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WEAK FORMAL SCHEMES

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0. Introduction

Throughout this paper, $(R, \mathfrak{m})$ denotes a (noetherian) local ring R with maximal ideal $\mathfrak{m}$.

In [5], Monsky and Washnitzer define *weakly complete* R-algebras with respect to $\mathfrak{m}$. In brief, an R-algebra $A^\dagger$ is weakly complete if

(1) $A^\dagger$ is $\mathfrak{m}$-adically separated (i.e. $\bigcap_{n\geq 1} \mathfrak{m}^n A^\dagger = 0$);

(2) If $f \in R[X_1, \cdots, X_n]^\wedge$, (where "$\wedge$" denotes the $\mathfrak{m}$-adic completion), i.e. if f is a power series with coefficients in R:

$$f = \sum_{0\leq i_1, \cdots, i_n} a_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n}$$

satisfying the special restriction that, for some constant c and all n-tuples of positive integers $(i_1,\ldots,i_n)$, $c[\mathrm{ord}_{\mathfrak{m}}(a_{i_1,\ldots,i_n}) + 1] \geq i_1 + \cdots + i_n$, then for each n-tuple $(x_1, \cdots, x_n)$ of elements of $A^\dagger$ the power series $f(x_1, \cdots, x_n)$ converges to an element of $A^\dagger$.

The weak completion of an R-algebra A is the smallest weakly complete subalgebra $A^\dagger$ of $\hat{A}$ containing the image of A. A weakly complete algebra $A^\dagger$ is called wcfg (weakly complete finitely generated) if there exists a finite collection of elements of $A^\dagger$ such that each element of $A^\dagger$ may be expressed as a power of series in these distinguished elements. The weak completion of a finitely generated R-algebra is a wcfg algebra.

In this paper we define in the obvious way the notion of a weak formal prescheme: namely a local ringed space $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ is a weak formal prescheme if it is locally isomorphic to affine weak formal schemes; and an affine weak formal scheme is a local ringed space $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ such that–for some wcfg R-algebra $A^\dagger$–the underlying topological space $\mathscr{X}$ is spec $(A^\dagger/\mathfrak{m}A^\dagger)$, and the sheaf $\mathcal{O}_{\mathscr{X}}$ is given on the basis of principal open subsets $\{\mathscr{X}_{\bar{f}}$:

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$\bar{f} \in A^\dagger/\mathfrak{m}A^\dagger\}$ of $\mathscr{X}$ as follows: $\Gamma(\mathscr{X}_{\bar{f}}, \mathscr{O}_{\mathscr{X}}) = (A^\dagger{}_f)^\dagger$, the weak completion of $A^\dagger{}_f$ for any preimage f of $\bar{f}$ in $A^\dagger$. Then we prove four main theorems:

(A) The presheaf $\mathscr{O}_{\mathscr{X}}$ associated to a wcfg algebra $A^\dagger$ on the topological space spec $(A^\dagger/\mathfrak{m}A^\dagger)$ (as described immediately above) is in fact a sheaf;

(B) If R is a complete discrete valuation ring and if $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is the affine wf scheme associated to a wcfg algebra $A^\dagger$, then the category of finitely generated $A^\dagger$-modules is equivalent to the category of coherent sheaves of $\mathscr{O}_{\mathscr{X}}$-modules.

(C) If $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is an (ordinary) scheme of R-algebras proper over spec R with weak completion the wf prescheme $(\mathscr{X}^\dagger, \mathscr{O}_{\mathscr{X}^\dagger})$, and if F is a coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-modules with weak completion the coherent sheaf of $\mathscr{O}_{\mathscr{X}^\dagger}$-modules $F^\dagger$, then the natural map

$$H^i(\mathscr{X}, F) \to H^i(\mathscr{X}^\dagger, F^\dagger)$$

is bijective, all $i \geq 0$.

(D) If R is a complete discrete valuation ring and $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is an (ordinary) scheme of R-algebras projective over spec R with weak completion the wf prescheme $(\mathscr{X}^\dagger, \mathscr{O}_{\mathscr{X}^\dagger})$, then the functor "weak completion" is an equivalence from the category of coherent $\mathscr{O}_{\mathscr{X}}$-modules to the category of coherent $\mathscr{O}_{\mathscr{X}^\dagger}$-modules.

Theorem A and C originally appeared in my thesis at Brandeis University, 1969. I am particularly indebted to Paul Monsky who directed this dissertation. I am also grateful to Saul Lubkin who suggested extending Theorem C above from projective R-schemes to proper R-schemes. (Undoubtedly Theorem D also admits such an extension). Our theorem (2.14) is proven in Lubkin's paper [7] by a somewhat different proof.

1. Weak Completions of Modules

Suppose A is a finitely generated R-algebra and M is a finite A-module. Let A^∞ denote the $\mathfrak{m}$-adic completion of A, and let $M^\infty = A^\infty \otimes_A M$. If $A^\dagger$ is the weak completion of A with respect to $\mathfrak{m}$, then $A^\dagger$ and A have the same $\mathfrak{m}$-adic completion.

Definition 1. The weak completion of M, denoted $M^\dagger$, is $M \otimes_A A^\dagger$.

PROPOSITION 2. *$(A^\dagger, \mathfrak{m}A^\dagger)$ is a Zariski ring.*

Proof. (5, Theorem 1.6)

PROPOSITION 3. *$A^\dagger$ is a flat A-module.*

Proof. It suffices to show first that $\frac{A^\dagger}{\mathfrak{m}^i A^\dagger}$ is flat over $\frac{A}{\mathfrak{m}^i A}$; and second that for any ideal $\mathfrak{a} \subset A$, $\mathfrak{a} \otimes_A A^\dagger$ is $\mathfrak{m}A$-separated (1, III. 5. Th. 1). In fact, the inclusion $\frac{A}{\mathfrak{m}^i A} \to \frac{A^\dagger}{\mathfrak{m}^i A^\dagger}$ is bijective (5, Th. 1.4), which proves the first part. For the second part, note that since A is noetherian, $\mathfrak{a} \otimes_A A^\dagger$ is a finite $A^\dagger$-module. Thus, by Proposition 2,

$$\bigcap_i \mathfrak{m}^i A(\mathfrak{a} \otimes_A A^\dagger) = \bigcap_i \mathfrak{m}^i A^\dagger(\mathfrak{a} \otimes_A A^\dagger) = 0.$$

COROLLARY 4. *Suppose A in a noetherian R-algebra whose weak completion $A^\dagger$ is also noetherian. Then $A^\dagger$ is a flat A-module.*

Proof. Exactly the same as for Proposition 3, noting that $(A^\dagger, \mathfrak{m}A^\dagger)$ is a Zariski ring whenever $A^\dagger$ is noetherian. (It is an easy consequence of the definition of weak completion that, for any R-algebra A, if $\alpha \in \mathfrak{m}A^\dagger$ then $1+\alpha$ is invertible in $A^\dagger$.)

2. Affine Weak Formal Schemes

Throughout this section, A is a wcfg R-algebra, M is a finite A-module, and $\bar{A}$ denotes $\frac{A}{\mathfrak{m}A}$. We will construct an affine weak formal scheme on spec $\bar{A}$ with global sections canonically isomorphic to A.

DEFINITION 1. If $f \in A$, then $A_{[f]}$ denotes the weak completion of A_f.

Note that if $\bar{f}$ is the image of f in $\bar{A}$, then $\bar{A}_{[f]} = \bar{A}_{\bar{f}}$.

LEMMA 2. *For any $f \in A$, $A_{[f]}$ is a flat A-module.*

Proof. A_f is a flat A-module. Moreover, since A is noetherian [10], A_f is noetherian. Also, $A_{[f]} = A_f^\dagger$ is noetherian, since $A_f^\dagger$ is a wcfg algebra with weak generators $\frac{1}{f}$ and the weak generators of A. Therefore $A_{[f]}$ is a flat A_f-module (1.4), and so $A_{[f]}$ is a flat A-module.

LEMMA 3. *Let $f, g \in A$ such that spec $\bar{A}_{\bar{f}} \supset$ spec $\bar{A}_{\bar{g}}$. Then the natural map $\bar{A}_{\bar{f}} \to \bar{A}_{\bar{g}}$ lifts to a unique A-homomorphism $A_{[f]} \to A_{[g]}$.*

Proof. Select $a \in A$ such that $af = g^n \bmod \mathfrak{m}A$. Let $g^n = af + \mu$. The unique A-homomorphism $A_f \to A_{af}$ extends uniquely to a map $A_{[f]} \to A_{[af]}$. The element $1 - \frac{\mu}{af + \mu}$ is invertible in $A_{[af+\mu]}$, so $(af)^{-1} = (af + \mu)^{-1}\left(1 - \frac{\mu}{af + \mu}\right)^{-1}$ is an element of $A_{[af+\mu]}$. By symmetry, $A_{[af]}$ is canonically isomorphic to $A_{[af+\mu]} = A_{[g^n]}$. Clearly, $A_{[g^n]}$ is canonically isomorphic to $A_{[g]}$, which concludes this proof.

Endow $\mathscr{X} = \operatorname{spec} \bar{A}$ with the Zariski topology. M induces a functor $\Gamma(\cdot, \tilde{M})$ on the principal open subsets of $\mathscr{X}$ as follows: if $U = \mathscr{X}_{\bar{f}}$, pull $\bar{f}$ back to $f \in A$, and let $\Gamma(U, \tilde{M}) = M \otimes_A A_{[f]}$. If $U = \mathscr{X}_{\bar{f}} \supset V = \mathscr{X}_{\bar{g}}$, Lemma 3 shows that there is a canonical A-homomorphism $\Gamma(U, \tilde{M}) \to \Gamma(V, \tilde{M})$.

DEFINITION 4. $\tilde{M}$ is the presheaf on principal open subsets of $\mathscr{X}$ with sections $\Gamma(\cdot, M)$ and restriction maps the A-homomorphisms described above.

PROPOSITION 5. *If $U \subset \mathscr{X}$ is a principal open set, then $M \rightsquigarrow \Gamma(U, \tilde{M})$ is an exact functor of finite A-modules.*

Proof. $A_{[f]}$ is a flat A-module (Lemma 2).

The remainder of this section is devoted to proving that the presheaf $\tilde{M}$ is a sheaf with trivial cohomology. It will be convenient to assume that A is the weak completion of a polynomial ring $R[X_1, \cdots, X_n]$, with R regular. In order to deduce the general case from this special one, choose a complete regular local ring $(S, \mathfrak{n})$ together with a surjection $\pi: S \to R$. As above, let A be any wcfg R-algebra, and let B be the weak completion of $S[X_1, \cdots, X_n]$, with $\mathfrak{n}$ chosen so that we may extend π to a surjection $\pi: B \to A$: If $Y = \operatorname{Spec} \frac{B}{\mathfrak{n}B}$, then $X \subset Y$ is closed. Viewing M as a finite B-module, M induces a presheaf $\tilde{M}$ on Y. $\operatorname{Supp} \tilde{M} \subset X$; in fact the presheaf $\tilde{M}$ on X when M is a B-module is canonically isomorphic to $\tilde{M}$ when M is considered an A-module via an isomorphism derived from the given homomorphism $\pi: B \to A$. Thus we may assume that $A = B = R[X_1, \cdots, X_n]^\dagger$, with R a complete local ring.

The proof that $\tilde{M}$ is a sheaf with trivial cohomology requires two steps. An intricate calculation shows that $\tilde{A}$ is such a sheaf; induction on $hd_A M$ extends this result to $\tilde{M}$. The proof that $\tilde{A}$ is a sheaf used the Cech cohomology.

LEMMA 6. *Suppose X is a noetherian space and F is a presheaf on X. Then the zero'th Cech cohomology functor on open subsets U of X:*

$$U \to \check{H}^0(U, F)$$

is the sheaf associated to F.
(cf: [2] for a definition of Cech cohomology.)

Proof. Let $U \subset X$ be open. The finite open covers of U are cofinal in the collection of all open covers of U. Consequently, if G is the sheaf associated to F, then $\Gamma(U, G) = \check{H}^0(U, F)$.

We return now to consider the particular case of the presheaf $\tilde{A}$. $\tilde{A}$ is defined only on the basis B of principal open subsets in the topology of $\mathscr{X}$. However, B is closed under intersection, and consequently if $\mathscr{U} = \{U_0, \dots, U_m\} \subset B$, then $C(\mathscr{U}, \tilde{A})$ is defined. Further, since B is a basis for the topology of $\mathscr{X}$, if $U \subset \mathscr{X}$ is open, then finite open covers of U by elements of B are cofinal in the set of all finite open covers of U. Thus we may define the Cech cohomology of $\mathscr{X}$ using only open covers consisting of principal open subsets of $\mathscr{X}$.

The next lemma is, technically, the most important of this section.

LEMMA 7. *Let $U \in B$ and let $\mathscr{U} = \{U_0, \cdots, U_m\}$ be an open cover of U by elements of B. Then:*

(1) *the natural map: $\Gamma(U, \tilde{A}) \to H^0(\mathscr{U}, \tilde{A})$ is bijective;*

(2) *$H^i(\mathscr{U}, \tilde{A}) = 0$ for all $i > 0$.*

Therefore $\check{H}^0(U, \tilde{A}) = \Gamma(U, \tilde{A})$, and $\check{H}^i(U, \tilde{A}) = 0$ for $i > 0$.

The proof of Lemma 7 will follow. Lemma 7 easily implies

THEOREM 8. *$\tilde{A}$ is a sheaf. For every principal open subset $U \subset \mathscr{X}$ and $i > 0$, $H^i(U, \tilde{A}) = 0$.*

Proof. Lemmas 6 and 7 together imply that $\tilde{A}$ is a sheaf. Lemma 7, combined with Cartan's criteria [2, II, 5.9.2], shows that $H^i(U, \tilde{A}) = \check{H}^i(U, \tilde{A}) = 0$ for $i > 0$.

Three lemmas will precede our proof of Lemma 7. These lemmas concern the ring $B = A_{[f]}$, where $f \in A \sim \mathfrak{m}A$. We may assume, without loss of generality, that $f \in R[X_1, \cdots, X_n]$.

LEMMA 9. (1) *B is a domain.*

(2) *If $g \in B \sim \mathfrak{m}B$, then $\mathfrak{m}^i B_{[g]} \cap B = \mathfrak{m}^i B$.*

Proof. (1) $\frac{B}{\mathfrak{m}B} = \left(\frac{A}{\mathfrak{m}A}\right)_{\bar{f}}$ is a domain, and B is R-flat. Consequently B is a domain [5, Lemma 6.1].

(2) The natural homomorphism $\frac{B}{\mathfrak{m}^i B} \to \frac{B_{[g]}}{\mathfrak{m}^i B_{[g]}}$ is injective, because

$$\frac{B_{[g]}}{\mathfrak{m}^i B_{[g]}} = \left(\frac{B}{\mathfrak{m}^i B}\right)_{\bar{g}} \text{ and } \bar{g} \text{ is not a zero divisor of } \frac{B}{\mathfrak{m}^i B}.$$

LEMMA 10. *Suppose* $s_1, \cdots, s_m \in B$, *and suppose* $P_i \in R[T_1, \cdots, T_m]$; $i = 0, 1, \cdots$; *is a sequence of polynomials satisfying*:

(1) $P_i(s) \in \mathfrak{m}^i B$

(2) $dg P_i \leq c(i+1)$ *for some constant* c.

Then $\sum_{i=0}^{\infty} P_i(s)$ *converges in* B.

Remark. In (3.1) we extend this lemma to any wcfg algebra under the additional hypothesis that R is a discrete valuation ring.

