elements of P_q lower than μ, if we allow the $b_{\mu\nu}$ to be possibly negative integers.

For $\mu \in P_q$, we see by induction that we can solve the equations (10) to obtain the s_μ in terms of the ψ_μ :

$$s_\mu = \psi_\mu + \sum_\nu c_{\mu\nu} \psi_\nu, \tag{11}$$

where the $c_{\mu\nu}$ are integers and the sum is taken over elements ν of P_q lower than μ. By (3B), we obtain the following result.

(3C) LEMMA. *The q^n Brauer characters ψ_μ, $\mu \in P_q$, form a basis for the vector space of complex valued functions on the p-regular classes of G_K. Hence every absolutely irreducible module of G_K over an extension field of K is a constituent of $\bar{U}(\mu)$, for some $\mu \in P_q$.*

We shall write σ for the highest member of P_q,

$$\sigma = (q-1) \sum_i \lambda_i.$$

(3D) LEMMA. *The G_K-module $\bar{U}(\sigma)$ is absolutely irreducible, of dimension q^N, where N is the number of positive roots in $\sum$. All other absolutely irreducible modules of G_K over an extension field of K have lower dimension. $\bar{U}(\sigma)$ is not a constituent of $\bar{U}(\mu)$, for $\mu \in P_q$, $\mu \neq \sigma$.*

Proof. We have Weyl's formula [12],

$$\dim_K \bar{U}(\mu) = \dim_L U(\mu) = \prod_{r>0} (r, \delta + \mu)/(r, \delta),$$

where $\delta = \sum_i \lambda_i$ and the product is taken over all positive roots r. For such an $r, (r, \lambda_i) = \frac{1}{2}(r, r)\lambda_i(H_r) \geqq 0$, and $(r, \lambda_i) \neq 0$ for some i, so that $(r, \delta) > 0$. If $\mu = \sum_i m_i \lambda_i$, then

$$(r, \delta + \mu) = \sum_i (m_i + 1)(r, \lambda_i).$$

Clearly, for $\mu \in P_q$, the largest dimension for $\bar{U}(\mu)$ is q^N, where N is the number of positive roots, and is obtained only when $\mu = \sigma$.

By a result of Steinberg [14], G_K has an absolutely irreducible module of dimension q^N in a field of characteristic p. It follows from (3C) that $\bar{U}(\sigma)$ must be absolutely irreducible. Since $\dim \bar{U}(\mu) < \dim \bar{U}(\sigma)$ when $\mu \in P_q$, $\mu \neq \sigma$, the other assertions follow.

(3E) THEOREM. *Let $|K| = q$. For each $\mu \in P_q$, there exists exactly one absolutely irreducible module $F(\mu)$ of G_K over an extension field of K whose Brauer character φ_μ has the expansion*

$$\varphi_\mu = s_\mu + \sum_\nu b_{\mu\nu} s_\nu, \tag{12}$$

where the $b_{\mu\nu}$ are integers, and the sum is taken over elements ν of P_q lower than μ. The $F(\mu)$ form a complete set of absolutely irreducible modules of G_K in characteristic p. For $\mu \in P_q$, $F(\mu)$ occurs as a constituent of $\bar{U}(\mu)$ exactly once. For any $\mu \in P^+$, all constituents $F(\nu)$ of $\bar{U}(\mu)$ correspond to elements ν of P_q such that $\nu \leqq \mu$.

Proof. Since the number of p-regular classes in G_K is q^n, there are q^n absolutely irreducible modules in characteristic p. Denote these as F_i, $1 \leqq i \leqq q^n$, and let φ_i be the Brauer character of F_i. By (3B), we have equations

$$\varphi_i = \sum_\nu d_{i\nu} s_\nu,$$

where the $d_{i\nu}$ are complex numbers, and the sum is taken over all $\nu \in P_q$.

Suppose that there exists an element μ of P_q such that $\bar{U}(\mu)$ has a constituent F_i for which $d_{i\nu} \neq 0$ for some ν in P_q higher than μ. Assume that ν is the highest member of P_q with this property. Of course, $\mu \neq \sigma$, and $\sigma - \nu \in P_q$. By (9), (10), (11),

$$\varphi_i \psi_{\sigma-\nu} = d_{i\nu} \psi_\sigma + \sum_\rho k_\rho \psi_\rho,$$

where the k_ρ are complex numbers and the sum is taken over elements ρ of P_q different from σ. Expanding both sides in terms of the irreducible Brauer characters, we see by (3D) that ψ_σ occurs as an irreducible part of the character $\varphi_i \psi_{\sigma-\nu}$ of the module $F_i \otimes \bar{U}(\sigma - \nu)$. Thus, $\bar{U}(\sigma)$ is a constituent of $F_i \otimes \bar{U}(\sigma - \nu)$, and hence of $\bar{U}(\mu) \otimes \bar{U}(\sigma - \nu)$. Now, by (9), (10), (11),

$$\psi_\mu \psi_{\sigma-\nu} = \psi_{\sigma-\nu+\mu} + \sum_\tau c_\tau \psi_\tau - \sum_\rho d_\rho \psi_\rho,$$

where the c_τ, d_ρ are positive integers and the sums are taken over certain τ, ρ in P_q lower than $\sigma - \nu + \mu$. Thus, the module

$$(\bar{U}(\mu) \otimes \bar{U}(\sigma - \nu)) \oplus \sum_\rho d_\rho \bar{U}(\rho)$$

has the same absolutely irreducible constituents as

$$\bar{U}(\sigma - \nu + \mu) \oplus \sum_\tau c_\tau \bar{U}(\tau).$$

Hence, $\bar{U}(\sigma)$ is a constituent of the latter module, contradicting (3D).

Thus, if $\mu \in P_q$, every irreducible part of ψ_μ can be expressed in terms of the s_ν with $\nu \in P_q$, $\nu \leq \mu$. In particular, there is an absolutely irreducible constituent of $\bar{U}(\mu)$ whose Brauer character has highest terms s_μ (with some non-zero coefficient). Since this is so for each of the q^n elements of P_q, it follows that, for $\mu \in P_q$, there is exactly one irreducible Brauer character with s_μ as highest term. We call this character φ_μ, and the corresponding module $F(\mu)$. Then the absolutely irreducible constituents of $\bar{U}(\mu)$ are $F(\mu)$, occurring with some multiplicity, and possibly certain $F(\nu)$, with $\nu < \mu$.

Suppose that $F(\mu)$ occurs in $\bar{U}(\mu)$ with multiplicity a_μ. Then we see that

$$\psi_\mu \psi_{\sigma-\mu} = a_\mu a_{\sigma-\mu} \varphi_\mu \varphi_{\sigma-\mu} + \cdots,$$

where the missing terms involve products $\varphi_\nu \varphi_\rho$, where $\nu, \rho \in P_q$, $\nu + \rho < \sigma$. Since s_σ occurs in the expansion of $\varphi_\mu \varphi_{\sigma-\mu}$ in terms of the s_τ, φ_σ occurs in its expansion in terms of the φ_τ. Thus, $\bar{U}(\sigma) = F(\sigma)$ occurs as a constituent of $\bar{U}(\mu) \otimes \bar{U}(\sigma - \mu)$ at least $a_\mu a_{\sigma-\mu}$ times. But we know that it occurs exactly once, from the expression of the Brauer characters by the s_τ. Hence, $a_\mu = a_{\sigma-\mu} = 1$.

We now have equations of the form

$$\psi_\mu = \varphi_\mu + \text{lower } \varphi' s,$$

which can be solved, giving

$$\varphi_\mu = \psi_\mu + \sum_\nu c_{\mu\nu} \psi_\nu,$$

where the $c_{\mu\nu}$ are integers and the sum is taken over elements ν of P_q lower than μ. Using (10), we obtain the expression (12).

Finally, for any element μ of P^{+}, let ν be the highest element of P_q such that $F(\nu)$ is a constituent of $\bar{U}(\mu)$. Using (12), we see that the expression for the Brauer character ψ_μ of $\bar{U}(\mu)$ in terms of the s_ρ, $\rho \in P_q$, has highest term s_ν (with some coefficient). Hence, $\nu \leqq \mu$, by (10) and (3B).

