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RELATIONS BETWEEN NON-COMPACT TRANSFORMATION GROUPS AND COMPACT TRANSFORMATION GROUPS

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§ 1. Introduction.

In this paper certain relations between non-compact transformation groups and compact transformation groups are studied. The notion of reducibility and separability of transformation groups is introduced, several necessary and sufficient conditions are established: (1) A separable transformation group to be locally weakly almost periodic, (2) A reducible and separable transformation group to be a minimal set and (3) A reducible and separable transformation group to be a fibre bundle. As applications we show, among other things, that (1) for certain reducible transformation groups its fundamental group is not trivial which is a generalization of a result in [4]. (2) Given a transformation group (Y, N, Π), where Y is compact Hausdorff and N is discrete and a group covering $\tilde{p}: T \to H$, where H is a compact group, T is a connected group, and the kernel of $\tilde{p}$ is N, then there is a transformation group (X, T, σ), where X is again compact Hausdorff, which is an extension of (Y, N, Π). Furthermore, if (T, N, Π) is minimal, so is (X, N, σ), if (Y, N, Π) is universal minimal so is (X, N, σ). Conversely if (X, T, σ) is a universal minimal set, where X is compact Hausdorff and if $f: T \to H$ is a group covering where H is a compact group, then for every $x \in X$ $cl(\sigma(x, N))$ must be a universal minimal under N, and (3) by using the conception of Whitney sum of two minimal sets, we find that the Cartesian product of two minimal, but not totally minimal, continuous flows, will never be minimal, if they have a same integer subgroup satisfying the property (A).

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§ 2. Some lemmas on locally weakly almost periodic transformation groups.

We may find the following definitions and lemmas in [2] and [8].

DEFINITION 1. Let (X, T, Π) be a transformation group. Let $x \in X$. We say that T is *locally weakly almost periodic at* x if U is a neighborhood of x, then there exist a neighborhood V of x and a compact subset K of T such that $y \in V$ and $t \in T$ imply $ytK \cap U \neq \emptyset$. We say the group T is *locally weakly almost periodic on* X if T is locally weakly almost periodic at every $x \in X$. Let (X, T, Π) be a transformation group where X is a uniform space. We say that T is weakly almost periodic at X if α is an uniform index of X, then there exists a compact subset K of T such that for each $x \in X$ and $t \in T$ we have $xtK \cap x\alpha \neq \emptyset$.

The following lemma is a consequence of these definitions:

LEMMA 1. *If T is locally weakly almost periodic on X and X is compact then T is weakly almost periodic.*

DEFINITION 2. Let X be a topological space. Let $\mathscr{A}$ be a partition of X. Let $E \subset X$. We say $E\mathscr{A}$ is a *star of* E in X if $E\mathscr{A} = \cup \{A \mid A \in \mathscr{A}, A \cap E \neq \emptyset\}$. We say the partition $\mathscr{A}$ is *star-open* (*star-closed*) if the star of open (closed) subset of X is open (closed) in X. A partition of X is called *decomposition* if every member of this partition is compact.

LEMMA 2 *Let (X, T, Π) be a locally weakly almost periodic transformation group, where X is a locally compact Hausdorff space. Then the class of all orbit-closures under T is a star-open and star-closed decomposition of X.*

We may find the proof of this lemma in [8].

Let (X, T, Π) be a transformation group where X is a locally compact Hausdorff space. Let S be a closed, normal, syndetic subgroup of T. Let (X, S, Π) be the induced transformation group of (X, T, Π). Assume that (X, S, Π) is locally weakly almost periodic. Let R be the relation on X defined by the orbit-closures of S, namely xRy if and only if $x \in cl(yS)$. It is clear that $x \in cl(yS)$ if and only if $y \in cl(xS)$ and R is an equivalence relation.

We may find the proof of the following lemma in [2].

LEMMA 3. *The relation R defined as an open and closed relation and the quotient space $X^* = X/R$, induced by the relation R, is Hausdorff and locally compact. If X is normal so is X^*.*

Let $T^* = T/S$ be the quotient group of T. Then T^* is a compact group. Denote $q : T \to T^*$ to be the natural projection. Define $\Pi^* \times T^* \to X^*$ by $\Pi^*(x^*, t^*) = (\Pi(x, t))^*$, where $x \in X$, $t \in T$, $x^* \in X^*$, $t^* \in X^*$, $p(x) = x^*$, $q(t) = t^*$ and p is the natural map of X onto X^*. It is well-defined and we have the following lemma.