Proof. Choose a constant d and polynomials $Q_{i,\alpha} \in \mathfrak{m}^i[T_1, \cdots, T_{n+1}]$ such that:

(3) $s_i = \sum_{\alpha=0}^{\infty} Q_{i,\alpha}\left(X_1, \cdots, X_n, \frac{1}{f}\right)$.

(4) $dg Q_{i,\alpha} \leq d(\alpha+1)$.

By (3) and (4), there exist polynomials $W_{i,\alpha} \in \mathfrak{m}^\alpha[T_1, \cdots, T_{n+1}]$ such that:

(5) $P_i(s) = \sum_{\alpha=0}^{\infty} W_{i,\alpha}\left(X_1, \cdots, X_n, \frac{1}{f}\right)$

(6) $dg \cdot W_{i,\alpha} \leq d\alpha + cd(i+1)$

By (1), $\sum_{\alpha=0}^{i} W_{i,\alpha}\left(X, \frac{1}{f}\right) \in \mathfrak{m}^i B$. Define $W_i = \sum_{j=0}^{i-1} W_{j,i} + \sum_{\alpha=0}^{i} W_{i,\alpha}$.

W_i satisfies the following two properties analogous to (1) and (2):

(8) $dg \cdot W_i \leq D(i+1)$ for $D = 2cd$;

(9) $W_i\left(x, \frac{1}{f}\right) \in \mathfrak{m}^i B$.

$\sum_{i=0}^{\infty} P_i(s)$ converges if and only if $\sum_{i=0}^{\infty} W_i\left(X, \frac{1}{f}\right)$ converges. We will prove the latter. Recall that $f \in R[X_1, \cdots, X_n]$. Let $\mathrm{dg}\, f = E$. Let $U_i \in R[X_1, \cdots,$

$X_n, Y]$ be defined by $U_i = f^{D(i+1)}W_i\left(X, \frac{1}{f}\right)Y^{D(i+1)}$. $U_i\left(X, \frac{1}{f}\right) = W_i\left(X, \frac{1}{f}\right)$, and $dg \cdot U_i \leq F(i+1)$ for some constant F. Furthermore, since $\mathfrak{m}^i B \cap R[X_1, \cdots, X_n] = \mathfrak{m}^i R[X_1, \cdots, X_n]$, $U_i \in \mathfrak{m}^i R[X_1, \cdots, X_n, Y]$, and so $\sum_{i=0}^{\infty} U_i$ converges in $R[X, Y]^\dagger$. The homomorphism $R[X, Y]^\dagger \to B$ sending $X_i \to X_i$ and $Y \to \frac{1}{f}$ also sends $\sum_{i=0}^{\infty} U_i \to \sum_{i=0}^{\infty} W_i\left(X, \frac{1}{f}\right)$. Consequently $\sum_{i=0}^{\infty} W_i\left(X, \frac{1}{f}\right)$ converges in B.

LEMMA 11. *Let $g_0, \cdots, g_m \in B$ generate the unit ideal of $\frac{B}{\mathfrak{m}B}$. Then the g_i generate the unit ideal of B. Further, there exist elements $r_0, \cdots, r_m \in B$ and polynomials $P_{i,\alpha} \in R[T_1, \cdots, T_{2m}]$ such that*

(1) $dg\, P_{i,\alpha} \leq 3m\alpha$

(2) $\sum_{i=0}^{m} P_{i,\alpha}(g, r) g_i^\alpha = 1$

$$\textit{for } 0 \leq i \leq m, \ \alpha \geq 1$$

Proof. Select $\bar{r}_i \in \frac{B}{\mathfrak{m}B}$ so that $\sum_{i=0}^{m} \bar{r}_i \bar{g}_i = 1$, and lift $\bar{r}_i$ back to r_i' in B. $\sum_{i=0}^{m} r_i' g_i = 1 + \mu$, $\mu \in \mathfrak{m}B$. Let $r_i = (1+\mu)^{-1} r_i'$. $\sum_{i=0}^{m} r_i g_i = 1$.

For the second part, note that $\left(\sum_{i=0}^{m} r_i g_i\right)^{(m+1)\alpha} = \sum_{i=0}^{m} P_{i,\alpha}(g, r) g_i^\alpha$, with polynomials P_i satisfying the lemma.

Proof of Lemma 7. Let $r \geq 0$ and let $B = \Gamma(U, \tilde{A})$, and select $f_i \in B$, $0 \leq i \leq m$, such that $U_i = U_{f_i}$. Let $B_{i_0, \ldots, i_r} = B_{[f_{i_0} \cdots f_{i_r}]}$. The $[f_i]$ generate the unit ideal of $\frac{B}{\mathfrak{m}B}$, so there exist elements $r_i \in B$ and polynomials $P_{i,\alpha}$ as in lemma 10. Let $r_{i,\alpha} = P_{i,\alpha}(f, r)$. Then $\sum_{i=0}^{m} r_{i,\alpha} f_i^\alpha = 1$ in B. Let

$$\begin{aligned} C^r = C^r(U, \tilde{A}) &= \bigoplus_{0 \leq i_0 < \cdots < i_r \leq m} \Gamma(U_{i_0} \cap \cdots \cap U_{i_r}, \tilde{A}) \\ &= \bigoplus_{0 \leq i_0 < \cdots < i_r \leq m} B_{i_0}, \cdots, i_r. \end{aligned}$$

It is convenient to define $C^{-1}(U, \tilde{A}) = C^{-1} = B$, and let $\delta \colon C^{-1} \to C^0$ be the sum of the restriction homomorphisms $B \to B_i$, $0 \leq i \leq m$. We must prove that the following in a long exact sequence:

$$0 \to C^{-1} \to C^0 \to \cdots \to C^m \to 0.$$

$C^{-1}\to C^0$ is injective if the restriction homomorphism $B\to B_i$ is injective. B is a domain (Lemma 9) and B_i is flat over B (1.3), so $B\to B_i$ is one-to-one.

Suppose $\sigma\in C^r$, $r\geq 0$, is a cocycle. σ has components $\sigma_{i_0,\dots,i_r}\in\Gamma(U_{i_0}\cap\cdots\cap U_{i_r}, \tilde{A})=B_{i_0,\dots,i_r}$, and each component may be expressed as a power series. In particular, there exist $b_1,\cdots,b_t\in B$ and polynomials $P_\alpha^{(i)}$ such that

(1) $\sigma_{i_0,\dots,i_r}=\sum_{\alpha=0}^{\infty}P_\alpha^{(i)}\left(b_1,\cdots,b_t,\frac{1}{f_{i_0}},\cdots,\frac{1}{f_{i_r}}\right)$

(2) $dg\,P_\alpha^{(i)}\leq c(\alpha+1)$ for some constant c

(3) the coefficients of $P_\alpha^{(i)}$ are in $\mathfrak{m}^i$.

We shall construct a cochain τ so that $\partial\tau=\sigma$

For each $k=1,2,\cdots$ the reduced complex $\frac{C^\cdot}{\mathfrak{m}^kC^\cdot}=C^\cdot\left(\mathscr{U},\frac{A}{\mathfrak{m}^kA}\right)$ is exact (3, III. 1.2.4). Using the exactness of this reduced complex, we will inductively construct a sequence of cochains

$$\tau_k=\sum_{0\leq i_0<\cdots<i_{r-1}\leq m}\tau_{k;\,i_0,\cdots i_{r-1}};\ k=0,1,\cdots$$

such that the sum $\sum_{k=0}^{\infty}\tau_k$ converges in C^{r-1} to a coboundary of σ. The τ_k are chosen to satisfy the following our conditions (where c is the constant of the above paragraph):

(1) $\partial\sum_{k=0}^{s-1}\tau_k=\sigma \mod \mathfrak{m}^{2^s-1}C^r$

(2) $\tau_{0;\,i_0,\cdots,i_{r-1}}\in B_{i_0,\cdots,i_{r-1}}$, and for $k\geq 1$, $\tau_{k;\,i_0,\cdots,i_{r-1}}\in\mathfrak{m}^{2^k-1}B_{i_0,\cdots,i_{r-1}}$

(3) $\tau_{k;\,i_0,\cdots,i_{r-1}}$ is a polynomial of degree $\leq 24\,mc\,(2^k)$ in the elements $\left\{b_1,\cdots,b_t,f_0,\cdots,f_m,r_0,\cdots,r_m,\frac{1}{f_{i_0}},\cdots,\frac{1}{f_{i_{r-1}}}\right\}$

(4) $f_{i_\alpha}^{c2^{k+1}}\tau_{k;\,i_0,\cdots,i_{r-1}}$ is a polynomial of degree $\leq c2^{k+1}+24mc2^k$ in the elements $\left\{b_i,f_i,r_i,\frac{1}{f_{i_0}},\cdots,\frac{\hat{1}}{f_{i_\alpha}},\cdots,\frac{1}{f_{i_{r-1}}}\right\}$

Lemma 10, together with (2) and (3) guarantee that $\sum_{k=0}^{\infty}\tau_k$ converges to a cochain $\tau\in C^{r-1}$. (1) shows that τ bounds σ. (4) is required to continue the inductive construction.

Define elements $\sigma_{s;\,i_0,\cdots,i_r}\in B_{i_0,\cdots,i_r}$, $s\geq 0$ by the formulae:

$$\sigma_{s;\,i_0\cdots,i_r}=\sum_{\alpha=0}^{2^{s+1}-1}P_\alpha^{(i)}\left(b_1,\cdots,b_t,\frac{1}{f_{i_0}},\cdots,\frac{1}{f_{i_r}}\right)$$

Then $\sigma_{s;\,(i)} = \sigma_{(i)} \bmod \mathfrak{m}^{2^{s+1}}$, and $dg\cdot\sigma_{s;\,(i)} \leq c2^{s+1}$. Define the cochain $\tau_0 \in C^{r-1}$ by

$$\tau_{0;\, i_0,\cdots i_{r-1}} = \sum_{i=0}^{m} r_{i,2c} f_i^{2c} \sigma_{0;\, i_0,\cdots i_{r-1},\, i}$$

Suppose now that for some integer $s>0$ we have constructed the cochains $\tau_k \in C^{r-1}$ for $0 \leq k < s$. We construct τ_s as follows: Let

$$\gamma_{s;\, i_0,\cdots,i_r} = \sigma_{s;i_0,\cdots,i_r} - \sigma\Big(\sum_{k=0}^{s-1} \tau_k\Big)_{i_0,\cdots,i_r}.$$

Then $(\gamma_{s;\, i_0,\cdots,i_r})_{0\leq i_0<\cdots<i_r\leq m} \in \mathfrak{m}^{2^s-1}C^r$ is a cocycle modulo $\mathfrak{m}^{2^{s+1}}C^r$. Moreover, $dg\cdot\gamma_{s:\, i_0\cdots,i_r} \leq 24mc2^{s-1}$. Define

$$\tau_{s;\, i_0\cdots,i_{r-1}} = \sum_{i=0}^{m} (r_i{}^{c2^{s+1}}) f_i^{c2^{s+1}} \gamma_{s;\, i_0,\cdots,i_{r-1},\, i}$$

By [3, III. 1.2.4], τ_s satisfies (1). Because

$$f_i^{c2^{s+1}} \gamma_{s;\, i_0,\cdots,i_{r-1},\, i} \in B_{i_0,\cdots i_{r-1}} \cap \mathfrak{m}^{2^s-1} B_{i_0,\cdots,i_r}$$

(by (4) and the power series expression of σ) and

$$B_{i_0\cdots i_{r-1}} \cap \mathfrak{m}^{2^s-1} B_{i_0\cdots i_r} = \mathfrak{m}^{2^s-1} B_{i_0\cdots i_{r-1}}$$

(Lemma 9), (2) is satisfied.

Moreover, $dg\,\tau_{s;\, i_0,\cdots,i_r} \leq dg\,\gamma_{s;\, i_0,\cdots,i_r} + dg\cdot r_{i,c2^{s+1}} + c2^{s+1}$

$$\leq 24mc2^{s-1} + 3mc^{s+1} + c2^{s+1} \leq 24mc2^s$$

Therefore τ_s satisfies condition (3).
Finally, if $0 \leq \alpha \leq r$, both of the elements

$$f_{i_\alpha}^{c2^{s+1}} \gamma_{s;\, i_0,\cdots,i_r} \text{ and } f_{i_\alpha}^{c2^{s+1}} \Big(\sum_{k=0}^{s-1} \tau_k\Big) i_0 \cdots i_r$$

are elements of $B_{i_0,\cdots,\hat{i}_\alpha,\cdots,i_r}$ of degree $\leq c2^{s+1} + 24mc2^{s-1}$. Therefore $f_{i_\alpha}^{c2^{s+1}} \gamma_{s;\, i_0,\cdots,i_r} \in B_{i_0\cdots\hat{i}_\alpha\cdots i_r}$, and $dg\cdot f_{i_\alpha}^{c2^{s+1}} \gamma_{s;\, i_0\cdots i_r} \leq c2^{s+1} + 24mc2^{s-1}$. Thus $f_{i_\alpha}^{c2^{s+1}} \tau_{k;\, i_0,\cdots,i_{r-1}} \in B_{i_0\cdots\hat{i}_\alpha \cdots i_{r-1}}$ for any α, and $dg\cdot f_{i_\alpha}^{c2^{s+1}} \tau_{k;\, i_0\cdots i_{r-1}} \leq c2^{s+1} + 24mc2^{s-1} + 3mc2^{s+1} + c2^{s+1}$

$$\leq c2^{s+1} + 24mc2^s$$

Therefore condition (4) is satisfied.

The calculation previously threatened has proven that $\tilde{A}$ is a sheaf with trivial cohomology. We shall next extend this result to the presheaf $\tilde{M}$ induced on Spec $\tilde{A}$ by the finite A-module M.

PROPOSITION 12. *If M is a finite, flat A-module, then $\tilde{M}$ is a sheaf and $H^i(U, \tilde{M}) = 0$ for $i > 0$ and U any principal open subset of $\mathscr{X}$.*

Proof. Let $U = \{U_0, \cdots, U_m\}$ be a covering of U by principal open subsets. Lemma 7 shows that the simplicial resolution

$$(1) \quad 0 \to \Gamma(U, \tilde{A}) \to C^0(\mathscr{U}, \tilde{A}) \to C^1(\mathscr{U}, \tilde{A}) \to \cdots$$

is exact. Tensoring (1) with M over A give the simplicial resolution of $\tilde{M}$ by

$$(2) \quad 0 \to \Gamma(U, \tilde{M}) \to C^0(\mathscr{U}, \tilde{M}) \to C^1(\mathscr{U}, \tilde{M}) \to \cdots$$

Because M is flat over A, (2) is exact. Consequently (as in the proof of Theorem 8) $\tilde{M}$ is a sheaf and $H^i(U, \tilde{M}) = 0$ for $i > 0$.

PROPOSITION 13. *A is a regular ring.*

Proof. (5, Lemma 6. 1)

THEOREM 14. *Suppose M is a finite A-module. Then $\tilde{M}$ is a sheaf. For every principal open subset $U \subset \mathscr{X}$ and $i > 0$, $H^i(U, \tilde{M}) = 0$.*

Proof. Because A is regular, $hd_A M < \infty$. By Proposition 14, the theorem is true for M if $hd_A M = 0$. We shall assume that $hd_A M > 0$ and proceed by induction. Suppose the theorem holds for all modules N such that $hd_A N < hd_A M$. Construct an exact sequence of A-modules

$$0 \to K \to F \to M \to 0$$

with F finite and free. By (2.5) this sequence induces an exact sequence of presheaves:

$$0 \to \tilde{K} \to \tilde{F} \to \tilde{M} \to 0.$$

Because $hd_A K = hd_A M - 1$, the induction hypothesis implies that $\tilde{K}$ is a sheaf with trivial cohomology. Let $\mathscr{M}$ denote the sheaf associated to $\tilde{M}$. Then we have an exact sequence of sheaves:

$$0 \to \tilde{K} \to \tilde{F} \to \mathscr{M} \to 0.$$

Since $H^i(U, \tilde{K}) = H^i(U, \tilde{F}) = 0$ for $i > 0$ and U a principal open subset of X, $\mathscr{M} = \tilde{M}$ and $H^i(U, \tilde{M}) = 0$, all $i > 0$.