(3F) COROLLARY. *All the absolutely irreducible modules for G_K in characteristic p can be realized over K. Every irreducible module for G_K over K is absolutely irreducible.*

Proof. The trace $\bar{\varphi}_\mu$ of the absolutely irreducible module $F(\mu)$ for G_K over Ω is obtained from (12) as

$$\bar{\varphi}_\mu = \bar{s}_\mu + \sum_\nu b_{\mu\nu} \bar{s}_\nu,$$

where, for each $\mu \in P_q$, $\bar{s}_\mu$ is the function from the p-regular calsses of G_K to Ω given by the corresponding formula to (7),

$$\bar{s}_\mu(g) = \sum_\nu \chi(\nu),$$

where g is conjugate in G_Ω to $h(\chi)$ $(\chi \in A)$, and the sum is taken over all ν in the orbit of μ under W. Since χ^q is conjugate to χ under W, we obtain

$$\bar{s}_\mu(g)^q = \sum_\nu \chi^q(\nu) = \sum_\nu \chi(\nu) = \bar{s}_\mu(g),$$

so that $\bar{s}_\mu(g) \in K$. Hence, all the traces $\bar{\varphi}_\mu$ have their values in K. By a theorem of Brauer [1], all the $F(\mu)$ can be realized over K. The second assertion follows from the first, since K is a perfect field [4].

(3G) COROLLARY. *Let $\mu, \nu, \rho \in P_q$. If $F(\rho)$ is a constituent of $F(\mu) \otimes F(\nu)$, then $\rho \leqq \mu + \nu$. If $\mu + \nu \in P_q$, then $F(\mu+\nu)$ occurs exactly once as a constituent of $F(\mu) \otimes F(v)$.*

Proof. For $\mu \in P_q$, we can solve the equations (12) to obtain

$$s_\mu = \varphi_\mu + \sum_\nu c_{\mu\nu} \varphi_\nu,$$

where the sum is taken over elements ν of P_q lower than μ, and the $c_{\mu\nu}$ are integers. Using (12), (9) and (3B), we can calculate an expression for $\varphi_\mu \varphi_\nu$ in terms of the φ_ρ to obtain the result.

4. An Irreducibility Criterion

We shall now determine a necessary and sufficient condition for the irreducibility of the G_K-module derived from an irreducible $\mathfrak{g}$-module whose highest weight lies in P_q. We shall use an argument which will be adapted to the construction of the irreducible G_K-modules $F(\mu)$ in general in the next section.

For $i = 1, \cdots, n$, let $V(i)$ denote a $\mathfrak{g}$-module whose highest weight is λ_i, occurring with multiplicity 1. For example, $V(i)$ can be taken as the irreducible module of highest weight λ_i. We choose an admissible lattice $V(i)_R$ on $V(i)$, and take a regular basis $\{x(i)_0, x(i)_1, \cdots\}$, where $x(i)_0$ has weight λ_i, so that the other $x(i)_j$ have lower weights. Choose a non-degenerate contravariant form on $V(i)$ and let $\{y(i)_0, y(i)_1, \cdots\}$ be the basis of $V(i)$ dual to $\{x(i)_0, x(i)_1, \cdots\}$ with respect to this form. Then the $y(i)_j$ are also weight vectors and we may assume that $y(i)_0 = x(i)_0$.

Now take any element μ of P^+, say

$$\mu = \sum_{i=1}^{n} m_i \lambda_i,$$

where the m_i are non-negative integers. Let $V(i)^{(m_i)}$ be the m_i-th symmetric power of $V(i)$, i.e., the space of all homogeneous polynomials over L of total degree m_i in $x(i)_0, x(i)_1, \cdots$. Then

$$V(\mu) = V(1)^{(m_1)} \otimes \cdots \otimes V(n)^{(m_n)}$$

is a $\mathfrak{g}$-module, consisting of homogeneous polynomials $v = f(x(i)_j)$ in the $x(i)_j$. Thus, $V(\mu)$ is also a G_R-module, the action of an element g of G_R being given by

$$vg = f(x(i)_j g),$$

where $x(i)_j g$ is the image of $x(i)_j$ under the action of g on $V(i)$, considered as a G_R-module.

We now form a regular basis of $V(\mu)$, consisting of the monomials

$$x_\beta = \prod_{i,j} x(i)_j^{\beta(i,j)}, \tag{13}$$

where, for each i, $\sum_j \beta(i,j) = m_i$, and β stands for the family of exponents $\beta(i,j)$. We shall similarly write

$$y_\beta = \prod_{i,j} y(i)_j^{\beta(i,j)},$$

and remark that these elements form another regular basis of $V(\mu)$. The monomial

$$x_0 = \prod_i x(i)_0{}^{m_i}$$

is a vector of weight μ and all other monomials x_β have lower weights. It follows that the $\mathfrak{g}$-submodule $U(\mu)$ of $V(\mu)$ generated by x_0 is irreducible of highest weight μ.

By (2E), $U(\mu)$ is the subspace generated by all the vectors x_0g, $g \in G_R$. We have equations

$$x(i)_0 g = \sum_j g(i)_j x(i)_j, \tag{14}$$

for $i = 1, \cdots, n$, where $g(i)_j \in R$. A calculation shows that, in terms of the basis (13),

$$x_0 g = \sum_\beta c_\beta g_\beta x_\beta \tag{15}$$

where g_β is the element of R obtained by substituting $g(i)_j$ for $x(i)_j$ in the expression for x_β,

$$g_\beta = \prod_{i,j} g(i)_j^{\beta(i,j)}, \tag{16}$$

and c_β is a product of polynomial coefficients,

$$c_\beta = \prod_i (m_i!/\beta(i,0)!\,\beta(i,1)!\cdots).$$

We remark that c_β is always an integer.

An element of $V(\mu)$,

$$v = \sum_\beta d_\beta x_\beta, \tag{17}$$

lies in $U(\mu)$ if and only if $\phi(v) = 0$ for all elements ϕ of the dual space of $V(\mu)$ which vanish on the elements (15). Setting $a_\beta = \phi(x_\beta)$, we see that $v \in U(\mu)$ if and only if

$$\sum_\beta a_\beta d_\beta = 0, \tag{18}$$

for all a_β such that

$$\sum_\beta c_\beta a_\beta g_\beta = 0, \tag{19}$$

for all $g \in G_R$.

For each β, define an operator $\varDelta_\beta$ on $V(\mu)$ by setting

$$\Delta_\beta = \prod_{i,j} (\partial/\partial x(i)_j)^{\beta(i,j)},$$

where the partial differentiations are carried out in the usual way on the elements of $V(\mu)$ regarded as polynomials in the $x(i)_j$. We see that for the element (17),

$$c_\beta \Delta_\beta v = m_1! \cdots m_n! d_\beta. \tag{20}$$

If an element w of $V(\mu)$ is expressed in terms of the $y(i)_j$ (rather than the $x(i)_j$) in the form

$$w = \sum_\beta e_\beta y_\beta,$$

we define an operator

$$\Delta_w = \sum_\beta e_\beta \Delta_\beta, \tag{21}$$

by replacing each $y(i)_j$ by $\partial/\partial x(i)_j$. Also, if g is an element of G_R giving coefficients $g(i)_j$, g_β as in (14), (16), we write

$$w(g) = \sum_\beta e_\beta g_\beta, \tag{22}$$

the element of $\underset{\cdot}{L}$ obtained by substituting $g(i)_j$ for $y(i)_j$ in the expression for w. We say that w *vanishes upon specialization* to elements of G_R if $w(g)=0$ for all $g \in G_R$, and set

$$W(\mu) = \{w \in V(\mu) \mid w(g) = 0, \quad \text{all} \quad g \in G_R\}. \tag{23}$$

Setting $e_\beta = c_\beta a_\beta$, and using (20), (21), (22), (23), we see that the conditions (18), (19) may be restated in the form

$$U(\mu) = \{v \in V(\mu) \mid \Delta_w(v) = 0, \quad \text{all} \quad w \in W(\mu)\}.$$

If we now set

$$(v, w) = \Delta_w(v), \tag{24}$$

we obtain a non-degenerate bilinear form on $V(\mu)$, the elements

$$(c_\beta / m_1! \cdots m_n!) w_\beta$$

forming a basis dual to the v_β. Thus, $U(\mu)$ and $W(\mu)$ are orthogonal complements in $V(\mu)$ with respect to this form.