LEMMA 4. *The following diagram:*

$$\begin{array}{ccccc} X & \times & T & \xrightarrow{\Pi} & X \\ p\downarrow & & q\downarrow & \Pi^* & \downarrow p \\ X^* & \times & T^* & \longrightarrow & X^* \end{array}$$

is commutative and the triple (X^*, T^*, Π^*) is a transformation group induced by (X, T, Π).

We may find the proof of this lemma in [2].

LEMMA 5. *Let (X, T, Π) be a transformation group, where X is a compact, Hausdorff, minimal set under T. Let S be a closed, syndetic, normal subgroup of T. Then X is locally weakly almost periodic undar S.*

LEMMA 6. *Let X be a locally compact T_2-space. Then X is locally weakly almost periodic under T if and only if the class of all orbit-closures under T is a star-closed decomposition of X.*

The above two lemmas are known (e.g., sec. [8]).

§ 3. Semi-reducible, reducible and separable transformation groups.

DEFINITION 3. Let (X, T, Π) be a transformation group. We say that (X, T, Π) is *semi-reducible* if there is a transformation group (X^*, H, Π^*), where X^* is a non-trivial compact Hausdorff space and H is a non-trivial compact group such that there is a continuous homomorphism $f : T \to H$ from T onto H and a continuous map $p : X \to X^*$ from X into X^* and for each $t \in T$ and each $x \in X$, we have $p\Pi(x, t) = \Pi^*(p(x), f(t))$. We say that (X, T, Π) is *reducible* if it is semi-reducible such that $X^* = H$ and Π^* is the multiplication in H.

Sometime we denote a semi-reducible transformation group by $(X, T, \Pi; X^*, H, \Pi^*, p, f)$ and a reducible transformation group by $(X, T, \Pi; H, p, f)$.

LEMMA 7. *Let $(X, T, \Pi; H, \Pi^*, p, f)$ be a semi-reducible transformation group. Let $Kev(f) = N$. Then N is a closed, normal, syndetic subgroup of T and for each $x \in X$, $p(cl(\Pi(x, N))) = p(x)$.*

Proof. Let $t \in N$. Then $f(t) = e, p(\Pi(x, t)) = \Pi^*(p(x), f(t)) = p(x)$. Since p is continuous and X^* is Hausdorff, we have $p(cl(\Pi(x, N))) = p(x)$.

DEFINITION 4. A semi-reducible transformation group $(X, T, \Pi; X^*, H, \Pi^*, p, f)$ is called *separable* if for any pair x, y in X such that $x \in cl(\Pi(y, N))$ then $p(x) \neq p(y)$.

LEMMA 8. *Let $(X, T, \Pi; X^*, H, \Pi^*, p, f)$ be a separable transformation group. Then for each $x \in X$, $cl(\Pi(x, N))$ is a minimal set under N in X.*

Proof. Let $x \in X$. Suppose that $cl(\Pi(x, N))$ is not a minimal set. There exists $y \in cl(\Pi(x, N))$ such that $cl(\Pi(y, N)) \subset cl(\Pi(x, N))$ but they are not co-incided. There is an element $z \in cl(\Pi(x, N))$ but not in $cl(\Pi(y, N))$. By the definition of separability, we have $p(z) \neq p(y)$. By lemma 7, we have $p(y) = p(x)$ and $p(z) = p(x)$. A contradiction!

THEOREM 1. *Let (X, T, Π) be a transformation group where X is a locally compact Hausdorff space. Let N be a closed normal syndetic subgroup of T. If the induced transformation group (X, N, Π) is locally weakly almost periodic, then (X, T, Π) is a separable transformation group and the map $p : X \to X^*$ is closed and $p^{-1}(x^*)$ for every x^* in X^* is compact. Conversely, if $(X, T, \Pi; X^*, H, \Pi^*, p, f)$ is a separable transformation group and the map P is closed, and $P^{-1}(x^*)$ for every x^* in X^* is compact, then the induced transformation group (X, N, Π) is locally weakly almost periodic, where $N = f^{-1}(e)$.*

Proof. The first part of this theorem is a direct consequence of Lemma 2, Lemma 3, and Lemma 4. The second part is a direct consequence of the definition of separability, Lemma 6 and Lemma 8.

Remark. Notice that the map p in Theorem 1 is proper.

§ 4. Minimal Sets.