DEFINITION 15. An *affine wf scheme* over R is a ringed space isomorphic to (Spec A, $\tilde{A}$) for some wcfg R-algebra A.

In the next section we study coherent sheaves over an affine wf scheme. By Theorem 14, if M is a finite A-module, $\tilde{M}$ is a sheaf of finite type over $\tilde{A}$. Using the exactness of the functor $M \to \tilde{M}$ and the fact that A is Noetherian we find that $\tilde{M}$ is coherent. In particular $\tilde{A}$ is coherent over itself; consequently the coherent $\tilde{A}$ sheaves are just those sheaves which are locally of finite presentation (i.e. locally look like $\tilde{M}$ for some finitely generated A-module M.)

3. Coherent Sheaves

In the previous section, we proved that a wcfg algebra A and finite A-module M give rise to an affine wf scheme and a coherent module with trivial cohomology on that scheme. In this section we will prove that every coherent sheaf over an affine *wf* scheme is generated by its global sections, *provided that the ground ring is a complete discrete valuation ring.* Throughout this section, (R, π) will denote a complete discrete valuation ring.

LEMMA 1. *Suppose A is a* wcfg *algebra and M is a finite A-module. Suppose further that $x_1, \cdots, x_n \in A$ and $\mu_1, \cdots, \mu_t \in M$. Let $P_{i,j}$ $i = 1, \cdots, t$; $j = 0, 1, 2, \cdots,$; be a collection of polynomials in n variables satisfying the following conditions*:

(1) $dg\, P_{i,j} \leq c(j+1)$ *for some constant* c;

(2) $\sum_{i=1}^{t} P_{i,j}(x)\mu_i \in \pi^j M$ *for all* j.

Then $\sum_{j=0}^{\infty} \sum_{i=1}^{t} P_{i,j}(x)\mu_i$ *converges in* M.

In particular, settling $t = 1$, $M = A$, *and* $\mu_1 = 1$, *then* $\sum_{j=0}^{\infty} P_j(x)$ *converges in* A.

LEMMA 2. *Let B be a noetherian ring, $I \subset B$ an ideal, $f \in B \sim I$, and M a finite B-module. There exists a constant N such that if $m \in M \cap I^p M_f$, then $f^{Np} m \in I^p M$.*

Proofs. The proof of Lemma 1 will be found at the end of this section. To prove Lemma 2, let $F \in G_I(B)$ be the leading form of f. The sequence of submodules of $G_I(M)$:

$$(0 : F) \subset (0 : F^2) \subset \cdots$$

has a maximal element, say $(0 : F^N)$. Thus if $m \in I^p M$ and $f^j m \in I^{p+1} M$, then $f^N m \in I^{p+1} M$.

Suppose $m \in M \cap I^p M_f$. There is an integer j such that $f^j m \in I^p M$. The above argument shows that $f^N m \in IM$; likewise $f^{aN} m \in I^a M$ for $a \leq p$. In particular, $f^{pN} m \in I^p M$.

THEOREM 3. *Suppose A is a* wcfg *algebra over the complete discrete valuation ring (R, π), and Suppose F is a coherent sheaf of $\tilde{A}$-modules on the affine wf scheme $(Spec\ A/\pi A, \tilde{A})$. Then there exists a finitely generated A-module M such that $F = \tilde{M}$; moreover, we may take $M = \Gamma(Spec\ A/\pi A, F)$.*

Proof. Let $\mathscr{X} = \text{Spec } A/\pi A$. First we show that, to prove Theorem 3, it suffices to show that the natural homomorphism $\Gamma(\mathscr{X}, F) \to \Gamma(\mathscr{X}, F/\pi F)$ is surjective. Suppose this map is surjective. Since $F/\pi F$ is a coherent sheaf of $A/\pi A$-modules on $\mathscr{X}$, there exist elements $\bar{f}_1, \cdots, \bar{f}_s \in \Gamma(\mathscr{X}, F/\pi F)$ which generate $\Gamma(\mathscr{X}, F/\pi F)$. Then $\bar{f}_1, \cdots, \bar{f}_s$ generate the coherent sheaf of modules $F/\pi F$ over the ordinary affine scheme $(\mathscr{X}, \widetilde{A/\pi A})$. Lift each element $\bar{f}_i$ back to an element $f_i \in \Gamma(\mathscr{X}, F)$. The stalk of the sheaf $\tilde{A}$ at any point x of $\mathscr{X}$, $\tilde{A}_x$, is a Zariski ring, and the stalk F_x of F is a finitely generated $\tilde{A}_x$-module (because F is locally the sheaf associated to a finitely presented $\tilde{A}$-module.) Moreover, $\bar{f}_1, \cdots, \bar{f}_s$ generate the $(\widetilde{A/\pi A})_x$-module $(F/\pi F)_x$, and $(\widetilde{A/\pi A})_x = \tilde{A}_x / \pi \tilde{A}_x$ and $(F/\pi F)_x = F_x/\pi F_x$. Therefore $f_1, \cdots, f_s$ generate F_x as an $\tilde{A}_x$-module. Thus there exists a surjective homomorphism of coherent sheaves of $\tilde{A}$-modules from the free module $\tilde{A}^s$ to F, $\alpha : \tilde{A}^s \to F$. Repeating the above argument for the coherent $\tilde{A}$-module (ker α), we see there exists an integer t and a homomorphism of coherent sheaves $\beta : \tilde{A}^t \to \tilde{A}^s$ such that $F =$ (coker β). The homomorphism β arise from a homomorphism $\beta_0 : A^t \to A^s$ of free A-modules. Since the function "$\sim$" is exact, (2.5), $F =$ (coker $\beta_0)^\sim$.

Next we prove that the natural homomorphism $\Gamma(\mathscr{X}, F) \to \Gamma(\mathscr{X}, F/\pi F)$ is surjective.

Select a covering of $\mathscr{X}$ by principal open sets $U_i = X_{f_i}$; $i = 0, \cdots, m$, such that $F|U_i = \Gamma(U_i, F)^\sim$. Let $A_i = A_{[f_i]}$, and $A_{i,j} = A_{[f_i f_j]}$. Let $U_{i,j} = U_i \cap U_j$. Set $F_i = \Gamma(U_i, F)$ and $F_{i,j} = \Gamma(U_{i,j}, F)$. Choose an integer N such that for each pair (i, j), if $x \in F_i \cap \pi^a F_{i,j}$, then $f_j^{aN} x \in \pi^a F_i$.

For each $i=0, \cdots, m$, choose generators $\mu_{i,1}, \cdots, \mu_{i,r}$ for F_i over A_i. For each ordered pair (i, j), choose elements

$$n_{\alpha,\beta}^{i,j} \in A_{i,j};\ \alpha, \beta = 1, \cdots, r;\ \text{such that}\ \mu_{j,\beta} = \sum_{\alpha=1}^{r} n_{\alpha,\beta}^{i,j} \mu_{i,\alpha}$$

and define matrices $N_{i,j} = (n_{\alpha,\beta}^{i,j})_{\alpha,\beta}$. Note that if $(f) = (f_1, \cdots, f_r)$ is a vector over $A_{i,j}$, and if $(g) = (g_1, \cdots, g_r) = N_{i,j}(f)$, then in $F_{i,j}$ we have

$$\sum_{\alpha=1}^{r} f_\alpha \mu_{j,\alpha} = \sum_{\alpha=1}^{r} g_\alpha \mu_{i,\alpha}.$$

For notational convenience, if (f) is a vector over $A_{i,j}$, then $(f\mu_i) = \sum_{\alpha=1}^{r} f_\alpha \mu_{i,\alpha}$. Thus, in $F_{i,j}$, $(f\mu_j) = ((N_{i,j}f)\mu_i)$. Also, $(N_{i,j}N_{j,k}f)\mu_i) = ((N_{i,k}f)\mu_i)$.

There exist elements $x_1, \cdots, x_n \in A$ such that $n_{\alpha,\beta}^{i,j}$ may be expressed as a power series

$$n_{\alpha,\beta}^{i,j} = \sum_{q=0}^{\infty} n_{\alpha,\beta,q}^{i,j};$$

with each $n_{\alpha,\beta,q}^{i,j} \in \pi^q A_{i,j}$ expressable as a polynomial of degree $\leq c(q+1)$ in the elements $\left\{x_1, \cdots, x_n, \frac{1}{f_i}, \frac{1}{f_j}\right\}$ (for some constant c).

Also, (as in (2.11)), there exist elements $r_i \in A$, $0 \leq i \leq m$, and polynomials $P_{i,j}$ of degree $\leq 3mj$, $0 \leq i \leq m$, $j \geq 1$, such that $\sum_{i=0}^{m} P_{i,j}(f, r) f_i^j = 1$. As before, we denote $P_{i,j}(f, r)$ by $r_{i,j}$.

Now we have sufficient machinery to permit us to lift a section $\tau \in \Gamma(\mathscr{X}, F/\pi F)$. Over each U_i, lift τ to a section of F_i; call this section $(g^{0,i}\mu_i)$, where $(g^{0,i}) = (g_1^{0,i}, \cdots, g_r^{0,i})$ is a vector over A_i. Replacing c by a larger constant if necessary, choose the lifting so that for all i, α; $f_i^c g_\alpha^{0,i} \in A$. Let $f_i^c g_\alpha^{0,i} = h_\alpha^i$.

Note that

$$((N_{i,j} g^{0,j} - g^{0,i})\mu_i) \in \pi F_{i,j}.$$

For each $i = 0, \cdots, m$, we will construct a sequence of vectors $(g^{s,i})$; $s = 1, 2, \cdots$, such that $\sum_{s=0}^{\infty} (g^{s,i}\mu_i)$ converges in F_i to a global section which reduces modulo π to τ. More precisely, we will construct $(g^{s,i}) = (g_1^{s,i}, \cdots, g_r^{s,i})$ over A_i such that:

(1) $(N_{i,j}(\sum_{s=0}^{h-1}(g^{s,j}) - \sum_{s=0}^{h-1}(g^{s,i}))\mu_i \in \pi^{2^h-1} F_{i,j}$;

(2) $(g^{s,i}\mu_i)\in\pi^{2^s-1}F_i$;

(3) $g^{s,i}_\alpha$ is a polynomial of degree $\leq k2^s$ in the elements

$$\{X_\beta,\ r_\gamma,\ f_\gamma,\ 1/f_i,\ f^c_\alpha g^{0,\alpha}_\delta\}\quad \begin{matrix}1\leq\beta\leq n\\ 0\leq\gamma\leq m\\ 1\leq\sigma\leq r\end{matrix}$$

all $1\leq\alpha\leq r$, where $k=25mc+2N$.

(4) $f_i^{c2^{s+2}}g^{s,i}_\alpha\in A\qquad 1\leq\alpha\leq r$

Condition (4) is necessary for the inductive construction of the vectors (g). Conditions (2) and (3), together with Lemma 1 guarantee that $\sum_{s=0}^{\infty}(g^{s,i}\mu_i)$ converges in F_i. Condition (1) proves that these vectors represent a global section of F. Since $\sum_{s=0}^{\infty}(g^{s,i}\mu_i)=(g^{0,i}\mu_i)$ mod πF_i, this section is a lifting of τ.

The vectors $(g^{0,i})$ satisfy (1), (2), (3) and (4). Suppose we have constructed $(g^{s,i})$ for $i=0,\cdots,m$ and $s=0,\cdots,h-1$. We will construct $(g^{h,i})$. Define

$$n^{i,j,h}_{\alpha,\beta}=\sum_{q=0}^{2^{h+1}-1}n^{i,j}_{\alpha,\beta,q}.$$

$n^{i,j,h}_{\alpha,\beta}\in A_{i,j}$ and $dg\cdot n^{i,j,h}_{\alpha,\beta}\leq c2^{h+1}$. Let $N_{i,j,h}=(n^{i,j,h}_{\alpha,\beta})$. $N_{i,j,h}=N_{i,j}$ mod $\pi^{2^{h+1}}$. Define a vector over $A_{i,j}$:

$$(w^{i,j,h})=N_{i,j,h}(\sum_{s=0}^{h-1}(g^{s,j}))-\sum_{s=0}^{h-1}(g^{s,i}).$$

By our inductive assumption (1), $(w^{i,j,h}\mu_i)\in\pi^{2^h-1}F_{i,j}$. Moreover, by (3), $(w^{i,j,h}_\alpha\alpha)$, the α'th coordinate of $(w^{i,j,h})$, is a polynomial in the elements

$$\{X_\beta,\ r_\gamma,\ f_\gamma,\ 1/f_i,\ 1/f_j,\ f^c_\gamma g^{0,\gamma}_\delta\}\quad \begin{matrix}1\leq\beta\leq n\\ 0\leq\gamma\leq m\\ 1\leq\delta\leq r\end{matrix}$$

of degree $\leq k2^{h-1}+c2^{h+1}$ and, by (4) and (5), $f_j^{c2^{h+1}+c2^{h+1}}w^{i,j,h}_\alpha\in A_i$, and $(f_if_j)^{c2^{h+2}}w^{i,j,h}_\alpha\in A$

Define vectors $y^{h,i,j}$ over A_i as follows, $0\leq i,\ j\leq m$:

$$y^{h,i,j}=r_{j,c2^{h+2}+N2^h}f_j^{c2^{h+2}}w^{i,j,h}.$$

Our preceeding arguments show

(a) $y^{h,i,j}\mu_i \in F_i \cap \pi^{2^h-1}F_{i,j}$

(b) $dg\, y_\alpha^{h,i,j} \leq 3m(c2^{h+2}+N2^h)+c2^{h+2}+c2^{h+1}+k2^{h-1}$

(c) $f_i^{c2^{h+2}} y_\alpha^{h,i,j} \in A$

where $y_\alpha^{h,i,j}$ is the α' th component of the vector $y^{h,i,j}$. Finally we define the desired vector $(g^{h,i})$ as follows:

$$(g^{h,i}) = \sum_{j=0}^{m} f_j^{N2^h} y^{h,i,j}.$$

Thus

(a′) $(g^{h,i}\mu_i) \in \pi^{2^h-1}F_i$

(b′) $dg\cdot g_\alpha^{h,i} \leq N2^h + 3m(c2^{h+2}+N2^h)+c2^{h+2}+k2^{h-1}+c2^{h+1} \leq K2^h$

(c′) $f_i^{c2^{h+2}} g_\alpha^h \in A$

where $g_\alpha^{h,i}$ is the α' th component of the vector $(g^{h,i})$. Thus we have verified condition (2) by (a′), (3) by (b′), and (4) by (c′).
To verify (1), we will prove that:

$$((N_{i,j}g^{h,j} - g^{h,i})\mu_i) = ((\sum_{s=0}^{h-1} g^{s,i} - N_{i,j}\sum_{s=0}^{h-1} g^{s,j})\mu_i) \qquad \bmod \pi^{2^{h+1}}F_{i,j}.$$

Let $C = 3m(c2^{h+2}+N2^h)$.
We arrive at this equation via:

$$\begin{aligned}
((N_{i,j}g^{h,j} - g^{h,i})\mu_i) &= ([N_{i,j}(\sum_{l=0}^{m} r_{l,C} f_l^C w^{j,l,h}) - \sum_{l=0}^{m} r_{l,C} f_l^C w^{i,l,h}]\mu_i) \\
&= (\sum_{l=0}^{m} r_{l,C} f_l^C [N_{i,j}N_{j,l}\sum_{s=0}^{h-1}(g^{s,l}) - N_{i,j}\sum_{s=0}^{h-1}(g^{s,j}) \\
&\quad - N_{i,l}\sum_{s=0}^{h-1}(g^{s,l}) + \sum_{s=0}^{h-1}(g^{s,i})]\mu_i) \qquad \bmod \pi^{2h+1}F_{i,j} \\
&= (\sum_{l=0}^{m} r_{l,C} f_l^C [\sum_{s=0}^{h-1}(g^{s,i}) - N_{i,j}\sum_{s=0}^{h-1}(g^{s,j})]\mu_i) \\
&= ((\sum_{s=0}^{h-1}(g^{s,i}) - N_{i,j}\sum_{s=0}^{h-1}(g^{s,j}))\mu_i) \bmod \pi^{2h+1}F_{i,j}.
\end{aligned}$$

QED for Theorem 3.