(4A) LEMMA. *Let* $v, w \in V(\mu)$, $g, h \in G_R$, *and let* g^* *be the image of* g *under the automorphism given by* (2H). *Then*

(a) $$\Delta_{wg^*}(vg) = \Delta_w(v),$$

(b) $$(wg^*)(h) = w(hg^{-1}).$$

Proof. As the calculations for both assertions are quite similar, we prove only (a). Express v, w as polynomials in the $x(i)_j$, $y(i)_j$ respectively,

$$v = f(x(i)_j), \quad w = h(y(i)_j).$$

By (4), if $x(i)_j' = x(i)_j g$, $y(i)_j' = y(i)_j g$, then

$$x(i)_j' = \sum_k g(i)_{jk} x(i)_k,$$

$$y(i)_j' = \sum_k g(i)^*_{jk} y(i)_k,$$

where, for each i, the matrices $(g(i)_{jk})$, $(g(i)^*_{jk})$ are contragredients of each other,

$$\sum_k g(i)^*_{jk} g(i)_{mk} = \delta_{jm}.$$

We now compute that

$$\partial/\partial x(i)_k = \sum_m g(i)_{mk}(\partial/\partial x(i)_m'),$$

$$\partial/\partial x(i)_j' = \sum_m \delta_{jm}(\partial/\partial x(i)_m') = \sum_k g(i)^*_{jk}(\partial/\partial x(i)_k).$$

Thus, substitution of $\partial/\partial x(i)_k$ for $y(i)_k$ in the expression for wg^* is tantamount to substitution of $\partial/\partial x(i)_j'$ for $y(i)_j'$. Since $wg^* = h(y(i)_j')$, $vg = f(x(i)_j')$, we see that

$$\Delta_{wg^*}(vg) = h(\partial/\partial x(i)_j') f(x(i)_j') = \Delta_w(v).$$

(4B) COROLLARY. $W(\mu)$ *is a* $\mathfrak{g}$-*submodule of* $V(\mu)$.

Proof. This is immediate from (4A) (b), (2E).

(4C) COROLLARY. $V(\mu) = U(\mu) \oplus W(\mu)$.

Proof. Using the assumption that $y(i)_0 = x(i)_0$, we calculate for the monomial x_0 of weight μ that

$$\Delta_{x_0}(x_0) = m_1! \cdots m_n! \neq 0,$$

so that $x_0 \notin W(\mu)$. Since $U(\mu)$ is an irreducible $\mathfrak{g}$-submodule of $V(\mu)$, $U(\mu) \cap W(\mu) = \{0\}$. Since $U(\mu)$ and $W(\mu)$ are orthogonal complements with respect to the form $(\ ,\ )$, we have the result.

For $i = 1, \cdots, n$, let $V(-i)$ be the dual space of $V(i)$, regarded as the $\mathfrak{g}$-module contragredient to $V(i)$, and let $\{x(-i)_0, x(-i)_1, \cdots\}$ be the (regular) basis of $V(-i)$ dual to the basis $\{x(i)_0, x(i)_1, \cdots\}$ of $V(i)$. The basis $\{y(-i)_0, y(-i)_1, \cdots\}$ of $V(-i)$ dual to the basis $\{y(i)_0, y(i)_1, \cdots\}$ of $V(i)$ is easily checked to be contragredient to $\{x(-i)_0, x(-i)_1, \cdots\}$ in the sense of Section 1. The $\mathfrak{g}$-module

$$V(-\mu) = V(-1)^{(m_1)} \otimes \cdots \otimes V(-n)^{(m_n)},$$

consisting of homogeneous polynomials in the $x(-i)_j$, has a regular basis consisting of the monomials

$$x_{-\beta} = \prod_{i,j} x(-i)_j^{\beta(i,j)}.$$

There is a vector space isomorphism of $V(\mu)$ with $V(-\mu)$, mapping y_β on $x_{-\beta}$, for all β. If $w \in V(\mu)$, we denote the corresponding element of $V(-\mu)$ by w^*. Then, if $g \in G_R$, $w \in V(\mu)$,

$$(wg^*)^* = w^*g.$$

If we now define, for $v, w \in V(\mu)$,

$$\langle v, w^* \rangle = \Delta_w(v)/(m_1! \cdots m_n!),$$

then, by (4A) (a), we obtain a non-degenerate bilinear pairing of $V(\mu)$ and $V(-\mu)$ which is invariant under G_R, i.e.,

$$\langle vg, w^*g \rangle = \langle v, w^* \rangle,$$

for $g \in G_R$. An argument as in the proof of (2E) shows that

$$\langle vX, w^* \rangle + \langle v, w^*X \rangle = 0$$

for $X \in \mathfrak{g}$. We note that the basis of $V(\mu)$ dual to the basis $\{x_{-\beta}\}$ of $V(-\mu)$ is the basis $\{c_\beta x_\beta\}$.

Now set

$$U(-\mu) = \{u^* \mid u \in U(\mu)\}, \quad W(-\mu) = \{w^* \mid w \in W(\mu)\}.$$

Then $U(-\mu)$ is the irreducible $\mathfrak{g}$-submodule of $V(-\mu)$ generated by the monomial

$$x_0^* = \prod_i x(-i)_0^{m_i}$$

of *lowest* weight $-\mu$ in $V(-\mu)$, and $W(-\mu)$ is the set of all elements of $V(-\mu)$ which vanish upon specialization to elements of G_R, in the obvious sense. Since $U(-\mu)$ is the subspace of $V(-\mu)$ orthogonal to $W(\mu)$ with respect to the form $\langle\ ,\ \rangle$, we have an induced non-degenerate invariant bilinear pairing between $U(-\mu)$ and $V(\mu)/W(\mu)$, which are thus contragredient $\mathfrak{g}$-modules.

We have admissible lattices

$$V(\mu)_R = \sum_\beta Rx_\beta,\quad V(-\mu)_R = \sum_\beta Rx_{-\beta},$$

on $V(\mu)$, $V(-\mu)$, and obtain admissible lattices

$$U(\mu)_R = U(\mu) \cap V(\mu)_R,\quad W(\mu)_R = W(\mu) \cap V(\mu)_R,$$
$$U(-\mu)_R = U(-\mu) \cap V(-\mu)_R,\quad W(-\mu)_R = W(-\mu) \cap V(-\mu)_R,$$

on $U(\mu)$, $W(\mu)$, $U(-\mu)$, $W(-\mu)$. We also have admissible lattices

$$V(\mu)_R' = \sum_\beta Rc_\beta x_\beta,\quad V(-\mu)_R' = \sum_\beta Rc_\beta x_{-\beta},$$

on $V(\mu)$, $V(-\mu)$. These are the dual R-modules to $V(-\mu)_R$, $V(\mu)_R$, with respect to the bilinear pairing $\langle\ ,\ \rangle$. We set

$$U(\mu)_R' = U(\mu) \cap V(\mu)_R',\quad U(-\mu)_R' = U(-\mu) \cap V(-\mu)_R',$$

admissible lattices on $U(\mu)$, $U(-\mu)$.

We now obtain derived G_K-modules $\bar{V}(\mu)$, $\bar{U}(\mu)$, $\bar{W}(\mu)$, $\bar{V}(\mu)'$, $\bar{U}(\mu)'$, etc. from the admissible lattices $V(\mu)_R$, $U(\mu)_R$, $W(\mu)_R$, $V(\mu)_R'$, $U(\mu)_R'$, etc., and, by (2G), we immediately have

(4D) LEMMA. *$\bar{V}(\mu)'$, $\bar{V}(\mu)$, $\bar{U}(\mu)'$, $\bar{V}(\mu)/\bar{W}(\mu)$ are contragredient to $\bar{V}(-\mu)$, $\bar{V}(-\mu)'$, $\bar{V}(-\mu)/\bar{W}(-\mu)$, $\bar{U}(-\mu)'$, respectively.*

Clearly, $V(\mu)_R' \subseteq V(\mu)_R$, $x_0 \in V(\mu)_R'$, $x_0 \notin \pi V(\mu)_R$.