DEFINITION 5. Let (X, T, Π) be a transformation group. Let N be a closed, normal syndetic subgroup of T. We say N has a *property* (A) if every $x \in X$, the group $\{t \in T \mid \Pi(cl(xN), t) = cl(xN)\}$ is equal to N.

LEMMA 9. *Let* $(X, T, \Pi; H, p, f)$ *be a reducible and separable transformation group. Then* X *is minimal, but not totally minimal, under* T, *and* $N = f^{-1}(e)$ *has the property* (A).

Proof. Let $x \in X$. We shall show that $cl(\Pi(x, T)) = X$. For each $y \in X$, we have $p(y) \in H$, which is a group. Since the homomorphism f is onto, there exists $t \in T$ such that $p(x)f(t) = p(y)$. By the reducibility of this transformation group we have $p(x)f(t) = p(\Pi(x, t))$. Consequently, $p(y) = p(\Pi(x, t)) = p(\Pi(xt, e))$. By the separability and Lemma 8, we have $cl(\Pi(y, N)) = cl(\Pi(x, tN)) \subset cl(\Pi(x, T))$. Hence $y \in cl(\Pi(x, T))$. This shows that X is a minimal set under T. Since H is not trivial, there are at least two distinct points, x and y, in X such that $p(x) \neq p(y)$. It follows that $cl(\Pi(x, N)) \cap cl(\Pi(y, N)) = \emptyset$ and X is not minimal under N. From Lemma 7, we know that N is a closed, normal, syndetic subgroup of T. Hence X is not totally minimal. For $x \in X$ let $S = \{t \in T \mid \Pi(cl(xN)), t) = cl(xN)\}$, for $t \in S$, we have $p(x) = p(cl(xN)) = p(\Pi(cl(xN), t)) = p(cl(xN)) \cdot f(t) = p(x)f(t)$ in H. Hence $f(t) = e$, $t \in N$ and $S = N$. This shows that N has the property (A).

LEMMA 10. *Let* (X, T, Π) *be a transformation group. Let* X *be a locally compact, Hausdorff, minimal set under* T. *Let* N *be a closed syndetic, proper normal subgroup of* T *with the property* (A) *such that for each* $x \in X$, xN *is compact. If* X *is not minimal under* N, *then* (X, T, Π) *is reducible and separable.*

Proof. By Lemma 5, we know that X is locally weakly almost periodic under N. Let R be the relation on X defined by the orbit-closures of N. Let $X^* = X/R$ be the quotient space of X, $T^* = T/N$ be the quotient group of T and $\Pi^* : X^* \times T^* \to X^*$ be defined by $\Pi^*(x^*, t^*) = (\Pi(x, t))^*$. Let p be the quotient map of X onto X^* and f the quotient homomorphism from T onto T^*. By Theorem 1, we know that $(X, T, \Pi; X^*, T^*, p, f)$ is separable transformation group.

Let $p(x_0) = x_0^*$ for $x_0 \in X$. We shall show that the isotopy subgroup $G_{x_0^*}$ of x_0^* is the identity in T^*. Suppose $\Pi^*(x_0^*, t^*) = x_0^*$ for $t^* \in T^*$. Choose $t \in T$, so that $f(t) = t^*$. We have $p(\Pi(cl(x_0N), Nt)) = p(cl(x_0N))$ and by the separability and Lemma 8 $\Pi(cl(x_0N), Nt) = \Pi(cl(x_0N), t) = cl(x_0N)$. Since N

has the property (A), we have $t \in N$ or $t^* =$ identity in T^* and $G_{x_0^*}$ is trivial for all x_0^* in X^*.

Let $x_0 \in X$ and $p(x_0) = x_0^* \in X^*$. Consider the map $\Pi^*_{x_0^*} : T^* \to X^*$, by $\Pi^*_{x_0^*}(t^*) = \Pi^*(x_0^*, t^*)$, which is continuous, one-to-one, and onto. Since both T^* and X^* are compact and Hausdorff the map $\Pi^*_{x^*}$ is a homeomorphism. Identify the topological group T^* with X^* by $\Pi^*_{x_0^*}$. Then X^* is a topological group, x_0^* is its identity and Π^* is the multiplication of the group. Consider the following diagram:

$$\begin{array}{ccccc} X & \times & T & \xrightarrow{\Pi} & X \\ h\big\downarrow & & f\big\downarrow & m & \big\downarrow h \\ T^* & \times & T^* & \longrightarrow & T^* \end{array}$$