It remains only to prove Lemma 1. The following notation will be necessary for the remainder of this section. We may assume that A is a

wcfg algebra with weak generators $\{x_1, \cdots, x_n\}$, and that M is a finite A-module spanned by $\{\varphi_1 \cdots, \varphi_r\}$. Let $B = R[X_1, \cdots, X_n]^\dagger$, and view A as a homomorphic image of B via the surjection $X_i \to x_i$.
Let $F = B^t$, and define K via the exact sequence:

$$0 \to K \to F \to M \to 0,$$

where $F \to M$ is defined by $(0 \cdots 1_i \cdots 0) \to \varphi_i$.

Define $L(a, b) \subseteq F$ as follows: $f = (f_1, \cdots, f_r) \in L(a, b)$ if and only if $f_i = \sum_{q=0}^{\infty} f_{i,q}$ with $f_{i,q} \in \pi^q[X_1, \cdots, X_n]$ and $dg \cdot f_{i,q} \leq a + bq$ for each i, $1 \leq i \leq r$.

Thus the sets $L(a, b)$ have the following properties:

(i) if $a' \geq a$, $b' \geq b$, then $L(a', b') \supset L(a, b)$

(ii) $h \in L(a, b) \Longleftrightarrow \pi^n h \in L(a+nb, b)$

all $n \geq 0$

(iii) If $h \in L(a, b)$, and if $\tau \in B$ is a finite polynomial of degree n, then $\tau h \in L(a + n, b)$

LEMMA 4. *Suppose M is R-flat. There exists a constant C satisfying the following condition for all i: if $f \in (\pi^i F + K) \cap L(c, d)$ and d is sufficiently large, then there exists $g \in \pi^i F \cap L(c + iC, d)$ with $f = g \bmod K$.*

Proof. Since M is R-flat, we have an exact sequence over $\dfrac{B}{\pi B}$.

$$0 \to \bar{K} \to \bar{F} \to \bar{M} \to 0.$$

Let $\bar{k}_1, \cdots, \bar{k}_t$ be a good basis for $\bar{K}$, in the sense of (14). (This means the following: If we define $dg \cdot \bar{f} = \max dg \cdot \bar{f}_i$ for $\bar{f} = (\bar{f}_1, \cdots, \bar{f}_t) \in \bar{F}$, there exists a constant l such that every $\bar{k} \in \bar{K}$ may be expressed as a sum $\bar{k} = \sum_{i=1}^{t} \bar{a}_i \bar{k}_i$, with $\bar{a}_i \in \dfrac{R}{\pi R}[X_1, \cdots, X_n]$ and $dg \cdot \bar{a}_i \leq dg \cdot \bar{k} + l$).

Lift each $\bar{k}_i$ to an element $k_i = (k_{i,1}, \cdots, k_{i,t}) \in K$. The set $\{k_i\}$ spans K over B by Nakayama's Lemma. Suppose, for every i, $k_i \in L(a, b)$. In particular, then $dg \cdot \bar{k}_i \leq a$ for each i. For the constant C required in the Lemma, we shall take $C = a + l$; and we shall prove that d is sufficiently large when $d \geq b$.

To prove that this constant C satisfies the lemma, we use induction i. The lemma is trivial for $i = 0$. Suppose $f \in (\pi^i F + K) \cap L(c, d)$. Suppose further that $d \geq b$. Then $f \in (\pi^{i-1} F + K) \cap L(c, d)$, and by induction there is an element $h \in \pi^{i-1} F \cap L(c + (i-1)C, d)$ such that $h = f \bmod K$. Thus $h \in \pi^i F$

$+K$, and $h=\pi^{i-1}h'$ for some h' in F. In fact, $h'\in L(c+(i-1)C+(i-1)d, d)$. Reducing modulo πF, $\bar{h}'\in\bar{K}$. Write $\bar{h}'=\sum_{i=1}^{r}\bar{r}_i\bar{k}_i$, with $\bar{\tau}_i\in\frac{B}{\pi B}$ and $dg\cdot\bar{\tau}_i\leq l+c+C(i-1)+d(i-1)$. Lift $\bar{\tau}_i$ to $\tau_i\in B$ with $dg\cdot\tau_i=dg\cdot\bar{\tau}_i$, and set $g'=h'-\sum_{i=1}^{t}\tau_i k_i$. Note that $g'\in\pi F\cap L(l+c+C(i-1)+d(i-1)+a, d)$, because $\sum_{i=1}^{t}\tau_i k_i\in L(l+c+C(i-1)+d(i-1)+a, b)$ and $b\leq d$. Define $g=\pi^{i-1}g'$. This finishes our induction, for $g=\pi^{i-1}g'=\pi^{i-1}h' \,(\mathrm{mod}\, K)=h(\mathrm{mod}\, K)=f(\mathrm{mod}\, K)$, and $g\in\pi^i F\cap L(l+c+C(i-1)+a, d)=\pi^i F\cap L(c+iC, d)$.

PROPOSITION 5. *Suppose M is R-flat, and suppose $g_j=(g_{j,1},\cdots,g_{j,r})\in F$; $j=0,1,\cdots,$; is a sequence of vectors such that:*

(1) $g_j\in\pi^j F+K$;

(2) $g_j\in L(c(j-1), d)$ *for some integers c, d independent of j.*

If m_j is the image of g_j in M, then $\sum_{j=0}^{\infty}m_j$ converges in M.

Proof. Choose C as in Lemma 4, and assume that d is so large that Lemma 4 applies. This assumption is innocuous, as $L(e, d)\subseteq L(e, d+1)$. Replace each g_j by some element $h_j\in\pi^j F\cap L(c+j(c+C), d)$ such that $h_j=g_j$ (mod K.) The image of h_j in M is m_j, and $\sum_{j=0}^{\infty}h_j$ converges to an element of $L(c, C+c+d)$. Consequenly, $\sum_{i=0}^{\infty}m_j$ converges in M.

Now we can prove Lemma 1. In the case that M is R-flat, Lemma 1 is a special case of Proposition 5. That is, for each $j\geq 0$, the polynomials $(P_{1,j}(x),\cdots,P_{t,j}(x))$ from Lemma 1 form a vector g_j in F which is an element of $L(c, c)$. Moreover, by the hypothese of Lemma 1, the image of g_j in M is $\sum_{i=1}^{t}P_{i,j}(x)\mu_i\in\pi^j M$, so $g_j\in\pi^j F+K$. By proposition 5, then $\sum_{j=0}^{\infty}(\sum_{i=1}^{t}P_{i,j}(x)\mu_i)$ converges in M.

If M is not flat over R, let T be the R-torsion submodule of M viewed as an A-module. Define N by the exact sequence:

$$0\rightarrow T\rightarrow M\rightarrow N\rightarrow 0$$

N is R-flat, and so the image of $\sum_{j=0}^{\infty}(\sum_{i=1}^{t}f_{i,j}\mu_i)$ converges in N. Since A is noetherian and T is finite type over A, there is a constant e such that $\pi^e T=0$. Thus $\pi^e M\cap T=0$, and so the map $\pi^e M\rightarrow N$ inherited from the projection $M\rightarrow N$ is injective. Moreover, the image $\pi^e M$ is closed in N, and the topology of $\pi^e M$ agrees with the one inherited from the topology of N. Consequently $\sum_{j=e}^{\infty}(\sum_{i=1}^{t} f_{ij}\mu_i)$ converges in $\pi^e M$, and so $\sum_{j=0}^{\infty}(\sum_{i=1}^{t} f_{i,j}\mu_i)$ converges in M.

4. Weak Formal Preschemes

Affine formal schemes can be patched together in much the same manner as affine algebraic schemes. The most interesting such construction is an analogue of projective space, which we will study in the next section. This introductory section contains elementary definitions and the construction of the weak completion of a finite type prescheme. The operation of weak completion will provide us with our best examples of weak formal preschemes.

DEFINITION 1. *A weak formal (wf) prescheme (over R)* is a ringed space $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ such that every point of $\mathscr{X}$ has a neighborhood isomorphic to an affine wf scheme (2.17). Open sets of $\mathscr{X}$ which are isomorphic to affine wf schemes are called *affine* wf *open sets.*

The sheaf $\dfrac{\mathscr{O}_{\mathscr{X}}}{\mathfrak{m}\mathscr{O}_{\mathscr{X}}} = \bar{\mathscr{O}}_{\mathscr{X}}$ is a scheme of finite type over $\dfrac{R}{\mathfrak{m}}$ whose underlying space is also $\mathscr{X}$. Often, for emphasis, we shall refer to $\mathscr{X}$ as $\overline{\mathscr{X}}$ when we want to consider $\mathscr{X}$ as the space underlying $\bar{\mathscr{O}}_{\mathscr{X}}$. Affine open sets of $\overline{\mathscr{X}}$ will be called simply affine open sets. Although affine wf open sets of $\mathscr{X}$ are affine open sets of $\overline{\mathscr{X}}$, it is not known is affine open sets are always affine wf open sets.

Suppose that $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is an R-prescheme of finite type and F is a coherent $\mathscr{O}_{\mathscr{X}}$-module. $\mathscr{O}_{\mathscr{X}}$ and F can be weakly completed to a wf prescheme and a coherent sheaf of modules over this wf prescheme. The operation of weak completion will be defined as an extension of the operation of weak completion for R-algebras and their modules.

To begin, let $\mathscr{X}^\dagger = \{x \in \mathscr{X} : \mathscr{O}_{\mathscr{X},x} \neq \mathfrak{m}\mathscr{O}_{\mathscr{X},x}\}$. Define a presheaf $\mathscr{O}^\dagger_{\mathscr{X}}$ on affine open sets of $\mathscr{X}$ as follows: if $U \subset V \subset \mathscr{X}$ are affine open sets, let $\Gamma(U, \mathscr{O}^\dagger_{\mathscr{X}}) = \Gamma(U, \mathscr{O}_{\mathscr{X}})^\dagger$, the weak completion; and let the morphism $\Gamma(V, \mathscr{O}^\dagger_{\mathscr{X}}) \to \Gamma(U, \mathscr{O}^\dagger_{\mathscr{X}})$ be the unique continuous extension of the restriction map $\Gamma(V, \mathscr{O}_{\mathscr{X}}) \to \Gamma(U, \mathscr{O}_{\mathscr{X}})$.

LEMMA 5. *Suppose $\mathscr{X}$ is affine. Then $\mathscr{O}^\dagger_{\mathscr{X}}$ is a sheaf on the principal open subsets of $\mathscr{X}$ which is concentrated on $\mathscr{X}^\dagger$. Further, $\mathscr{O}^\dagger_{\mathscr{X}}$ induces an affine wf scheme,* *also denoted* $\mathcal{O}^{\dagger}_{\mathscr{X}}$, *on* $\mathscr{X}^{\dagger}$.

Proof. If A is a finitely generated R-algebra and $f \in A$, then $(A_f)^{\dagger} = A^{\dagger}_{[f]}$. Consequensly $\mathcal{O}^{\dagger}_{\mathscr{X}}$ can be identified with the extension by zero of $(\mathscr{X}^{\dagger}, \Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})^{\dagger\sim})$ to $\mathscr{X}$. This proves the Lemma.

Lemma 5 shows that $\mathcal{O}^{\dagger}_{\mathscr{X}}$ induces a sheaf, also denoted $\mathcal{O}^{\dagger}_{\mathscr{X}}$, concentrated on $\mathscr{X}^{\dagger}$, and that $(\mathscr{X}^{\dagger}, \mathcal{O}^{\dagger}_{\mathscr{X}})$ is a wf prescheme whose affine wf open sets include the intersections of $\mathscr{X}^{\dagger}$ with affine open sets of $\mathscr{X}$. We shall say that the wf prescheme $(\mathscr{X}^{\dagger}, \mathcal{O}^{\dagger}_{\mathscr{X}})$, together with its canonical morphism $(\mathscr{X}^{\dagger}, \mathcal{O}^{\dagger}_{\mathscr{X}}) \to (\mathscr{X}, \mathcal{O}_{\mathscr{X}})$, is the weak completion of $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$.

The weak completion, $F^{\dagger}$, of F is defined analogously. If $U \subset V \subset \mathscr{X}$ are open affine sets, let $\Gamma(U, F^{\dagger}) = \Gamma(U, F) \otimes_{\Gamma(U, \mathcal{O}_{\mathscr{X}})} \Gamma(U, \mathcal{O}^{\dagger}_{\mathscr{X}})$; and let the morphism $\Gamma(V, F^{\dagger}) \to \Gamma(U, F^{\dagger})$ be the unique continuous extension of the restriction map $\Gamma(V, F) \to \Gamma(U, F)$. The action of $\mathcal{O}_{\mathscr{X}}$ on $F^{\dagger}$ extends continuously to make $F^{\dagger}$ into an $\mathcal{O}^{\dagger}_{\mathscr{X}}$-module.

LEMMA 6. Suppose $\mathscr{X}$ is affine. Then $F^{\dagger}$ induces a sheaf concentrated on $\mathscr{X}^{\dagger}$ which is a coherent $\mathcal{O}^{\dagger}_{\mathscr{X}}$-module.

Proof. Let A be a finitely generated R-algera, M a finite A-module, and $f \in A$. Then $M_f \otimes_{A_f} (A_f)^{\dagger} = M^{\dagger} \otimes_A + A^{\dagger}_{[f]}$. Consequently, $F^{\dagger}$ can be identified with the extension by zero of the sheaf $(\mathscr{X}^{\dagger}, \Gamma(\mathscr{X}, F)^{\dagger\sim})$. This establishes the lemma.

Thus, for any coherent module F over an R-prescheme of finite type $\mathscr{X}, \mathcal{O}_{\mathscr{X}})$, $F^{\dagger}$ is a coherent $\mathcal{O}^{\dagger}_{\mathscr{X}}$-module. $(\mathscr{X}^{\dagger}, F^{\dagger})$, together with its canonical morphism $(\mathscr{X}^{\dagger}, F^{\dagger}) \to (\mathscr{X}, F)$, is the weak completion of F.

PROPOSITION 7. *The functor* (*weak completion*) *of coherent* $\mathcal{O}_{\mathscr{X}}$*-modules is exact.*

Proof. Let $F \to G \to H$ be an exact sequence of coherent $\mathcal{O}_{\mathscr{X}}$-modules. For $x \in X$, select an open affine neighborhood $U \subset X$ of x; then

$$\Gamma(U, F) \to \Gamma(U, G) \to \Gamma(U, H)$$

is exact. Because $\Gamma(U, \mathcal{O}^{\dagger}_{\mathscr{X}})$ is a flat $\Gamma(U, \mathcal{O}_{\mathscr{X}})$-module (1.3),

$$\Gamma(U, F^{\dagger}) \to \Gamma(U, G^{\dagger}) \to \Gamma(U, H^{\dagger})$$

is exact. Thus $F^{\dagger} \to G^{\dagger} \to H^{\dagger}$ is exact.