By (2C), (2F), we obtain

(4E) LEMMA. *$\bar{U}(\mu)$, $\bar{U}(\mu)'$ and $\bar{V}(\mu)/\bar{W}(\mu)$ have the same irreducible constituents; and $\bar{U}(\mu)'$ has a non-trivial quotient module isomorphic with a submodule of $\bar{U}(\mu)$ containing $\bar{x}_0$.*

(4F) LEMMA. *Let $\mu, \nu \in P^+$. Then,*

(a) $V(\mu)/W(\mu)$ is isomorphic with a submodule of

$$(V(\mu+\nu)/W(\mu+\nu)) \otimes V(-\nu).$$

(*b*) *$U(\mu)$ is isomorphic with a quotient module of*

$$U(\mu+\nu) \otimes V(-\nu).$$

Proof. Let $\nu = \sum n_i\lambda_i$. The $\mathfrak{g}$-module $V(\mu+\nu) \otimes V(-\nu)$ consists of homogeneous polynomials

$$w = h(y(i)_j,\ x(-i)_j)$$

of certain degrees in the $y(i)_j$, $x(-i)_j$, and $W(\mu+\nu) \otimes V(-\nu)$ consists of those for which

$$f(g(i)_j,\ x(-i)_j) = 0,$$

for all g in G_R, where the left side is the polynomial in the $x(-i)_j$ obtained by substituting the appropriate $g(i)_j$ for the $y(i)_j$.

Now set

$$z = \prod_i \left(\sum_j x(i)_j x(-i)_j\right)^{n_i}.$$

Since the $x(-i)_j$ form the basis of $V(-i)$ dual to the basis of $V(i)$ formed by the $x(i)_j$, with respect to an invariant bilinear form, z is annihilated by the action of $\mathfrak{g}$. Hence, the map

$$\eta : V(\mu) \to V(\mu+\nu) \otimes V(-\nu),$$
$$\eta(v) = vz,$$

is an injective $\mathfrak{g}$-homomorphism.

Let $g \in G_R$. For a given i, if $\sum_j x(i)_j x(-i)_j$ is expressed in terms of the $y(i)_j$ and $x(-i)_j$, and then $g(i)_j$ is substituted for $y(i)_j$, the resulting linear polynomial in the $x(-i)_j$ is non-zero, since the $g(i)_j$ are not all 0, and the $x(i)_j$ are related to the $y(i)_j$ by an invertible linear transformation. Hence the same procedure applied to z gives a non-zero polynomial in the $x(-i)_j$. Thus, for $v \in V(\mu)$, $vz \in W(\mu+\nu) \otimes V(-\nu)$ if and only if $v \in W(\mu)$. In other words,

$$\eta(W(\mu)) = \eta(V(\mu)) \cap (W(\mu+\nu) \otimes V(-\nu)).$$

Hence, $V(\mu)/W(\mu)$ is isomorphic with a submodule of $V(\mu+\nu) \otimes V(-\nu))/(W(\mu+\nu) \otimes V(-\nu))$, which in turn is isomorphic with $(V(\mu+\nu)/W(\mu+\nu)) \otimes V(-\nu)$, and (a) is proved.

The same argument with i and $-i$ interchanged proves the same result with μ, ν replaced by $-\mu$, $-\nu$. Taking contragredients, we obtain (b).

Of course, since $U(\mu) \simeq V(\mu)/W(\mu)$, and $U(\mu+\nu) \simeq V(\mu+\nu)/W(\mu+\nu)$, (b) follows immediately from (a) by complete reducibility. However, we give the above argument because of its application to the situation in characteristic p (see (5D)).

We shall use a rather special result about contragredient modules for a group.

(4G) LEMMA. *Let X, Y, Z be modules for a group G over a field K, and let X^* denote the contragredient of X.*

a) *X and Y have non-trivial quotient modules which are contragredients of each other, if and only if $X \otimes Y$ has a quotient module isomorphic with the trivial G-module K.*

(*b*) *If Y is irreducible and Z is isomorphic with a submodule of $X^* \otimes Y$, then Y is isomorphic with a quotient module of $X \otimes Z$.*

(c) *If Y is irreducible and Z is isomorphic with a quotient module of $X^* \otimes Y$, then Y is isomorphic with a submodule of $X \otimes Z$.*

Proof. If $X \otimes Y$ has a quotient module isomorphic with K then there is a non-trivial bilinear pairing (,) of X and Y into K which is invariant under G. If

$$X_0 = \{x \in X \mid (x, Y) = \{0\}\}, \; Y_0 = \{y \in Y \mid (X, y) = \{0\}\},$$

then X_0, Y_0 are proper submodules of X, Y, and (,) induces a non-degenerate invariant bilinear pairing of X/X_0 and Y/Y_0, so that these are contragredients. The reverse argument proves the converse. This proves (a).

If Z is isomorphic with a submodule of $X^* \otimes Y$, then Z^* is isomorphic with a quotient module of $X \otimes Y^*$, so that, by (a), $(X \otimes Y^*) \otimes Z$ has a quotient module isomorphic with the trivial module K. Since

$$(X \otimes Y^*) \otimes Z \simeq (X \otimes Z) \otimes Y^*,$$

and $Y \simeq (Y^*)^*$, it follows from (a) that, if Y is irreducible, then Y is isomorphic with a quotient module of $X \otimes Z$. This proves (b), and (c) follows by taking contragredients.

We can now prove the following irreducibility criterion.

(4H) THEOREM. *Let* $\mu \in P_q$, i.e., $0 \leqq m_i \leqq q-1$ *for all* i, *and let* $U(\mu)$ *be the irreducible* $\mathfrak{g}$*-module of highest weight* μ. *Then the derived* G_K*-module* $\bar{U}(\mu)$ *is irreducible if and only if* $U(\mu)$ *is* $\mathfrak{p}$*-unimodular.*

Proof. By (2D) we need prove only that $\bar{U}(\mu)$ is irreducible if $U(\mu)$ is $\mathfrak{p}$-unimodular. In this case the derived module $\bar{U}(\mu)$ is independent of the choice of admissible lattice, by (2C).

Suppose that $\bar{U}(\mu)$ has an irreducible submodule isomorphic with $F(\rho)$, $\rho \in P_q$. By (3E), $\rho \leqq \mu$. By (4E), (4F) and (2G), we see that $F(\rho)$ is isomorphic with a submodule of $(\bar{V}(\sigma)/\bar{W}(\sigma)) \otimes \bar{V}(-(\sigma-\mu))$, where σ is the highest element $(q-1)\sum\lambda_i$ of P_q. By (3D) and (4E), $\bar{V}(\sigma)/\bar{W}(\sigma)$ is irreducible, $\bar{V}(\sigma)/\bar{W}(\sigma) \simeq F(\sigma)$. By (4D) and (4G), $F(\sigma)$ is isomorphic with a quotient module of $\bar{V}(\sigma-\mu)' \otimes F(\rho)$. Every irreducible constituent of the $\mathfrak{g}$-module $V(\sigma-\mu)$ is isomorphic with some $U(\tau)$, where $\tau \leqq \sigma-\mu$. By (2C) and (3E), every irreducible constituent of the G_K-module $\bar{V}(\sigma-\mu)'$ is isomorphic with some $F(\nu)$, where $\nu \in P_q$, $\nu \leqq \sigma-\mu$. By (3G),

$$\sigma \leqq \sigma - \mu + \rho,$$

so that $\rho \geqq \mu$, and so $\rho = \mu$. Thus every irreducible submodule of $\bar{U}(\mu)$ is isomorphic with $F(\mu)$.

Using (4F) (b) and (4G) (c), we prove in the same way that every irreducible quotient module of $\bar{U}(\mu)$ is isomorphic with $F(\mu)$. Since $F(\mu)$ occurs in $\bar{U}(\mu)$ with multiplicity 1, by (3E), it follows that $\bar{U}(\mu)$ is irreducible, $\bar{U}(\mu) \simeq F(\mu)$.

EXAMPLE. We illustrate (4H) by applying it to the fundamental irreducible $\mathfrak{g}$-module $U(\lambda_i)$, in the case that $\mathfrak{g}$ is simple of type B_n.