Remark. Henceforth, we shall use $F^\dagger$ to denote the weak completion of F on $\mathscr{X}^\dagger$ and $i_* F^\dagger$ to denote $F^\dagger$ extended by zero to $\mathscr{X}$. Since $\mathscr{X}^\dagger$ is closed in $\mathscr{X}$, there is a natural map $H^i(\mathscr{X}, F) \to H^i(\mathscr{X}, i_* F^\dagger) \to H^i(\mathscr{X}^\dagger, F^\dagger)$. The second homomorphism of this sequence is bijective.

5. The Comparison Theorem

Throughout this section, $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ will be Proj $R[X_0, \cdots, X_m]$, and F will be a coherent $\mathcal{O}_{\mathscr{X}}$-module. $(\mathscr{X}^\dagger, \mathcal{O}^\dagger_{\mathscr{X}})$ and $F^\dagger$ are then the weak completions of $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ and F respectively (cf. § 4). We will prove that the natural map $F \to F^\dagger$ induces a cohomology isomorphism $H^i(\mathscr{X}, F) \to H^i(\mathscr{X}^\dagger, F^\dagger)$.

The first two theorems of this section are special cases of this cohomology isomorphism.

Theorem 1. $H^i(\mathscr{X}^\dagger, F^\dagger) = 0$ for $i > m$.

Proof. Affine wf open sets in $\mathscr{X}^\dagger$ are cohomologically trivial (2.14). Consequently, the cohomology of $F^\dagger$ may be computed using the singular cochains of an open affine covering $\mathscr{U}^\dagger$ of $\mathscr{X}^\dagger$, provided that the intersection of any finite collection of elements of $\mathscr{U}^\dagger$ is affine wf (2, **II.** 5.4.1). In particular, we may take $\mathscr{U}^\dagger$ to be the open sets $\mathscr{X}_{X_i} \cap \mathscr{X}^\dagger$. This open covering contains $m+1$ elements, so $H^i(\mathscr{X}^\dagger, F^\dagger) = 0$ for $i > m$.

Our second special case deals with certain invertible $\mathcal{O}_{\mathscr{X}}$-modules. We need some notation. $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ has a homogeneous coordinate ring $A = R[X_0, \cdots, X_m]$. Let $U_i = \mathscr{X}_{X_i}$, and let $\mathscr{U} = \{U_0, \cdots, U_m\}$. Define $U_{i_0,\dots,i_r} = U_{i_0} \cap \cdots \cap U_{i_r}$. Then we make the identification:

$$\Gamma(U_{i_0,\dots,i_r}, \mathcal{O}_{\mathscr{X}}) = R\left[\frac{X_0}{X_{i_0}}, \cdots, \frac{X_m}{X_{i_0}}, \frac{X_{i_0}}{X_{i_1}}, \cdots, \frac{X_{i_0}}{X_{i_r}}\right].$$

Let $$U^\dagger_{i_0,\dots,i_r} = U_{i_0,\dots i_r} \cap \mathscr{X}^\dagger \text{ and } \mathscr{U}^\dagger = \{U^\dagger_0, \dots, U^\dagger_m\}.$$

We are going to analyze the invertible modules $\mathcal{O}(n)$, $n \in Z$, given on the affine open subsets U_i by the cyclic sub-$\Gamma(U_i, \mathcal{O}_{\mathscr{X}})$-module of $R[X_0, \cdots, X_m, X_0^{-1}, \cdots, X_m^{-1}]$:

$$\Gamma(U_i \mathcal{O}(n)) = X_i^n \Gamma(U_i, \mathcal{O}_{\mathscr{X}}),$$
$$0 \le i \le m.$$

The natural map $\mathcal{O}(n) \to i_* \mathcal{O}(n)^\dagger$ induces a differential homomorphism of the simplicial complexes

$$C^{\cdot}(\mathscr{U}, \mathscr{O}(n)) \to C^{\cdot}(\mathscr{U}^{\dagger}, \mathscr{O}(n)^{\dagger}).$$

Which in turn induces a homomorphism

$$\varphi : H^{\cdot}(\mathscr{U}, \mathscr{O}(n)) \to H^{\cdot}(\mathscr{U}^{\dagger}, \mathscr{O}(n)^{\dagger}).$$

In addition, the following diagram commutes:

$$\text{(A)} \qquad \begin{array}{ccc} H^{\cdot}(\mathscr{U}, \mathscr{O}(n)) & \xrightarrow{\varphi} & H^{\cdot}(\mathscr{U}^{\dagger}, \mathscr{O}(n)^{\dagger}) \\ \downarrow & & \downarrow \\ H^{\cdot}(\mathscr{X}, \mathscr{O}(n)) & \longrightarrow & H^{\cdot}(\mathscr{X}^{\dagger}, \mathscr{O}(n)^{\dagger}) \end{array}$$

where the vertical arrows are the usual natural transformations from simplicial cohomology to sheaf cohomology, and the bottom arrow is the homomorphism induced by weak completion.

LEMMA 2. *φ is bijective.*

The proof of Lemma 2 will be given at the end of this section.

THEOREM 3. *The homomorphism $H^{\cdot}(\mathscr{X}, \mathscr{O}(n)) \to (H^{\cdot}(\mathscr{X}^{\dagger}, \mathscr{O}(n)^{\dagger})$ induced by weak completion is bijective.*

Proof. Leray's Theorem (2, II. 5. 4. 1) and (2. 14) prove that the natural transformations from $H^{\cdot}(\mathscr{U}, \mathscr{O}(n))$ (resp. $H^{\cdot}(\mathscr{U}^{\dagger}, \mathscr{O}(n)^{\dagger})$) to $H^{\cdot}(X, \mathscr{O}(n))$ (resp. $H^{\cdot}\ \mathscr{X}^{\dagger}, \mathscr{O}(n)^{\dagger})$) are bijective. Thus Lemma 2, together with the diagram (A) establish the desired result.

Now we are ready to prove our main theorem.

THEOREM 4 (THE COMPARISON THEOREM). *The natural map $H^{\cdot}(\mathscr{X}, F) \to H^{\cdot}(\mathscr{X}^{\dagger}, F^{\dagger})$ induced by weak completion is an isomorphism.*

Proof. Construct an exact sequence of sheaves

$$0 \to H \to G \to F \to 0$$

such that G is a finite direct sum of sheaves $\mathscr{O}(n_{\alpha})$, $n_{\alpha} \in Z$ (3, II. 2. 7. 9). From this exact sequence, derive a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{i}(\mathscr{X}, H) & \longrightarrow & H^{i}(\mathscr{X}, G) & \longrightarrow & H^{i}(\mathscr{X}, F) & \longrightarrow \cdots \\ & {\scriptstyle h_i}\downarrow & & {\scriptstyle g_i}\downarrow & & {\scriptstyle f_i}\downarrow & \\ \cdots \longrightarrow & H^{i}(\mathscr{X}^{\dagger}, H^{\dagger}) & \longrightarrow & H^{i}(\mathscr{X}^{\dagger}, G^{\dagger}) & \longrightarrow & H^{i}(\mathscr{X}^{\dagger}, F^{\dagger}) & \longrightarrow \cdots \end{array}$$

For $i>m$, f_i is bijective (Theorem 1 and 3, III. 2. 2. 2.). Suppose f_i is bijective for all coherent modules F and all $i>r$. Theorem 3 proves that g_i is bijective for all i; then the diagram shows that f_r is surjective for any F. In particular, h_r is surjecitve, and so f_r is bijective. By descending induction, the theorem is proven.

Remark. The proof of Theorem 4 is copied from (8, pp. 21).

COROLLARY 5. *$(\mathscr{X}^\dagger, F^\dagger)$ as above. Then:*

(1) *$H^i(\mathscr{X}^\dagger, F^\dagger)$ is a finite R-module, all i.*

(2) *$H^i(\mathscr{X}^\dagger, F^\dagger)=0$ for $i>m$.*

Proof. The second assertion is just Theorem 1. The first statement follows from Theorem 4 and 3, III. 3. 2. 3.

We conclude this section with a proof of Lemma 2. As in the proof of (2. 8), it is convenient to augument the simplicial complexes $C^{\cdot}(\mathscr{U}, \mathscr{O}(n))$ and $C^{\cdot}(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger)$ by a term $C^{-1}(\mathscr{U}, \mathscr{O}(n))=C^{-1}(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger)=$ the n-th homogeneous component of A. Recall that for $r\geq 0$, $C^r(\mathscr{U}, \mathscr{O}(n))$ and $C^r(\mathscr{U}, \mathscr{O}(n)^\dagger)$ are defined as follows

$$C^r(\mathscr{U}, \mathscr{O}(n))=\bigoplus_{0\leq i_0<\cdots<i_r\leq m} X^n_{i_0}\Gamma(U_{i_0\ldots i_r}, \mathscr{O}_{\mathscr{X}})$$

$$C^r(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger)=\bigoplus_{0\leq i_0<\cdots<i_r\leq m} X^n_{i_0}\Gamma(U^\dagger_{i_0,\ldots,i_r}, \mathscr{O}^\dagger_{\mathscr{X}})$$

where $X^n_{i_0}\Gamma(U_{i_0\ldots i_r}, \mathscr{O}_{\mathscr{X}})$ is a cyclic submodule of the $\Gamma(U_{i_0,\ldots,i_r}, \mathscr{O}_{\mathscr{X}})$-module $R[X_0, \cdots, X_m, X_0^{-1},\ldots,X_m^{-1}]$, and $X^n_{i_0}\Gamma(\mathscr{U}^\dagger_{i_0,\ldots,i_r}, \mathscr{O}^\dagger_{\mathscr{X}})$ is a cyclic submodule of the $\Gamma(U^\dagger_{i_0\ldots i_r}, \mathscr{O}^\dagger_{\mathscr{X}})$-module $R[X_0, \cdots, X_m, X_0^{-1},\ldots,X_m^{-1}]^\dagger$. The components of $C^r(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger)$ inherit their topology from the topology on $\Gamma(U_{i_0,\ldots,i_r}, \mathscr{O}^\dagger_{\mathscr{X}})$. In either the algebraic or the weakly complete case, the boundary map $\delta: C^{-1}\to C^0$ is the sum of the inclusion maps of C^{-1} into each component of C^0.

LEMMA 6. *Let $S=\dfrac{R}{\mathfrak{m}^k}$, $B=\dfrac{A}{\mathfrak{m}^k A}$, and $F=\dfrac{\mathscr{O}(n)}{(\mathfrak{m})^k}$. Suppose for some integer r, $\sigma\in C^r(\mathscr{U}, F)$ is a coboundary such that:*

(1) *$\sigma\in\mathfrak{m}^h C^r(\mathscr{U}, F)$;*

(2) *for all $(i_0, \cdots, i_r)$ and some fixed s, $(X_{i_0}\cdots X_{i_r})^s(\sigma_{i_0,\ldots,i_r}X_{i_0}^{-n})\in B$.*

(Note that for each $(i_0, \cdots, i_r)$, there exists an integer s such that (2) holds, because $\sigma_{(i)} X_{i_0}^{-n} \in \Gamma\Big(U_{(i)}, \frac{\mathcal{O}\mathcal{X}}{(\mathfrak{m}^k)}\Big)$. We are assuming that s is sufficiently large to work for every component of σ.) Then there is an element $\tau \in C^{r-1}(\mathcal{U}, F)$ such that $\delta\sigma = \tau$ and:

(3) $\tau \in \mathfrak{m}^h C^{r-1}(\mathcal{U}, F)$;

(4) for all $(i_0, \cdots, i_{r-1})$, $(X_{i_0} \cdots X_{i_{r-1}})^s (\tau_{i_0, \ldots, i_{r-1}} X_{i_0}^{-n}) \in B$.

Proof. The lemma is clearly true for $r \leq 0$. The following proof works for $r > 0$. Copying Grothendieck (3, III. 2. 1), we define a double complex $K^{\cdot}(X^{\cdot})$ as follows: $K^{r+1}(X^t)$ is the direct sum of certain free S-modules $K^{r+1}(X^t)_{i_0, \ldots, i_r}$ indexed by all sets $0 \leq i_0 < \cdots < i_r \leq m$. Namely, take $K^{r+1}(X^t)_{(i)}$ to be the $n + t(r+1)$-homogeneous component of B. The maps $p_t : K^{r+1}(X^t) \to K^{r+1}(X^{t+1})$ given by $P_t(\eta_{i_0, \ldots, i_r} = (X_{i_0} \cdots X_{i_r})\eta_{i_0, \ldots, i_r}$ permit us to define $K^{r+1}((X)) = \varinjlim_t K^{r+1}(X^t)$. The homomorphisms $q_t : K^{r+1}(X^t) \to C^r(\mathcal{U}, F)$ given by $q_t(\eta)_{i_0, \ldots, i_r} = \frac{\eta_{i_0, \ldots, i_r}}{(X_{i_0} \cdots X_{i_r})^t}$ induces an isomorphism $K^{r+1}((X)) \to C^r(\mathcal{U}, F)$ for $r \geq 0$ (3, III. 2. 1. 3).

The double complex $K^{\cdot}(X^{\cdot})$ has a second map, $\sigma : K^{r+1}(X^t) \to K^{r+2}(X^t)$, given by $\delta(\eta)_{i_0, \ldots, i_{r+1}} = \sum_{\alpha=0}^{r+1} (-1)^\alpha X_{i_\alpha}^t \eta \delta_{i0, \ldots, i_\alpha, \ldots, i_{r+1}}$. The homomorphism δ gives $K^{\cdot}(X^t)$ the structure of a differential complex with the following two properties:

(5) $q_t : K^{\cdot}(X^t) \to C^{\cdot}(\mathcal{U}, F)$ is an injection of differential complexes;

(6) the induced maps $\bar{q}_t : H^{\cdot}(X^t) \to H^{\cdot}(\mathcal{U}, F)$ are also injective.

(3, III. 1. 1. 6, III. 2. 1. 9)

Suppose now that $\sigma \in C^r(\mathcal{U}, F)$, $r > 0$, is as given in the statement of the lemma. Then $q_s^{-1}(\sigma)$ is defined, and —by (5) and (6) above— $q_s^{-1}(\sigma) = \delta\gamma$ for some $\gamma \in K^r(X^t)$. Moreover, γ is a coboundary modulo $\mathfrak{m}^h$, because $\delta\gamma = 0 \bmod \mathfrak{m}^h$ and $H^r(X^{\cdot}) = 0$ for $r \leq m$ (3, III. 1. 1. 4). Let $\nu = \delta\rho \bmod \mathfrak{m}^h$. The element $\tau = q_s(\nu - \delta\rho)$ satisfies the requirements of the lemma.