We may take $\mathfrak{g}$ as the orthogonal Lie algebra of a $(2n+1)$-dimensional vector space V with a non-degenerate symmetric bilinear form of maximal Witt index [8, p. 138]. We may assume that the form has matrix

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot} & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & \cdot^{\cdot} & & & & & \\ 1 & & & & & & \end{pmatrix}$$

with respect to a basis $\{x_{-n}, \cdots, x_{-1}, x_0, x_1, \cdots, x_n\}$ of V. The positive roots of $\mathfrak{g}$ may be expressed in the form

$$\omega_j, \ 1 \leqq j \leqq n,$$
$$\omega_j \pm \omega_k, \ 1 \leqq j < k \leqq n,$$

and then the fundamental roots are

$$\alpha_1 = \omega_1 - \omega_2, \ \ \alpha_2 = \omega_2 - \omega_3, \cdots, \alpha_{n-1} = \omega_{n-1} - \omega_n, \ \ \alpha_n = \omega_n.$$

Denote by e_{jk} the linear transformation on V such that

$$x_m e_{jk} = \delta_{mj} x_k.$$

Then root vectors X_r satisfying Chevalley's conditions may be chosen as in the following table

r	ω_j	$\omega_j + \omega_k$	$\omega_j - \omega_k$
X_r	$2e_{0,j} - e_{-j,0}$	$e_{-j,k} - e_{-k,j}$	$e_{kj} - e_{-j,-k}$
X_{-r}	$e_{j0} - 2e_{0,-j}$	$e_{k,-j} - e_{j,-k}$	$e_{jk} - e_{-k,-j}$

For $i = 1, \cdots, n-1$, the irreducible module $U(\lambda_i)$ is the exterior power $\overset{i}{\wedge} V$, acted on by $\mathfrak{g}$ in the natural way. The vector $v_0 = x_1 \wedge \cdots \wedge x_i$ is of the highest weight λ_i, and one checks easily that $v_0 \mathfrak{U}_R$ has an R-basis consisting of all vectors

$$v = x_{k_1} \wedge \cdots \wedge v_{k_i}, \quad -n \leqq k_1 < k_2 < \cdots < k_i \leqq n,$$

Further, this is an orthogonal basis of $U(\lambda_i)$ with respect to the contravariant form, and with the normalization $(v_0, v_0) = 1$, the value of (v, v) for the above vector v is 2 if 0 occurs as one of the indices $k_1, \cdots, k_i$, and 1 otherwise. Thus the discriminant of $U(\lambda_i)$ is

$$2^{\binom{2n}{i-1}}$$

Hence, $\bar{U}(\lambda_i)$ is irreducible if and only if $p \neq 2$.

The spin module $U(\lambda_n)$ has discriminant 1 and so $\bar{U}(\lambda_n)$ is irreducible, whatever p is (see (7E)).

These results may be compared with that of Springer [13, Thm. 4.2], which implies in this case that, for $i = 1, \cdots, n-1$, a sufficient condition for $\bar{U}(\lambda_i)$ to be irreducible is that p should not divide

$$(2n-2i+2)(2n-2i+3)\cdots(2n-i+1).$$

We remark that Springer's condition is computed just from a knowledge of the weight structure of the $\mathfrak{g}$-module concerned.

A similar calculation in the case that $\mathfrak{g}$ is simple of type D_n shows that the two spin modules $U(\lambda_{n-1})$, $U(\lambda_n)$ and the module $U(\lambda_1)$ corresponding to the representation of $\mathfrak{g}$ as an orthogonal Lie algebra of a $2n$-dimensional space all have discriminant 1, so that $\bar{U}(\lambda_1)$, $\bar{U}(\lambda_{n-1})$, $\bar{U}(\lambda_n)$ are irreducible. For $i=2,\cdots,n-2$, $\bar{U}(\lambda_i)$ is irreducible if and only if $p\neq 2$.

Calculation of the discriminants of the $U(\lambda_i)$ in the case when $\mathfrak{g}$ is simple of type C_n appears to be difficult. Numbering the fundamental weights λ_i in the way corresponding to the numbering of fundamental roots given in [8, p. 135], we find that the discriminants of $U(\lambda_1)$, $U(\lambda_2)$, $U(\lambda_3)$ are $1, n, (n-1)^{2n}$, respectively. For $i>3$, we have not computed the discriminant of $U(\lambda_i)$, but we can show that it is divisible

$$(n-i+2)^{2^{i-2}\binom{n}{i-2}}.$$

Springer's condition implies that it is a divisor of some power of

$$(n-i+2)(n-i+3)\cdots(n-i+j+1),$$

where j is the integer part of $\frac{1}{2}i$.

In the case when $\mathfrak{g}$ is simple of type A_n, all the fundamental modules $U(\lambda_i)$ have discriminant 1 (see (7E)), and so the $\bar{U}(\lambda_i)$ are always irreducible.

5. The Irreducible Modules

In order to give a description of the irreducible G_K-modules, we shall assume that $\mathfrak{g}$ satisfies the following property:

(*) For $i=1,\cdots,n$, $\mathfrak{g}$ has a module $V(i)$ whose highest weight is λ_i, occurring with multiplicity 1, such that $V(i)$ has an admissible lattice $V(i)_R$ and a contravariant form which is integral and unimodular on $V(i)_R$.

In general, the irreducible $\mathfrak{g}$-module of highest weight λ_i does not fulfil the above condition. However, it seems likely that the property (*) is always satisfied. In the last section we shall indicate how it is verified for the cases when $\mathfrak{g}$ is simple of type A_n, B_n, C_n, D_n, E_6, F_4, G_2.

We now suppose that, in the situation set up in Section 4, the modules $V(i)$ are chosen in accordance with (*). The significance of the property is that the contravariant form on $V(i)$ induces a non-degenerate form on the derived module $\bar{V}(i)$. Equivalently, both the basis $\{x(i)_0, x(i)_1, \cdots\}$ and the contragredient basis $\{y(i)_0, y(i)_1, \cdots\}$ are bases of $V(i)_R$ over R, so that we have corresponding bases $\{\bar{x}(i)_0, \bar{x}(i)_1, \cdots\}$, $\{\bar{y}(i)_0, \bar{y}(i)_1, \cdots\}$ of $\bar{V}(i)$ over K.

The derived module $\bar{V}(\mu)$ of $V(\mu)$ may be identified with

$$\bar{V}(1)^{(m_1)} \otimes \cdots \otimes \bar{V}_n{}^{(m_n)},$$

and has a basis consisting of monomials

$$\bar{x}_\beta = \prod_{i,j} \bar{x}(i)_j^{\beta(i,j)},$$

and another basis consisting of monomials

$$\bar{y}_\beta = \prod_{i,j} \bar{y}(i)_j^{\beta(i,j)}.$$

If $\bar{g} \in G_K$, we write equations analogous to (14), (16),

$$\bar{x}(i)_0 \bar{g} = \sum_j \bar{g}(i)_j \bar{x}(i)_j,$$

$$\bar{g}_\beta = \prod_{i,j} \bar{g}(i)_j^{\beta(i,j)},$$

and if $\bar{w} = \sum_\beta \bar{e}_\beta \bar{y}_\beta$ is an element of $\bar{V}(\mu)$, we set

$$\bar{w}(\bar{g}) = \sum_\beta \bar{e}_\beta \bar{g}_\beta,$$

analogously to (22). We say that $\bar{w}$ vanishes upon specialization to elements of G_K if $\bar{w}(\bar{g}) = 0$ for all $\bar{g} \in G_K$, and set

$$\tilde{W}(\mu) = \{\bar{w} \in \bar{V}(\mu) \mid \bar{w}(\bar{g}) = 0, \quad \text{all} \quad \bar{g} \in G_K\}.$$

(5A) LEMMA. *$\tilde{W}(\mu)$ is a G_K-submodule of $\bar{V}(\mu)$, containing $\bar{W}(\mu)$.*

Proof. The analogue of (4A) (b) holds, and the first statement follows immediately. If $w \in V(\mu)_R$, $g \in G_R$, and $\bar{w}$, $\bar{g}$ are the corresponding elements of $\bar{V}(\mu)$, G_K, under the natural maps, then $\bar{w}(\bar{g})$ is the image of $w(g)$ under the residue class map $R \to K$. If also $w \in W(\mu)$, it follows that $\bar{w} \in \tilde{W}(\mu)$, since the natural map $G_R \to G_K$ is an epimorphism. Hence $\bar{W}(\mu) \subseteq \tilde{W}(\mu)$.

In an exactly similar way, we obtain a submodule $\tilde{W}(-\mu)$ of $\bar{V}(-\mu)$ consisting of the elements vanishing upon specialization to elements of G_K, in the appropriate sense.