LEMMA 7. *The natural projections* $\mathcal{O}(n)^\dagger \to \frac{\mathcal{O}(n)^\dagger}{(\mathfrak{m}^s)}$ *induce a homomorphism* $\rho : H^{\cdot}(\mathcal{U}^\dagger, \mathcal{O}(n)^\dagger) \to \varprojlim H^{\cdot}\Big(\mathcal{U}^\dagger, \frac{\mathcal{O}(n)^\dagger}{(\mathfrak{m}^s)}\Big)$. *The map* ρ *is injective.*

Proof. Let $\sigma \in C^r(\mathcal{U}^\dagger, \mathcal{O}(n)^\dagger)$ be a cocycle which is a coboundary modulo $\mathfrak{m}^s$ for all $s > 0$. We must prove that σ is a coboundary. Each component $\sigma_{i_0, \ldots, i_r}$ of σ can be expressed as a power series:

$$\sigma_{i_0,\dots,i_r} = \sum_{j=0}^{\infty} \sigma^j_{i_0,\dots,i_r}) X^n_{i_0},$$

where $\sigma^j_{i_0,\dots,i_r} \in \mathfrak{m}^j \Gamma(U_{i_0,\dots,i_r}, \mathscr{O}_x)$ has degree $\leq c(j+1)$ in the elements $\left\{\frac{X_0}{X_{i0}}, \cdots, \frac{X_m}{X_{i_0}}, \frac{X_{i_0}}{X_{i_1}}, \cdots, \frac{X_{i_0}}{X_{i_r}}\right\}$ for some constant c. Thus:

$$(X_{i_0} \dots X_{i_r}) c^{(j+1)} \sigma^j_{i_0,\dots,i_r} \in A.$$

To complete the proof of Lemma 7, we shall construct a sequence of cochains $\tau_k \in C^{r-1}(\mathscr{U}, \mathscr{O}(n))$, $k = 0, 1, \cdots$, such that:

(1) $\tau_k \in \mathfrak{m}^k C^{r-1}(\mathscr{U}, \mathscr{O}(n))$;

(2) $(X_{i_0} \cdots X_{i_{r-1}})^{c(k+1)} (\tau_{k;\, i_0,\dots,i_{r-1}} X^{-n}_{i_0}) \in A$ for each component $\tau_{k;\, i_0,\dots,i_{r-1}}$ of τ_k;

(3) $\sigma - \sum_{h=0}^{k} \tau_h \in \mathfrak{m}^{k+1} C^r(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger)$.

Conditions (1) and (2), together with (2.10), imply that $\sum_{k=0}^{\infty} \tau_k$ converges in $C^{r-1}(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger)$; condition (3) and the fact that $C^r(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger)$ is $\mathfrak{m}$-separated imply that $\sum_{k=0}^{\infty} \tau_k$ bounds σ.

We shall construct the τ_k inductively. Suppose we have already constructed τ_h for $h = 0, \cdots, k-1$. First we construct a cochain μ_k by defining its components;

$$\mu_{k;\, i_0,\dots,i_r} = \left(\sum_{j=0}^{k} \sigma^j_{i_0,\dots,i_r}\right) X^n_{i_0} - \sum_{h=0}^{k-1} \tau_{h;\, i_0,\dots,i_r}.$$

μ_k is a coboundary modulo $\mathfrak{m}^{k+1}$, $\mu_k = 0 \bmod \mathfrak{m}^k C^r$, and $(X_{i_0} \cdots X_{i_r})^{c(k+1)} (\mu_{k;\, i_0,\dots,i_r} X^n_{i_0}) \in A$. Lemma 6 proves that there exists a cochain τ_k such that τ_k is congruent to μ_k modulo $\mathfrak{m}^{k+1} C^r(\mathscr{U}, \mathscr{O}(n))$ which satisfies (1) and (2) above, and (3) follows easily.

Proof of Lemma 2. The projections $\mathscr{O}_{\mathscr{X}} \to \frac{\mathscr{O}_{\mathscr{X}}}{(\mathfrak{m}^s)}$ and $\mathscr{O}^\dagger_{\mathscr{X}} \to \frac{\mathscr{O}_{\mathscr{X}}}{(\mathfrak{m}^s)}$ commute with the natural injection $\mathscr{O}_{\mathscr{X}} \to \mathscr{O}^\dagger_{\mathscr{X}}$. Consequently we have the commutative triangle of cohomology groups:

$$\begin{array}{ccc} H^\cdot(\mathscr{U}, \mathscr{O}(n)) & \xrightarrow{\varphi} & H^\cdot(\mathscr{U}^\dagger, \mathscr{O}(n)^\dagger) \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & \varprojlim_s H^\cdot\left(\mathscr{U}, \frac{\mathscr{O}(n)}{(\mathfrak{m}^s)}\right) & \end{array}$$

Since R is complete, ψ is bijective (3, **III.** 2. 1. 12). We have established in Lemma 7 that ρ is injective; consequently φ is bijective.

Remark. The comparison theorem may be extended to the case of sheaves over a proper R-scheme. In fact, more generally, we have:

THEOREM A. *Suppose* $\mathscr{X}$ *and* $\mathscr{Y}$ *are finite type preschemes over* R, *and* $f : \mathscr{X} \to \mathscr{Y}$ *is a proper morphism (over* R*). Let* F *be a coherent sheaf on* $\mathscr{X}$. *This data may be weakly completed to a morphism* $f^\dagger : \mathscr{X}^\dagger \to \mathscr{Y}^\dagger$ *and a coherent* $\mathscr{O}^\dagger_{\mathscr{X}}$*-module* $F^\dagger$. *For all* n, $R^n f*(F)$ *is a coherent sheaf on* $\mathscr{X}$, *and we have the natural map on* $\mathscr{Y}^\dagger$:

$$\varphi : (R^n f*(F))^\dagger \to R^n f^\dagger *(F^\dagger).$$

For all n, φ *is bijective.*

In order to prove Theorem A, we first present a special case.

THEOREM B. *Suppose* A *is a finitely generated* R*-algebra and* $\mathscr{X}$ *is an* R*-prescheme projective over spec* A. *Let* F *be a coherent sheaf on* $\mathscr{X}$. *Then the natural map*

$$H^i(\mathscr{X}, F) \otimes_A A^\dagger \to H^i(\mathscr{X}^\dagger, F^\dagger)$$

is bijective for all i.

Proof. We may assume that $A = R[T, \cdots, T_s]$ and that $\mathscr{X} = \text{Proj}\, A[X_0, \cdots, X_m] = P^m(A)$. By the proof of Theorem 4, we may assume that $F = \mathscr{O}_{\mathscr{X}}(n)$, which we write as $\mathscr{O}_m(n)$. Suppose the theorem is true whenever $m = 0$ or $n = 0$. $\mathscr{X} = P^m(A) \supset P^{m-1}(A)$ for some imbedding, and this imbedding leads to an exact sequence of sheaves on X:

$$0 \to \mathscr{O}_m(n-1) \to \mathscr{O}_m(n) \to \mathscr{O}_{m-1}(n) \to 0.$$

Thus we have a commutative diagram

$$\begin{array}{ccccc}
\cdots \to H^i(\mathscr{X}, \mathscr{O}_m(n-1)) \otimes_A A^\dagger & \to & H^i(\mathscr{X}, \mathscr{O}_m(n)) \otimes_A A^\dagger & \to & H^i(\mathscr{X}, \mathscr{O}_{m-1}(n)) \otimes_A A^\dagger \to \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \to H^i(\mathscr{X}^\dagger, \mathscr{O}_m(n-1)^\dagger) & \to & H^i(\mathscr{X}^\dagger, \mathscr{O}_m(n)^\dagger) & \to & H^i(\mathscr{X}^\dagger, \mathscr{O}_{m-1}(n)^\dagger) \to \cdots
\end{array}$$

Suppose the theorem is true whenever $m = 0$ or $n = 0$; that is, all the vertical arrows are bijections whenever $m = 0$ or $n = 0$. Then the diagram, a two way induction on n, and an ascending induction on m prove that the vertical arrows are bijections for all m, n, i, which in turn proves Theorem B.

That the theorem holds whenever $m=0$ is obvious. To prove the theorem for arbitrary m and $n=0$, we imitate the proof of Lemma 2. Let $B=A[X_0, \cdots, X_m]$, and let $B_{i_0\ldots i_n}=A\left[\frac{X_0}{X_{i_0}}, \cdots, \frac{X_m}{X_{i_0}}, \frac{X_{i_0}}{X_{i_1}}, \cdots, \frac{X_{i_0}}{X_{i_r}}\right]$ $=\Gamma(U_{i_0,\ldots,i_r}, \mathcal{O}_{\mathcal{X}})$. As before, let $C^{-1}(\mathcal{U}, \mathcal{O}_{\mathcal{X}})=\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})=A$. Let $C^{-1}(\mathcal{U}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})=C^{-1}(\mathcal{U}, \mathcal{O}_{\mathcal{X}})\otimes_A A^\dagger=A^\dagger$. Consider the cochain complex $0\rightarrow C^{-1}(\mathcal{U}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})\rightarrow \cdots \rightarrow C^m(\mathcal{U}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})\rightarrow 0$. Suppose we have shown that the cohomology modules for this complex are separatde in the $\mathfrak{m}$-adic topology. Then $H^i(\mathcal{U}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})\subset H^i(\mathcal{U}^\dagger, \hat{\mathcal{O}}_{\mathcal{X}})=0$ for all $i\geq -1$, where "$\wedge$" indicates the formal completion of $\mathcal{O}_{\mathcal{X}}$ at $\mathfrak{m}$. Thus, $H^i(\mathcal{U}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})=0$ for all $i\geq -1$, and consequently (as in Theorem 3), $H^0(\mathcal{X}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})=A^\dagger=\dot{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})\otimes_A A^\dagger$; and $H^i(\mathcal{X}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})=0=H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})\otimes A^\dagger$, $i>0$.

In order to prove that the cohomology modules $H^i(\mathcal{U}^\dagger, \mathcal{O}^\dagger_{\mathcal{X}})$ are $\mathfrak{m}$-separated, we need

LEMMA C. *Define $D=A/\mathfrak{m}^p$ for some $p>0$. Let $X=P^m(D)=\operatorname{Proj} D[X_0, \cdots, X_m]$ and let $\mathcal{U}$ be the usual covering of $\mathcal{X}$. Suppose $\sigma\in C^r(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$ is a coboundary satisfying the following conditions:*

(1) $\sigma\in\mathfrak{m}^q C^r(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$

(2) $(X_{i_0}\cdots X_{i_r})^c\sigma_{i_0\ldots i_r}\in D[X_0, \cdots, X_m]$ *for all* $i_0\cdots i_r$.

(3) $dg_T\sigma\leq d$, *where $dg_T\sigma$ is defined to be the maximum of the degrees of the elements $\sigma_{i_0\ldots i_r}$ considered as polynomials in $T_1, \cdots, T_s$ over the ring $R/\mathfrak{m}^p[X_0/X_{i_0}, \cdots, X_m/X_{i_0}, X_{i_0}/X_{i_1}, \cdots, X_{i_0}/X_{i_r}]$. Then there is a cochain $\tau\in C^{r-1}(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$ satisfying*

(4) $\delta\tau=\sigma$;

(5) $\tau\in\mathfrak{m}qC^{r-1}(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$:

(6) $(X_{i_0}\cdots X_{i_r})^c\tau\in D[X_0, \cdots, X_m]$;

(7) $dg_T\tau\leq d$.

Proof. A close examination of the proof of Lemma 6 shows that, if $K^{r+1}(X^t)$ are taken to be free modules over D, then the T-degree of the cochain bounding the given coboundary σ which is constructed there may be controlled as required. Thus (7). Statements (4), (5) and (6) are proven in Lemma 6.

To prove that $H^r(\mathscr{U}^\dagger, \mathcal{O}_{\mathscr{X}})$ is $\mathfrak{m}$-separated, suppose $\sigma \in C^r(\mathscr{U}^\dagger, \mathcal{O}^\dagger_{\mathscr{X}})$ is a cocycle which is a coboundary modulo $\mathfrak{m}^p$, all $p > 0$. We will conctruct a coboundary for σ by choosing cochains $\tau_q \in C^{r-1}(\mathscr{U}^\dagger, \mathcal{O}^\dagger_{\mathscr{X}}), q = 0, 1, 2, \cdots$ satisfying

(8) $\delta(\sum_{q=0}^{t} \tau_q) = \sigma \bmod \mathfrak{m}^{t+1}$;

(9) $\tau_q \in \mathfrak{m}^q C^{r-1}(\mathscr{U}^\dagger, \mathcal{O}^\dagger_{\mathscr{X}})$;

(10) $(X_{i_0} \cdots X_{i_r})^{c(q+1)} \tau_{q;\, i_0 \dots i_{r-1}} \in B$ for some constant c;

(11) $dg_T \tau_q \leq d(q+1)$ for some constant d.

First, write

$\sigma_{i_0 \dots i_r} = \sum_{\alpha=0}^{\infty} \sigma^\alpha_{i_0 \dots i_r}$, where $\sigma^\alpha_{i_0 \dots i_n} \in \mathfrak{m}^\alpha B_{i_0 \dots i_r}$, $dg_T \sigma^\alpha_{(i)} \leq d(\alpha+1)$ and $(X_{i_0} \cdots X_{i_r})^{c(\alpha+1)} \sigma_{(i)} \in B$ for all α. Using Lemma C, the proof proceeds exactly as the proof of Lemma 7, only noting that the extension of Lemma 6 and Lemma C permits us to construct τ_q satisfying (11) as well as (8), (9), and (10). $\sum_{q=0}^{\infty} \tau_q = \tau$ is the desired cochain bounding σ; (9), (10), and (11) prove that the sum converges, and (8) shows that it bounds σ.

Theorem A is a direct consequence of Theorem B and Chow's Lemma. (The basic argument going from Theorem B to Theorem A was shown to me be S. Lubkin.) Since Theorm A is local on Y, we may assume that $Y = \operatorname{Spec} A$, where A is a finitely generated R-algebra. In this case, it suffices to prove that if F is a coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-modules, then the natural homomorphism:

$$(1) \qquad H^p(\mathscr{X}, F) \otimes_A A^\dagger \xrightarrow{\cong} H^p(\mathscr{X}^\dagger, F)^\dagger,$$

is bijective, all $p > 0$.

$\mathscr{X}$ is a noetherian scheme proper over $\operatorname{Spec} A$. Let K be the category of coherent $\mathcal{O}_{\mathscr{X}}$-modules on $\mathscr{X}$, and let K' be the subcategory of coherent $\mathcal{O}_{\mathscr{X}}$-modules satisfying (1). It is easy to see that K' is an exact subcategory of K, and that if F is an object of K' and if F' is a direct summand of F, then F' is an object of K'. (cf. (3, III. 3.1) for the definition of exact subcategory. Our proof that Theorem B implies Theorem A follows (3, III. 3.2).) Thus, by Grothendieck's "lemme de Devissage" (3, III. 3.1.3), to prove

Theorem A it suffices to show that, if $x \in \mathscr{X}$ is the generic point of an irreducible component of $\mathscr{X}$, there exists a coherent sheaf F of $\mathcal{O}_{\mathscr{X}}$-modules satisfying Theorem A such that $F_x \neq 0$.

Replacing $\mathscr{X}$ by a closed subscheme of $\mathscr{X}$ which contains the point x, we may assume that $\mathscr{X}$ is integral and irreducible. By Chow's Lemma, choose a scheme Z and a morphism $g : z \to \mathscr{X}$ such that g is surjective and projective, and $f \circ g : z \to \operatorname{Spec} A$ is also projective. By (3 : III. 3. 2. 1), for n sufficiently large, the coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-modules, $F = g_*(\mathcal{O}_Z(n))$, does not vanish at the point x; and the higher derived images of $\mathcal{O}_Z(n)$ vanish on $\mathscr{X}$: i.e., $R^p g_*(\mathcal{O}_Z(n)) = 0$, all $p > 0$.