We now let $\tilde{U}(\mu)$, $\tilde{U}(-\mu)$ be the submodules of $\bar{V}(\mu)'$, $\bar{V}(-\mu)'$ orthogonal to $\tilde{W}(-\mu)$, $\tilde{W}(\mu)$ with respect to the non-degenerate invariant pairings of $\bar{V}(\mu)'$ with $\bar{V}(-\mu)$, and $\bar{V}(-\mu)'$ with $\bar{V}(\mu)$, obtained from the pairings of $V(\mu)'_R$ with $V(-\mu)_R$, and $V(-\mu)'_R$ with $V(\mu)_R$, given in Section 4. Let $\tilde{x}_0$, $\tilde{x}_0^*$ be the elements of $\bar{V}(\mu)'$, $\bar{V}(-\mu)'$ corresponding to the elements x_0, x_0^* of $V(\mu)'_R$, $V(-\mu)'_R$. Using calculations like those at the beginning of Section 4, we see that the following holds.

(5B) LEMMA. *$\tilde{U}(\mu)$, $\tilde{U}(-\mu)$ are the G_K-submodules of $\bar{U}(\mu)'$, $\bar{U}(-\mu)'$ generated by $\tilde{x}_0$, $\tilde{x}_0^*$ respectively. $\tilde{U}(\mu)$, $\bar{V}(\mu)/\tilde{W}(\mu)$ are contragredient to $\bar{V}(-\mu)/\tilde{W}(-\mu)$, $\tilde{U}(-\mu)$ respectively.*

In the G_K-homomorphism of $\bar{U}(\mu)'$ into $\bar{U}(\mu)$ given in (4E), the element $\tilde{x}_0$ maps on $\bar{x}_0$, as may be seen by the proof of (2F). Hence, we have

(5C) LEMMA. *$\tilde{U}(\mu)$ has a non-trivial quotient module isomorphic with a submodule of $\bar{U}(\mu)$ containing $\bar{x}_0$.*

Now a proof completely analogous to that of (4F) gives the following result.

(5D) LEMMA. *Let μ, $\nu \in P^+$. Then*

(a) *$\bar{V}(\mu)/\tilde{W}(\mu)$ is isomorphic with a submodule of*

$$(\bar{V}(\mu+\nu)/\tilde{W}(\mu+\nu)) \otimes \bar{V}(-\nu).$$

(b) *$\tilde{U}(\mu)$ is isomorphic with a quotient module of*

$$\tilde{U}(\mu+\nu) \otimes \bar{V}(-\nu)'.$$

The argument of the proof of (4H) may be applied to give the next result.

(5E) LEMMA. *Let $\mu \in P_q$, i.e., $0 \leqq m_i \leqq q-1$ for all i. Then $\bar{V}(\mu)/\tilde{W}(\mu)$ has a unique irreducible submodule, which is isomorphic with $F(\mu)$, and $\tilde{U}(\mu)$ has a unique irreducible quotient module, which is also isomorphic with $F(\mu)$.*

(5F) THEOREM. *Assume that $\mathfrak{g}$ has the property $(*)$. Let $\mu \in P_q$, and let $\tilde{X}(\mu)$ be the G_K-submodule of $\bar{V}(\mu)$ generated by $\bar{x}_0$. Then $\tilde{X}(\mu) \cap \tilde{W}(\mu)$ is the unique maximal submodule of $\tilde{X}(\mu)$, and*

$$\tilde{X}(\mu)/\tilde{X}(\mu) \cap \tilde{W}(\mu)) \simeq F(\mu).$$

Proof. Since $\bar{x}_0 \notin \tilde{W}(\mu)$, $\tilde{X}(\mu)/(\tilde{X}(\mu) \cap \tilde{W}(\mu))$ is isomorphic with a non-trivial G_K-submodule of $\bar{V}(\mu)/\tilde{W}(\mu)$, and so, by (5E), it has a unique irreducible submodule $Z/(\tilde{X}(\mu) \cap \tilde{W}(\mu))$, which is isomorphic with $F(\mu)$. By (5C), $\tilde{U}(\mu)$ has a quotient module isomorphic with a submodule Y of $\bar{U}(\mu)$ such that

$$Y \supseteq \tilde{X}(\mu) \supseteq Z.$$

By (5E), Y has a unique irreducible quotient module, isomorphic with $F(\mu)$. If $Y \neq Z$, $F(\mu)$ would occur with multiplicity at least 2 in $\bar{U}(\mu)$, contradicting (3E). Hence, $\tilde{X}(\mu) = Z$, so that we have the asserted isomorphism, and $\tilde{X}(\mu) = Y$, so that $\tilde{X}(\mu) \cap \tilde{W}(\mu)$ is the unique maximal submodule of $\tilde{X}(\mu)$.

6. Tensor Product Theorem

We continue to assume that $\mathfrak{g}$ has the property (*), and proceed to give a proof of Steinberg's tensor product theorem, which expresses the irreducible modules $F(\mu)$ in terms of those for which $0 \leqq \mu(H_i) \leqq p-1$ for all i.

Every automorphism γ of the field K induces an automorphism of G_K, which we also denote by γ, carrying each $\bar{x}_{r,K}(\bar{t})$ into $\bar{x}_{r,K}(\bar{t}^{\gamma})$. Then, any G_K-module X is made into a new G_K-module X^{γ} by defining the new action

$$v \circ g = vg^{\gamma} \qquad (v \in X, g \in G_K).$$

Clearly X^{γ} is irreducible if X is. In particular, if γ is the automorphism $\bar{t} \to \bar{t}^p$ of K, we write X^{p^i} for X^{γ^i}.

Clearly γ extends to an automorphism of the algebraic closure Ω of K which induces an automorphism of G_Ω transforming an element $h(\chi)$ of $\mathfrak{H}_\Omega$ into $h(\chi^p)$. Hence, if an element g of G_K is conjugate in G_Ω to $h(\chi)$, then g^{γ} is conjugate to $h(\chi^p)$. Now, if $\mu \in P^+$, it is clear from (7) that

$$s_\mu(g^{\gamma}) = s_{p\mu}(g).$$

If $\mu \in P_q$ and $p\mu \in P_q$, then a computation of Brauer characters shows that $F(p\mu)$ is a constituent of $F(\mu)^p$. Since the latter is irreducible,

$$F(\mu)^p \simeq F(p\mu).$$

This situation can occur with some non-zero μ only if $q = p^r$, $r > 1$, and we now assume this is the case. Set

$$\sigma_1 = (p-1)\sum_i \lambda_i, \ \sigma_2 = (p^{r-1}-1)\sum_i \lambda_i,$$

so that $\sigma_1, \sigma_2 \in P_q$, and

$$\sigma_1 + p\sigma_2 = \sigma,$$

the highest element of P_q.

(6A) LEMMA. $\bar{U}(\sigma_1) \otimes \bar{U}(\sigma_2)^p \simeq \bar{U}(\sigma)$.

Proof. The left side contains as constituent the module $F(\sigma_1) \otimes F(\sigma_2)^p = F(\sigma_1) \otimes F(p\sigma_2)$, which in turn has as constituent $F(\sigma_1 + p\sigma_2) = F(\sigma) = \bar{U}(\sigma)$. By Weyl's dimension formula, $\bar{U}(\sigma_1) \otimes \bar{U}(\sigma_2)^p$ has the same dimension as $\bar{U}(\sigma)$, and so is isomorphic with it.

We notice that the proof shows that $\bar{U}(\sigma_1)$ is irreducible. A similar argument shows that $\bar{U}(\mu)$ is irreducible when μ has the form $(p^s - 1)\sum\lambda_i, 1 \leqq s \leqq r$.

If $\mu_1 \in P_p$, $\mu_2 \in P_{q/p}$, i.e.

$$\mu_1 = \sum_i m_i \lambda_i, \quad 0 \leqq m_i \leqq p-1, \quad \text{all } i,$$

$$\mu_2 = \sum_i n_i \lambda_i, \quad 0 \leqq n_i \leqq p^{r-1} - 1, \quad \text{all } i,$$

then clearly $\mu_1 + p\mu_2 \in P_q$, $\sigma_1 - \mu_1 \in P_p$, $\sigma_2 - \mu_2 \in P_{q/p}$.

(6B) LEMMA. *If* $\mu_1 \in P_q$, $\mu_2 \in P_{q/p}$, *then*

$$F(\mu_1) \otimes F(\mu_2)^p \simeq F(\mu_1 + p\mu_2).$$

Proof. We know that the left side, which is isomorphic with $F(\mu_1) \otimes F(p\mu_2)$, has highest constituent $F(\mu_1 + p\mu_2)$, occurring with multiplicity 1.

Suppose that $F(\tau)$ is isomorphic with a submodule of $F(\mu_1) \otimes F(\mu_2)^p$, so that $\tau \leqq \mu_1 + p\mu_2$. By (5E), (5D), $F(\tau)$ is isomorphic with a submodule of

$$(\bar{V}(\sigma_1)/\tilde{W}(\sigma_1)) \otimes \bar{V}(-(\sigma_1 - \mu_1)) \otimes (\bar{V}(\sigma_2)/\tilde{W}(\sigma_2))^p \otimes V(-(\sigma_2 - \mu_2))^p.$$

By (5A), (4E), (6A), this is isomorphic with

$$\bar{U}(\sigma) \otimes \bar{V}(-(\sigma_1 - \mu_1)) \otimes \bar{V}(-(\sigma_2 - \mu_2))^p.$$

Since $\bar{U}(\sigma)$ is irreducible, we may apply (4G), (4D) to see that $\bar{U}(\sigma)$ is isomorphic with a quotient module of

$$F(\tau) \otimes \bar{V}(\sigma_1 - \mu_1)' \otimes (\bar{V}(\sigma_2 - \mu_2)')^p.$$

The highest irreducible constituents of $\bar{V}(\sigma_1 - \mu_1)'$ and $(\bar{V}(\sigma_2 - \mu_2)')^p$ are $F(\sigma_1 - \mu_1)$ and $F(p\sigma_2 - p\mu_1)$ respectively. By (3G), we must have

$$\sigma \leqq \tau + (\sigma_1 - \mu_1) + (p\sigma_2 - p\mu_2),$$

so that $\tau \geqq \mu_1 + p\mu_2$, and so $\tau = \mu_1 + p\mu_2$.

Similarly, we may use (5E), (5D) (b) and (4G) (c) to show that an irreducible quotient module of $F(\mu_1) \otimes F(\mu_2)^p$ must be isomorphic with $F(\mu_1 + p\mu_2)$. Since this occurs just once as a constituent, we have the asserted isomorphism.

(6C) THEOREM. *Assume that* $\mathfrak{g}$ *has the property* (*). *Let* $|K| = q = p^r$. *Then, every irreducible* G_K*-module can be written uniquely in the form*

$$F(\mu_0) \otimes F(\mu_1)^p \otimes F(\mu_2)^{p^2} \otimes \cdots \otimes F(\mu_{r-1})^{p^{r-1}},$$

where $\mu_j \in P_p$, $j = 0, 1, \cdots, r-1$. *Conversely, every such* G_K*-module is irreducible.*

Proof. Since the number of such tensor product expressions is

$$|P_p|^r = p^{nr} = q^n,$$

which is the number of irreducible G_K-modules, it is enough to show that every irreducible G_K-module $F(\mu)$ can be written in the desired form.

If $\mu = \sum_i m_i \lambda_i \in P_q$, we can expand each coefficient m_i p-adically

$$m_i = \sum_{j=0}^{r-1} m_{ij} p^j, \qquad 0 \leqq m_{ij} \leqq p - 1.$$

Setting $\mu_j = \sum_i m_{ij} \lambda_i$, we have

$$\mu_j \in P_p, \quad \mu = \sum_{j=0}^{r-1} p^j \mu_j.$$

Repeated application of (5B) now shows that $F(\mu)$ is isomorphic with the tensor product of the asserted form.

We remark that the theorems (3E), (5F), (6C) were proved by Brauer and Nesbitt in the case when $\mathfrak{g}$ is simple of type A_1, by Mark in the case of type A_2, and by the author in the case of types A_n, C_n [3], [10], [16]. Completely explicit descriptions of the modules $F(\mu)$ and formulas for their dimensions and Brauer characters were found by Brauer and Nesbitt, and Mark, in the cases A_1, A_2.

7. The Property (*)

For brevity we shall say that a $\mathfrak{g}$-module V is *suitable* if it has an admissible lattice V_R and a contravariant form which is integral and unimodular on V_R.

(7A) LEMMA. *If V is a suitable $\mathfrak{g}$-module, then so is every exterior power* $\overset{k}{\wedge} V$.

Proof. Take an admissible lattice V_R on V, and a contravariant form $(\ ,\ )$ which is integral and unimodular on V_R. With $X_{r,m}$ denoting $X_r^m/m!$ as in Section 2, we have the formula

$$(v_1\wedge\cdots\wedge v_k)X_{r,m} = \sum(v_1X_{r,m_1})\wedge\cdots\wedge(v_kX_{r,m_k}),$$

the sum being taken over all k-tuples $(m_1,\cdots,m_k)$ of non-negative integers such that $m_1+\cdots+m_k=m$. This shows that the exterior power $\overset{k}{\wedge} V_R$, naturally embedded in $\overset{k}{\wedge} V$, is an admissible lattice. The form $(\ ,\ )$ on V induces naturally a bilinear form on $\overset{k}{\wedge} V$, which we denote also by $(\ ,\ )$, such that

$$(v_1\wedge\cdots\wedge v_k, w_1\wedge\cdots\wedge w_k) = \sum\varepsilon(\sigma)(v_1,w_{\sigma(1)})(v_2,w_{\sigma(2)})\cdots(v_k,w_{\sigma(k)}),$$

the sum being taken over all permutations σ of $\{1,2,\cdots,k\}$, with $\varepsilon(\sigma)$ being the sign of σ. This form is easily checked to be contravariant. We can choose two bases $\{x_i\}$, $\{y_i\}$ for V_R which are dual with respect to the original form $(\ ,\ )$, i.e., $(x_i,y_j)=\delta_{ij}$. Then the bases $\{x_{i_1}\wedge\cdots\wedge x_{i_k}\}$, $\{y_{i_1}\wedge\cdots\wedge y_{i_k}\}$ $(i_1<\cdots<i_k)$ are dual bases of $\overset{k}{\wedge} V_R$, so that the contravariant form we have defined is integral and unimodular on $\overset{k}{\wedge} V_R$.

We recall that weights of a $\mathfrak{g}$-module V which are conjugate under the Weyl group W occur with the same multiplicity. A weight μ for V is called a *frontier* weight if there is no root r such that $\mu+r$, $\mu-r$ are both weights of V. If V is irreducible, the frontier weights are precisely the conjugates of the highest weight under W, and so occur with multiplicity 1.

(7B) LEMMA. *Let μ be a frontier weight of a $\mathfrak{g}$-module V which occurs with multiplicity* 1, *and let ν be a conjugate of μ under the Weyl group W. Let v_μ, v_ν be the vectors of a regular basis of V, of weights μ, ν. Then, if $(\ ,\ )$ is a contravariant form on V,*

$$(v_\mu, v_\mu) = u^2(v_\nu, v_\nu),$$

for some unit u *of* R.

Proof. Clearly, we may assume that ν is obtained from μ by reflection with respect to a root r. Replacing r by $-r$ if necessary, we then have

$$\nu = \mu - mr,$$

for some positive integer m. Setting

$$\mathfrak{g}' = \mathfrak{h} \oplus LX_r \oplus LX_{-r},$$

and using the fact that μ is a frontier weight, we see as in [8, p. 113], that the $\mathfrak{g}'$-submodule of V generated by v_μ has a basis $y_0, y_1, \cdots, y_m$, where $y_0 = v_\mu$ and

$$y_i X_{-r} = y_{i+1},\ y_m X_{-r} = 0,\ i = 0, \cdots, m-1,$$
$$y_0 X_r = 0,\ y_i X_r = i(m-i+1) y_{i-1},\ i = 1, \cdots, m.$$

Since y_m has weight $\mu - mr = \nu$ and ν occurs with multiplicity 1, we have

$$y_m = a v_\nu,$$

for some $a \in L$, $a \neq 0$. A calculation shows that

$$v_\mu X_{-r,m} = y_0 X_{-r,m} = (m!)^{-1} y_m = (m!)^{-1} a v_\nu.$$

Thus, $(m!)^{-1} a \in R$, so that $a = m!u$, for some u in R.