Since $A^\dagger$ is a flat A-modules, we have a spectral sequence:

$$E_2^{pq} = H^p(\mathscr{X}, R^q g_*(\mathcal{O}_Z(n)) \otimes_A A^\dagger \Longrightarrow H^n(Z, \mathcal{O}_Z(n)) \otimes_A A^\dagger$$

(The Leray spectral sequence for the morphism g tensored with $A^\dagger$ over A). Since $R^q g_*(\mathcal{O}_Z(n)) = 0$ for $g > 0$, this spectral sequence degenerates to natural isomorphisms:

$$H^p(\mathscr{X}, F) \otimes_A A^\dagger \cong H^p(Z, \mathcal{O}_Z(n)) \otimes_A A^\dagger \qquad \text{all } p > 0. \tag{2}$$

Since the morphism g is projective, Theorem B implies that the derived images of the sheaf $\mathcal{O}_Z(n)^\dagger$ via the morphism $g^\dagger : Z^\dagger \to \mathscr{X}^\dagger$ of wf schemes may be described as follows:

$$R^p g^\dagger{}_*(\mathcal{O}_Z(n)^\dagger) = [R^p g_*(\mathcal{O}_Z(n)]^\dagger \qquad \text{all } p > 0. \tag{3}$$

Thus $R^q g^\dagger(\mathcal{O}_Z(n)^\dagger) = 0$ if $q > 0$, and the Leray spectral sequence for $g^\dagger : Z^\dagger \to \mathscr{X}^\dagger$:

$$E_2^{pq} = H^p(X^\dagger, R^q g^\dagger{}_*(\mathcal{O}_Z(n)^\dagger)) \Longrightarrow H^n(Z^\dagger, \mathcal{O}_Z(n)^\dagger)$$

degenerates into natural isomorphisms

$$H^p(\mathscr{X}^\dagger, g^\dagger{}_*(\mathcal{O}_Z(n)^\dagger) \cong H^p(Z^\dagger, \mathcal{O}_Z(n)^\dagger). \quad \text{all } p > 0. \tag{4}$$

Combining (2), (3) and (4), we have that the natural homomorphism

$$H^p(\mathscr{X}, F) \otimes_A A^\dagger \to H^p(\mathscr{X}^\dagger, F^\dagger) \tag{5}$$

is bijective all $p > 0$. This establishes (1) for the particular sheaf F and thus proves (1) in general.

6. The Existence Theorem

Throughout this section (R, π) is a complete discrete valuation ring.

Our purpose is this section is to prove that every coherent module on $(P_R^n)^\dagger$ is the weak completion of a unique coherent module on P_R^n. We shall begin with a lemma about cochain complexes over P_k^n, where k is a field. Let $A = k[X_0, \cdots, X_n]$ be the homogeneous coordinate ring for P_k^n, and let $\mathscr{U} = \{U_0, \cdots, U_n\}$ be the usual covering of P_k^n. Suppose F is a coherent $\mathscr{O}_{P_k^n}$ module. Fix generators $\varphi = \{\varphi_1^{(i)}, \cdots, \varphi_t^{(i)}\}$ for each $\Gamma(U_{i_0,\dots,i_r}, F)$ over $\Gamma(U_{(i)}, \mathscr{O}_{P_k^n})$. If $s \in C^r(\mathscr{U}, F)$, we define $dg_\varphi s \leq d$ if and only if for each $(i_0, \cdots, i_r)$ there exist coefficients $a_\alpha^{(i)} \in \Gamma(U_{(i)}, \mathscr{O}_P)$, $\alpha = 1, \cdots, t$, such that the (i)'th component s, $s^{(i)} = \sum_{\alpha=1}^{t} \alpha_\alpha^{(i)} \varphi_\alpha^{(i)}$ and $(X_0 \cdots X_n)^d a_\alpha^{(i)} \in A$ for all (i) and all α.

LEMMA 1. *Retaining the above notation, suppose there exist $\mathscr{O}_{P_k^n}$-modules F_i, $0 \leq i \leq r$, and a long exact sequence:*

$$F_r \to F_{r-1} \to \cdots \to F_0 \to F \to 0,$$

where $r < n$ and each $F_i = \sum_{j=1}^{c_i} \mathscr{O}(n_{i,j})$ with $n_{i,j} \geq 0$ and $\mathscr{O}(n) = \mathscr{O}_R n_k(n)$. Then there exists an integer N such that for all cocycles $s \in C^{n-r}(\mathscr{U}, F)$, $s = \delta s'$ with $dg \cdot_\varphi s' \leq dg \cdot_\varphi s + N$.

SUBLEMMA 1. *Suppose $F' \subset F$, and $\psi = (\psi_1^{(i)}, \cdots, \psi_u^{(i)})$ are generators of $\Gamma(U_{(i)}, F')$. There exists a constant N' such that for all $s \in C^r(\mathscr{U}, F') \subset C^r(\mathscr{U}, F)$, $dg \cdot_\psi s \leq dg \cdot_\varphi s + N'$. in particular, if the lemma holds for one set of generators of F it holds for all sets of generators of F.*

Proof. For all $(i_0, \cdots, i_r)$, $\alpha = 1, \cdots, u$, $\beta = 1, \cdots, t$, choose $\psi_\alpha^{(i)} = \sum_{\beta=1}^{t} v_{\alpha\beta}^{(i)} \varphi_\beta^{(i)}$. Then choose N' sufficiently large so that

$$(X_0 \cdots X_n)^{N'} v_{\alpha,\beta}^{(i)} \in A.$$

N' satisfies the sublemma.

Proof of the lemma. The lemma holds for $F = \mathscr{O}(n')$ for any $n' \geq 0$ and all $r < n$ by (5.6). In fact, by that lemma, we may choose $N = 0$ for the 'natural' basis of $\mathscr{O}(n')$. Consequently, we may choose a free basis $T = \{T_1^{(i)}, \cdots, T_u^{(i)}\}$ of $\Gamma(U_{(i)}, F_0)$ satisfying the lemma for $r < n$. By the sublemma, it suffices to prove the lemma when φ is the image of T in $C^{\cdot}(\mathscr{U}, F)$.

We will prove the lemma by induction on r. If $r = 0$, let $s \in C^n(\mathscr{U}, F)$. We may pull a back to $S \in C^n(\mathscr{U}, F_0)$ such that $dg \cdot_\varphi s = dg \cdot_T S$. By (5.6) and our choice of T, $S = \delta S'$ with $dg \cdot_T S' = dg \cdot_T S$. If s' is the image of S' in $C^{n-1}(\mathscr{U}, F)$, then $\delta s' = s$ and $dg \cdot_\varphi s' \leq dg \cdot_T S'$. Thus $dg \cdot_\varphi s' \leq dg \cdot_\varphi s$, and we may take $N = 0$.

Suppose the lemma holds for $r - 1 < n - 1$. Let

$$0 \to G \to F_0 \to F \to 0$$

be exact. G satisfies the conditions of the lemma for $r - 1$.

SUBLEMMA 2. *There exists a constant N_2 and a set of generators' $\psi = \{\psi_\alpha^{(i)}\}$ for the sheaf G such that if $s \in C^{\cdot}(\mathscr{U}, G) \subset C^{\cdot}(\mathscr{U}, F)$, Then*

$$dg \cdot_\psi s \leq dg_T S + N_2$$

Proof. Fix an r-tuple $(i_0, \cdots, i_r) = (i)$, $0 \leq i_0 < \cdots < i_r \leq n$. We will exhibit a finite set of generators $\{\psi_\alpha^{(i)}\}$ $1 \leq \alpha \leq t$ for the $\Gamma(\mathscr{U}_{(i)}, \mathscr{O}_{P_k^n})$-module $\Gamma(U_{(i)}, G)$ such that, if $s \in \Gamma(U_{(i)}, G) \subset \Gamma(U_{(i)}, F_0)$;

$$s = \sum_{\alpha=1}^{t} a_\alpha \psi_\alpha^{(i)} = \sum_{\beta=1}^{u} b_\beta T_\beta^{(i)}; \text{ and if}$$

$(X_{i_0} \cdots X_{i_r})^d b_\beta A$, $1 \leq \beta \leq u$, then $(X_{i_0} \cdots X_{i_r})^{d+N} a_\alpha \in A$, $1 \leq \alpha \leq t$. Let $B = \Gamma(U_{(i)}, \mathscr{O}_{P_k^n})$, $M = \Gamma(U_{(i)}, G)$, and $F = \Gamma(U_{(i)}, F_0)$ M is a B-submodule of the free B-module F. Choose a polynomial ring $C = k[Y_1, \cdots, Y_q]$ over k and a surjection $C \to B$ of k-algebras such that for $a \in B : (X_{i_0} \cdots X_{i_r})^d a \in A$, if and only if a may be pulled back to a polynomial $a' \in C$ of degree $\partial a' \leq d$. (For example, one could take C to be the polynomial ring generated over k by the symbols $y_{j_0, \ldots, j_d}$, where $(j_0, \cdots, j_d)$ is a d-tuple of integers such that $0 \leq j_0 \leq \cdots \leq j_d \leq n$. Then map $y_{j_0} \cdots {}_{j_d}$ onto $(X_{j_0} \cdots X_{j_d})(X_{i_0} \cdots X_{i_d})^{-1} \in B$.)

Let F' be a free C-module on the symbols $\{T_1^{(i)}, \cdots, T_u^{(i)}\}$. The surjection $C \to B$ induces a surjection $F' \to F$ such that, if $s \in F$, then s may be pulled back to an element $s' = \sum_{\beta=1}^{u} b_\beta T_\beta^{(i)} \in F'$ such that $dg_T\, s \in \max_\beta (\partial b_\beta)$, where ∂b_β is the degree of the polynomial $b_\beta \in C$.

Let M' be the preimage of M in F'. By the Lemma in (14) there exists a set of generators $\psi' = (\psi_1', \cdots, \psi_t')$ for the C-module M' and an integer N_2 such that, if $s' \in M' \subset F'$:

$$S' = \sum_{\alpha=1}^{t} a'_\alpha \phi'_\alpha = \sum_{\beta=1}^{u} b'_\beta T^{(i)}_\beta$$

Then we may choose the elements $a'_\alpha \in C$ to satisfy the relationship

$$\max_\alpha (\partial a'_\alpha) \leq \max_\beta (\partial b'_\beta) + N_2.$$

Let $\{\phi^{(i)}_1, \cdots, \phi^{(i)}_t\}$ be the image of ϕ' in M. The set of generators $\{\phi^{(i)}_\alpha\}_{1 \leq \alpha \leq t}$ for M and the integer N_2 satisfy the requirements of Sublemma 2: if $s \in M \subset F$, then

$$s = \sum_{\beta=1}^{u} b_\beta T^{(i)}_\beta$$

where for some integer d, $(X_{i_0} \cdots X_{i_r})^d b_\beta \in A$. Thus s pulls back to an element of F':

$$s' = \sum_{\beta=1}^{u} b'_\beta T^{(i)}_\beta$$

where $\partial b'_\beta \leq d$, $1 \leq \beta \leq u$.
Thus, as an element of M', s' may be written:

$$s' = \sum_{\alpha=1}^{t} a'_\alpha \phi'_\alpha$$

where $\partial a'_i \leq d + N_2$. If we set a_α = the image of a'_α in B then—in M—

$$s = \sum_{\alpha=1}^{t} a_\alpha \phi^{(i)}_\alpha$$

and $(X_{i_0} \cdots X_{i_r})^{d+N_2} a_\alpha \in A$, $1 \leq \alpha \leq t$. QED for the Sublemma

By Sublemma 1 and Sublemma 2, we may assume that we have chosen N_2 sufficiently large to satisfy the relationships: if $s \in C^{\cdot}(\mathscr{U}, G) \subset C^{\cdot}(\mathscr{U}, F_0)$, then

$$dg_\phi\, s \leq dg_T\, s + N_2$$
$$dg_T\, s \leq dg_\phi\, s + N_2$$

Let N_1 be the constant which, by our induction assumption, the Lemma assigns to the sheaf G.

If $s \in C^{n-r}(\mathscr{U}, F)$ is a cocycle, pull s back to a cochain $S \in C^{n-r}(\mathscr{U}, F_0)$ such that $dg_T S = dg_\phi s$. Note that $\delta S \in C^{n-r+1}(\mathscr{U}, G) \subset C^{n-r+1}(\mathscr{U}, F_0)$ is a cocycle. By the induction hypothesis then, $\delta S = \delta S'$, with $S' \in C^{n-r}(\mathscr{U}, G)$ and $dg_\phi S' \leq dg_\phi(\delta S) + N_1$. Thus $dg_\phi S' \leq dg_\Psi(\delta S) + N_1 \leq dg_T(\delta S) + N_1 + N_2 \leq dg_T S + N_1$

$+ N_2 \leq dg_\varphi s + N_1 + N_2$. Therefore $dg_T S' \leq dg_\varphi S' + N_2 \leq dg_\varphi s + N_1 + 2N_2$.

Consequently $dg_T(S - S') \leq dg_\varphi s + N_1 + 2N_2$. Also $S - S'$ is a cocycle of $C^{n-r}(\mathscr{U}, F_0)$. Consequently, there exists $S'' \in C^{n-r-1}(\mathscr{U}, F_0)$ such that $\delta S'' = S - S'$ and $dg\cdot_T S'' \leq dg\cdot_T(S - S') \leq dg\cdot_\varphi s + N_1 + 2N_2$.

Set s'' to be the image of S'' in $C^{n-r-1}(\mathscr{U}, F)$. Then $s = \delta s''$ and $dg\cdot_\varphi s'' \leq dg\cdot_T s'' \leq dg\cdot_\varphi s + N_1 + 2N_2$. Let $N = N_1 + 2N_2$. QED.

LEMMA 2. *If F is coherent on $P_r^{n\dagger} = \mathscr{X}$ and torsion free over R, and if $\bar{F} = \frac{F}{\pi F}$ satisfies Lemma 1 for $r = n - 1$, then $H^1(\mathscr{X}, F) = 0$.*

Proof. Choose sets of generators φ_α^i, $\varphi_\alpha^{i,j}$ for $\Gamma(U_i, F)$ and $\Gamma(U_{i,j}, F)$ over $\Gamma(U_i, \mathscr{O}\mathscr{X})$ and $\Gamma(U_{i,j}, \mathscr{O}\mathscr{X})$. If $s \in C^r(\mathscr{U}, F)$, $r = 0$ or 1, and $s^{(i)} = \sum_\alpha a_\alpha^{(i)} \varphi_\alpha^{(i)}$, we shall say that $dg\cdot_\varphi s \leq d$ if and only if $(X_0 \cdots X_n)^d a_\alpha^{(i)} \in A$ for all (i) and all α. An element $s \in C^r(\mathscr{U}, F)$ has finite degree if and only if each of the coefficients $a_\alpha^{(i)}$ is an element of $R[X_0, \cdots, X_n, X_0^{-1}, \cdots, X_n^{-1}]$. If $s_i \in C^r(\mathscr{U}, F)$ $i = 0, 1, \cdots$ is a sequence of elements, and if $dg\cdot_\varphi s_i \leq c(i+1)$ for some constant c, then $\sum_{i=0}^{\infty} \pi^i s^i$ converges in C^r (3.1).

We must establish two constants to be used in the proof. Let N_1 be an integer such that if $s \in C^1(\mathscr{U}, F)$ is a cocycle, then $s = \delta s'$ with $dg\cdot_\varphi s' = dg_\varphi s + N_1$ (Lemma 1). Let N_2 be a constant such that if $s \in \pi C^1\left(\mathscr{U}, \frac{F}{\pi^2 F}\right)$, then $s = \pi s'$ with $dg\cdot_\varphi s' \leq dg\cdot_\varphi s + N_2$.

(To show that the constant N_2 exists, we use (3.4). Fix a pair (i_0, i_1), $0 \leq i_0 < i_1 \leq n$).

Let $B = R\left[\frac{X_0}{X_{i_0}}, \cdots, \frac{X_n}{X_{i_0}}, \frac{X_{i_0}}{X_{i_1}}\right]$ and let C be a polynomial ring over R such that B is a quotient of C and an element $a \in B$ satisfies $(X_{i_0} X_{i_1})^d a \in A$ if and only if a pulls back to a polynomial $a' \in C$ with degree $\partial a' \leq d$. (cf. Sublemma 2 above). Let $M = \Gamma(U_{i_0, i_1}, F)$. M is a $B^\dagger = \Gamma(U_{i_0,i_1}, \mathscr{O}^\dagger_{\mathscr{X}})$-module which is flat as on R-module.

We have already chosen a set of generators $\{\varphi_\alpha^{i_0, i_1}\}_{1 \leq \alpha \leq t}$ for M. Let H be a free module over $C^\dagger$ with basis the symbols $\{T_\alpha\}_{1 \leq \alpha \leq t}$, and construct a surjection of $C^\dagger$ modules $H \to M$ by sending T_α to φ^{i_0, i_1}.

Since U_{i_0, i_1} is a affine, the natural map

$$M \to \Gamma(U_{i_0, i_1}, F/\pi^2 F)$$

is surjective (2.14 and 3.3). Suppose $s \in \pi \Gamma(U_{i_0, i_1}, F/\pi^2 F)$ is such that $dg_\varphi s = d$. Then

(1) s pulls back to an element s' of M such that $dg_\varphi s'' = d$.

(2) $s'' \in \pi M$.

Therefore as an element of the $C^\dagger$ module M, s'' may be expressed as a linear combination

$$s'' = \sum_{\alpha=1}^{t} a''_d \varphi_\alpha^{(i_0, i_1,)}$$

where $a''_\alpha \in C^\dagger$. In fact a''_α is a polynomial in C of degree $\partial a''_\alpha \leq d$, $1 \leq \alpha \leq t$. Using the notation of (3.4), s'' pulls back to an element $s''' \in F \cap L(d, 0)$. However since $s'' \in \pi M$, by (3.4) there exists s'''', a preimage of s'', which is contained in $\pi F \cap L(d + C, D)$ for some constants C and D which depend only on M. Thus there exists an element $w' \in F$ such that

(3) $\pi w' = s'''' \bmod \pi^2 F$

(4) w may be expressed as a linear combination

$$w' = \sum_{\alpha=1}^{t} b'_\alpha T_\alpha$$

where elements $b'_\alpha \in C^\dagger$ are polynomials of C of degree $\partial b'_\alpha \leq d + D + C$, $1 \leq \alpha \leq t$. (w' is, so to speak, the first term of s'''' divided by π).

If w is the image of w' in M, then w may be expressed as a linear combination over $B^\dagger$:

$$w = \sum_{\alpha=1}^{t} b_\alpha \varphi_\alpha^{(i_0, i_1)}$$

where b_α is the image of b'_α in $B^\dagger$.

Thus b_α is actually an element of B, and $dg_\varphi w \leq d + C + D$. Setting s' equal to the image of w in $\Gamma(U_{i_0, i_1}, F/\pi^2 F)$, see that

(5) $\pi s' = s$

(6) $dg_\varphi s' \leq d + D + C$.

Let $N_2^{(i_0, i_1)} = D + C$, and let $N_2 = \max_{(i_0, i_1)} \{N_2^{(i_0, i_1)}\}$. N_2 is the required constant.)

Suppose $s \in C^1(\mathscr{U}, F)$ is a cocycle. We may express s as an infinite sum:

$$s = \sum_{t=0}^{\infty} \pi^i s_i$$

with $dg_\varphi s_i \leq c(i + 1)$ for some constant c. To prove the lemma, we will construct a coboundary for s.

Suppose we have constructed cochains t_i, $i \geq 0$, and u_i, $i \geq 0$, satisfying the following three conditions:

(1) $\delta(\sum_{i=0}^{h-1} \pi^i t_i) = s \bmod \pi^h$ for all h;

(2) $dg \cdot_\varphi t_i \leq C(i+1)$ for $C = 2(N_1 + N_2)$.

(3) $s - \delta \sum_{i=0}^{h-1} \pi^i t_i = \pi^h u_h \bmod \pi^{h+1}$, with $dg \cdot_\varphi u_h \leq C'(h+1)$ for $C' = N_1 + N_2$ and all $h \geq 0$.

Condition (3) is necessary for the inductive construction of the t_i. Conditions (1) and (2) guarantee that $\sum_{i=0}^{\infty} t_i$ converges to a cochain with coboundary s.

Assume N_1 is so large that $c \leq N_1$. Set $u_0 = s_0$. Suppose we have constructed t_i for $i < h$ and u_i for $i \leq h$. The proof will be finished once we show how to construct t_h and u_{h+1}.

Because $\pi^h u_h$ is a cocycle module π^{h+1} and F is torsion free, u_h is a cocycle modulo π. Thus there exists $t_h \in C^0(\mathscr{U}, F)$ bounding u_h modulo π such that $dg \cdot_\varphi t_h \leq dg \cdot_\varphi u_h + N_1 \leq C'(h+1) + N_1 \leq C(h+1)$. Thus t_h satisfies (2). t_h also saitsfies (1), for

$$\pi^h u_h - \pi^h \delta(t_h) = s - \delta \sum_{i=0}^{h} \pi^i t_i = 0 \qquad \bmod \pi^{t+1}.$$

To construct u_{h+1}, note that $u_h - \delta t_h \in \pi C^1(\mathscr{U}, F)$. Take $u_{h+1} \in C^1(\mathscr{U}, F)$ such that $\pi u_{h+1} = \delta t_h \bmod \pi^2$ and $dg \cdot_\varphi u_{h+1} \leq dg \cdot_\varphi (u_h - \delta t_h) + N_2 \leq C'(h+1) + N_1 + N_2 = C'(h+2)$.

COROLLARY 3. *If F is coherent on $P_R^{n\dagger} = \mathscr{X}$ and torsion free over R, and if $\bar{F} = F/\pi F$ satisfies Lemma 1 for $r = n-1$, then the natural homomorphism*:

(1) $$\Gamma(\mathscr{X}, F) \to \Gamma(\mathscr{X}, \bar{F})$$

is surjective.

Proof. The endomorphism "multiplication by π" from F to itself is injective since F is torsion free over R. Thus we have an exact sequence of coherent sheaves on $\mathscr{X}$:

$$0 \to F \xrightarrow{\pi} F \to F/\pi F \to 0.$$

By Lemma 2, $H^1(\mathscr{X}, F) = 0$, so the natural homomorphism (1) is surjective.

If F is coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-moule, $\mathscr{X}=P_R^{n\dagger}$, then we define the "twistings" of F, $F(m)$, as follows. If n is an integer and $F=\mathcal{O}_{\mathscr{X}}$, then $F(m)=\mathcal{O}p_R^n(m)^\dagger$. In general, $F(m)=F\otimes_{\mathcal{O}_{\mathscr{X}}}\otimes\mathscr{X}(m)$. $F(m)$ is a coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-module, and the functor $F\rightsquigarrow F(m)$ is an exact functor from the category of coherent $\mathcal{O}_{\mathscr{X}}$-modules to itself. If, for some integer t, $\pi^t F=0$, then of course F may be viewed as a coherent sheaf of modules over the (ordinary projective) scheme $P^n(R/\pi^t R)=\mathscr{X}_t$. In this case the sheaves $F(m)$ are canonically isomorphism to the sheaves $F\otimes_{\mathcal{O}_{\mathscr{X}_t}}\otimes\mathscr{X}_t(m)$; i.e. in this special case our definition of $F(m)$ corresponds to the usual definition of the "twisted sheaves" $F(m)$. Also if $F=G^\dagger$ for some coherent sheaf G on $P_R^n=\mathscr{X}$, then $F(m)=G(m)^\dagger$.

PROPOSITION 4. *Suppose F is a coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-modules, where $\mathscr{X}=P_R^{n\dagger}$. Then for all sufficient by lagre integers in*

(1) $H^1(\mathscr{X}, F(m))=0$

(2) *The sheaf F is generated by its global sections: for each point $x\in\mathscr{X}$, the image of $\Gamma(\mathscr{X}, F(m))$ in the stalk $F(m)_x$ is a set of generators for the $\mathcal{O}_{\mathscr{X},x}$-module $F(m)_x$.*

Proof. If F is torsion free over F, then $F(m)$ is torsion free over R for all integers m. Moreover, for all sufficiently large integers m, the sheaves $\frac{F(m)}{\pi F(m)}=(F/\pi F)(m)$ satisfy Lemma 1 for $r=n-1$. Thus, by Lemma 2 and Corollary 3, $H^1(\mathscr{X}, F(m))=0$ and the natural map $\Gamma(\mathscr{X}, F(m))\to\Gamma\left(\mathscr{X}, \frac{F(m)}{\pi F(m)}\right)$ is surjective for all sufficiently large integers m. But for m sufficiently large, $F(m)/\pi F(m)$ is generated by its global sections [3, III. 2. 2. 2]. Therefore, by Nakayama's lemma and (1. 2), $F(m)$ is generated by its global sections.

If F is not torsion free, let T be the torsion submodule of F: i.e. T is the sheaf associated to the presheaf $U\rightsquigarrow$ (R-torsion elements of $\Gamma(U, F)$) for each open subset U of $\mathscr{X}$. T is a coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-modules, and for some integer N $\pi^N T=0$.

(*Proof.* The problem is local, so we may assume that $\mathscr{X}$ is affine; i.e. we may assume that $\mathscr{X}=Spf\ B$ for some wcfg algebra B, and thus (3. 3) $F=\tilde{M}$ for a finitely generated B-module M. The R-torsion submodule of M, $M'=\{m\in M:\pi^N m=0 \text{ for some is } N\}$, a B-submodule of M. Since B is noetherian and M is finitely generated, there exists an integer N such that $\pi^N M' = 0$. We claim also that $T = \tilde{M}'$, which will complete the proof of our assertion. It is equivalent to prove that the sheaf $Q = (M/M')^{\sim}$ associated to the quotient module M/M' is torsion free over R. Assuming that B is the weak completion of a polynomial ring, we have that for each principal open subset $U = \mathscr{X}_f$ of $\mathscr{X}$, $\Gamma(U, \mathcal{O}_{\mathscr{X}}) = B_{\{f\}}$ is torsion free over R. Therefore $\Gamma(U, Q) = M/M' \otimes_B B_{\{f\}}$ is torsion free over R, and so, for each point $x \in \mathscr{X}$, the stalk of Q at x, Q_x, is torsion free over R, (it is the direct limit of torsion free modules.) Therefore Q is torsion free over R.)

Let Q be the quotient sheaf F/T. Q is a coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-modules, and Q is torsion free over R. Thus, for all sufficiently large integers m, $H^1(\mathscr{X}, T(m)) = H^1(\mathscr{X}, Q(m)) = 0$. Therefore, for all sufficiently large integers m, $H(\mathscr{X}, F(m)) = 0$, proving (1).

The subsheaf πF of F is also a sheaf of coherent $\mathcal{O}_{\mathscr{X}}$-modules, so for m sufficiently large, $H^1(\mathscr{X}, \pi F(m)) = 0$. Also, for m sufficiently large, $(F/\pi F)$ is generated by its global sections.

Thus, for all sufficiently large integers m, the natural map $\Gamma(\mathscr{X}, F(m)) \to \Gamma(\mathscr{X}, (F/\pi F)(m))$ is surjective and $F/\pi F(m)$ is generated by its global sections. By Nakayama's lemma and (1.2), for all such m, $F(m)$ is generated by its global sections.

THEOREM 5. *The functor $F \to F^\dagger$ taking coherent modules on P^n_R into coherent modules on $P^{n\dagger}_R$ is an equivalence of categories.*

Proof. First we will show that this functor is fully faithful, and then we will prove that every $P^\dagger$-module is the weak completion of a P-module.

If F and G are P-modules, we must show that $\mathrm{Hom}(F, G) = \mathrm{Hom}(F^\dagger, G^\dagger)$. Since $\mathrm{Hom}(F, G) = \Gamma(P, \mathscr{H}om(F, G))$ and $\mathrm{Hom}(F^\dagger, G^\dagger) = \Gamma(P^\dagger, \mathscr{H}om(F^\dagger, G^\dagger))$, by (5.4) it suffices to prove that $\mathscr{H}om(F, G)^\dagger = \mathscr{H}om(F^\dagger, G^\dagger)$. To establish this last equality, we need only check the affine analogue. Let A be a finitely generated R-algebra, and M, N be two finite A-modules. We will prove that $\mathrm{Hom}_A(M, N)^\dagger = \mathrm{Hom}_{A^\dagger}(M^\dagger, N^\dagger)$. Let $F_1 \to F_0 \to M \to 0$ be a finite presentation of M by free A-modules. We have a commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_A(M, N)^\dagger & \to & \mathrm{Hom}_A(F_0, N)^\dagger & \to & \mathrm{Hom}_A(F_1, N)^\dagger \\ & \downarrow a & & \downarrow b & & \downarrow c \\ 0 \to & \mathrm{Hom}_A(M^\dagger, N^\dagger) & \to & \mathrm{Hom}_{A^\dagger}(F_0^\dagger, N^\dagger) & \to & \mathrm{Hom}_{A^\dagger}(F_1^\dagger, N^\dagger) \end{array}$$

The top row is exact because weak completion is an exact functor of finite A-modules (4.7). We will show that b and c are bijective. Then the snake lemma proves a is bijective. If F is a free A-module, then $\mathrm{Hom}_A(F, N)^\dagger = \mathrm{Hom}_A(F, N) \otimes_A A^\dagger = \mathrm{Hom}_{A^\dagger}(F^\dagger, N^\dagger)$, which shows b and c bijective.

It remains to prove only that every coherent module G over $P^\dagger$ is the weak completion of a coherent P-module. First we will show that there exist locally free $P^\dagger$-modules:

$$F_0 = \sum_{i=1}^{r} \mathcal{O}(n_i)^\dagger, \text{ and } F_1 = \sum_{j=1}^{s} \mathcal{O}(m_j)^\dagger$$

so that G may be finitely presented:

$$F_1 \xrightarrow{f} F_0 \to G \to 0.$$

Select an integer N' so large for $N \geq N'$, $G(N)$ is generated by its global section (Proposition 4). Thus G is the image of a locally free sheaf F_0 as required. Since $\mathcal{O}_{P^\dagger}$ is coherent over itself, F_0 is coherent, the kernel of the chosen projection $F_0 \to G$ is also coherent, and by the foregoing argument this kernel is the image of a locally free sheaf F_1, as required.

Let $E_0 = \sum_i \mathcal{O}(n_i)$ and $E_1 = \sum_j \mathcal{O}(m_j)$. That is, $F_i = E_i^\dagger$. Since the natural map $\mathrm{Hom}_P(E_1, E_0) \to \mathrm{Hom}_{P^\dagger}(F_1, F_0)$ is bijective, there is a homomorphism $e : E_1 \to E_0$ such that $e^\dagger = f$. Let $H = \mathrm{coker}\ (e)$. The natural map $E_0^\dagger \to F_0$ induces a bijection $H^\dagger \to G$.

COROLLARY 6. *Suppose F is a coherent sheaf $\mathcal{O}_{\mathscr{X}}$-modules, where $\mathscr{X} = P_R^{n\dagger}$. Then for all sufficiently large integers m,*

(a) $H^i(\mathscr{X}, F(m)) = 0,\ i > 0$

(b) $F(m)$ is generated by its global sections.

Proof Corollary 4, Theorem 5, and (5.4).

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