Also, we calculate that

$$v_\nu X_{r,m} = a^{-1} y_m X_{r,m} = a^{-1} m! y_0 = u^{-1} v_\mu,$$

so that $u^{-1} \in R$. Hence, u is a unit of R.

Now the contravariance property (3) shows that

$$(v_\mu, v_\mu) = (u v_\nu X_{r,m}, v_\mu) = (u v_\nu, v_\mu X_{-r,m}) = (u v_\nu, u v_\nu),$$

proving the assertion of the lemma.

(7C) COROLLARY. *If* V *is an irreducible* $\mathfrak{g}$*-module whose weights are all frontier weights, then* V *is suitable.*

Proof. Let v_0 be the vector of highest weight in a regular basis of V and choose the contravariant form on V so that $(v_0, v_0) = 1$. Since all frontier weights are conjugate under the Weyl group, and since vectors of different weights are orthogonal with respect to (,), the result follows immediately from (7B).

(7D) LEMMA. *If V is an irreducible $\mathfrak{g}$-module such that* 0 *is the only weight which is not a frontier weight, then there exists a trivial $\mathfrak{g}$-module W such that $V \oplus W$ is a suitable $\mathfrak{g}$-module.*

Proof. We take a vector v_0 of highest weight in V and a contravariant form on V such that $(v_0, v_0) = 1$. If V_R is the smallest admissible lattice on V containing v_0, then, by (2A), the form is integral on V_R. We have an orthogonal decomposition

$$V = S \oplus T,$$

where S is the subspace spanned by vectors of non-zero weights and T is the zero weight space. Correspondingly,

$$V_R = S_R \oplus T_R$$

where $S_R = S \cap V_R$, $T_R = T \cap V_R$. We now take the dual space T^* of T, with the dual module T_R^* to T_R naturally embedded in it, and form

$$U = V \oplus T^* = S \oplus T \oplus T^*,$$
$$U_R = V_R \oplus T_R^* = S_R \oplus T_R \oplus T_R^*.$$

We extend the bilinear form (,) on V to the whole of U by defining, for $s \in S$, $t \in T$, t^*, $t_1^* \in T^*$,

$$(s, t^*) = (t^*, s) = 0, \quad (t^*, t_1^*) = 0,$$
$$(t, t^*) = (t^*, t) = \text{value of } t^* \text{ on } t.$$

From (7B), the form is unimodular on S_R. For a suitable basis of $T_R \oplus T_R^*$ the form has matrix of the form

$$\begin{pmatrix} A & I \\ I & 0 \end{pmatrix}.$$

Thus, the form is integral and unimodular on U_R.

Now let W be the orthogonal complement of V in U. (Since the form is symmetric by (1B), W is well-defined). By the non-degeneracy of the form on V, we have

$$U = V \oplus W.$$

We now make U into a $\mathfrak{g}$-module by requiring the action of $\mathfrak{g}$ on W to be trivial,

$$wX = 0, \qquad w \in W, \ X \in \mathfrak{g}.$$

Now it is easily checked that the bilinear form on U is contravariant.

Finally, we show that U_R is an admissible lattice on U. It is enough to show that if $v \in T_R^*$ then $vX_{r,m} \in U_R$ for all roots r, and all integers $m > 0$. Since $T \oplus T^* = T \oplus W$, we can write

$$v = t + w, \qquad t \in T, \ w \in W.$$

Now, $(T_R, t) = (T_R, v) \subseteq R$, so that

$$t \in \hat{T}_R,$$

where $\hat{T}_R$ is the R-submodule of T dual to T_R with respect to the form $(\ ,\ )$. Since the form is unimodular on S_R, the R-submodule of V dual to V_R with respect to the form is

$$\hat{V}_R = S_R \oplus \hat{T}_R.$$

This is another admissible lattice on V, by the argument of (2B). Hence.

$$vX_{r,m} = tX_{r,m} \in \hat{V}_R.$$

Since $tX_{r,m}$ is 0 or a weight vector of weight mr, we must have

$$vX_{r,m} \in S_R \subseteq U_R.$$

Thus, U_R is an admissible lattice, and the lemma is proved.

(7E) THEOREM. *If* $\mathfrak{g}$ *is simple of type* A_n, B_n, C_n, D_n, E_6, F_4 *or* G_2, *then* $\mathfrak{g}$ *satisfies the property* (*).

Proof. We use the descriptions of the fundamental irreducible $\mathfrak{g}$-modules and their weights given by Cartan [5, pp. 369–398]. We number the fundamental weights λ_i in the order corresponding to the numbering of the fundamental roots of the Dynkin diagram as given in [8, pp. 134–135]. We denote the irreducible module of highest weight λ_i as V_i.

For $\mathfrak{g} = A_n$, all weights of V_i are frontier weights. By (7C), (*) is satisfied, with $V(i) = V_i$.

For $\mathfrak{g} = B_n$, all non-zero weights of V_1 are frontier weights. By (7D), there exists a trivial $\mathfrak{g}$-module W such that $V(1) = V_1 \oplus W$ is suitable. For $i = 2, \cdots, n-1$, set $V(i) = \overset{i}{\wedge} V(1)$. The spin module $V(n) = V_n$ has only frontier weights. Then these $V(i)$ satisfy $(*)$.

For $\mathfrak{g} = C_n$, all weights of V_1 are frontier weights. For $i = 1, \cdots, n$, set $V(i) = \overset{i}{\wedge} V_1$. Then $(*)$ is satisfied.

For $\mathfrak{g} = D_n$, all weights of V_1, V_{n-1}, V_n are frontier weights. Set $V(i) = \overset{i}{\wedge} V_1$, $i = 1, \cdots, n-2$, $V(n-1) = V_{n-1}$, $V(n) = V_n$. Then $(*)$ is satisfied.

For $\mathfrak{g} = E_6$, the weights of V_1, V_5 are all frontier weights while all non-zero weights of V_6 are frontier weights. Set $V(1) = V_1$, $V(2) = \overset{2}{\wedge} V_1$, $V(3) = \overset{3}{\wedge} V_1$, $V(4) = \overset{2}{\wedge} V_5$, $V(5) = V_5$, and let $V(6)$ be the suitable module obtained from V_6 by the process of (7D). Then $(*)$ is satisfied.

For $\mathfrak{g} = F_4$, the non-zero weights of V_1 are all frontier weights. Let $V(1)$ be the suitable module obtained from V_1 by the process of (7D), and let $V(2) = \overset{2}{\wedge} V(1)$. Now, F_4 may be realized as the derivation algebra of the exceptional Jordan algebra J of 3×3 Hermitian matrices over the Cayley numbers, and is a subalgebra of E_6, regarded as a certain algebra of linear transformations on J [8, pp. 144–145]. Then the suitable module $V(6)$ for E_6 constructed above is a suitable module $V(4)$ for F_4. Set $V(3) = \overset{2}{\wedge} V(4)$. Then $(*)$ is satisfied.

For $\mathfrak{g} = G_2$, the non-zero weights of V_2 are frontier weights. Let $V(2)$ be the suitable module obtained from V_2 by the process of (7D), and let $V(1) = \overset{2}{\wedge} V(2)$. Then, $(*)$ is satisfied. This proves (7E).

For $\mathfrak{g} = E_7$, V_1 has only frontier weights while the non-zero weights of V_6 are all frontier weights. Using (7D), we obtain suitable $V(1)$, $V(6)$, and then $V(2) = \overset{2}{\wedge} V(1)$, $V(3) = \overset{3}{\wedge} V(1)$, $V(5) = \overset{2}{\wedge} V(6)$, $V(4) = \overset{3}{\wedge} V(6)$ are suitable. However, V_7 has non-zero weights which are not frontier weights, so that our methods do not apply for the construction of a suitable $V(7)$.

For $\mathfrak{g} = E_8$, the non-zero weights of V_1 are all frontier weights so that we can form $V(1)$ by (7D) and then take $V(i) = \overset{i}{\wedge} V(1)$, $i = 2, 3, 4, 5$. Our methods do not apply for the construction of suitable $V(7)$, $V(8)$. (A suitable $V(6)$ would be obtained as $\overset{2}{\wedge} V(7)$.)

It seems likely that the property $(*)$ is always satisfied, and further that there is a proof that this is so which does not require the detailed knowledge of the fundamental irreducible modules used in (7E).

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