where $h = (\Pi^*_{x_0})^{-1}p : X \xrightarrow{p} X^* \xrightarrow{(\Pi^*_{x_0})^{-1}} T^*$ and m is the multiplication in T^*. Let $x \in X$, $t \in T$. Let $h(x) = s^*$ and $f(t) = t^*$. Then $\Pi^*_{x_0^*}(s^*) = p(x)$. We show that $h(\Pi(x, t)) = h(x) \cdot f(t)$. We know that from the preceding commutative diagram, we have $\Pi^*(p(x), f(t)) = p(\Pi(x, t))$ and $h\Pi(x, t) = (\Pi^*_{x_0^*})^{-1}p\Pi(x, t) = (\Pi^*_{x_0^*})^{-1}\Pi^*(p(x), f(t)) =$

$$= (\Pi^*_{x_0^*})^{-1}\Pi^*(\Pi_{x_0^*}(s^*), t^*) = (\Pi^*_{x_0^*})^{-1}\Pi^*(\Pi^*(x^*, s^*), t^*) =$$

$$= (\Pi^*_{x_0^*})^{-1}\Pi^*(x_0^*, s^*t^*) = s^*t^* = h(x)f(t).$$

This shows that the transformation group (X, T, Π) is reducible. The lemma is proved.

LEMMA 11. *Let (X, T, Π) be a transformation group where T is an abelian topological group. Let X be a compact, Hausdorff, minimal, but not totally minimal set, under T. Then T has a proper, closed, syndetic subgroup with the property (A).*

Proof. Since X is not totally minimal set, there is a proper, closed, syndetic subgroup N of T such that $cl(\Pi(x, N)) \neq X$ for every $x \in X$. If N does not have the property (A) for some $x_0 \in X$, let $N' = \{t \in T \mid \Pi(cl(x_0N), t) = cl(x_0N)\}$. It is clear that N' is a proper, closed, syndetic subgroup of T, $N' \supset N$ and $cl(x_0N) = cl(x_0N')$. Let R be the relation on X defined by the orbit-closures of N'. Let $X^* = X/R$ be the quotient space of X, $T^* = T/N'$ be the quotient group of T and $\Pi^* : X^* \times T^* \to X^*$ be defined by $\Pi^*(x^*, t^*) = (\Pi(x, t))^*$. Let p be the quotient map from X onto X^* and q the quotient homomorphism from T onto T^*. By Lemma 5 and Theorem 1, we have $(X, T, \Pi; X^*, T^*, \Pi^*; p, f)$ is a separable transformation group. It is clear that $\{t \in T \mid \Pi(cl(x_0N'), t) = cl(x_0N')\} = N'$ and $p(N')$ is the identity in T^*. For each $y \in X$, we have $p(y) = y^* \in X^*$. Since X is minimal under T, we have the map $\Pi^*_{x_0^*}: T^* \to X^*$ by $\Pi^*_{x_0^*}(t^*) = \Pi^*(x_0^*, t^*)$ is onto, there is $t_0^* \in T^*$ such that $\Pi^*(x_0^*, t_0^*) = y^*$. Suppose $\Pi(cl(yN'), s) = cl(yN')$ for $s \in T$, then $\Pi^*(y^*, s^*) = y^*$, where $s^* = f(s)$ and $\Pi^*(x_0, t_0(s^*)(t_0)^{-1}) = x_0$. It follows that $\Pi(cl(x_0N'), t_0st_0^{-1}) = cl(x_0N')$, where $t_0 \in f^{-1}(t_0^*)$. Hence $t_0st_0^{-1} = e$ in T or $s = e$. The lemma is proved.

THEOREM 2. *Let (X, T, Π) be a transformation group where X is compact Hausdorff. Then X is minimal under T but not minimal under a proper, closed, normal subgroup having the property (A) if and only if (X, T, Π) is reducible and separable.*

Proof. It is a direct consequence of Lemma 9 and Lemma 10.

COROLLARY. *Let (X, T, Π) be a transformation group where X is compact Hausdorff and T is abelian. Then X is minimal but not totally minimal if and only if (X, T, Π) is reducible and separable.*

Proof. By Lemma 11 and Theorem 2.

§ 5. Fundamental Groups.

The following lemma is well-known in Homotopy Theory (e.g. see p. 86, Lemma 15.1, [9]).

LEMMA 12. *Let $f: T \to H$ be a fiber space with discrete fiber. For each path $\sigma: I \to H$ joining b_0 to b_1 and for each $e_0 \in f^{-1}(b_0)$ there exists one and only one covering path $\sigma^*: I \to T$ such that $\sigma^*(0) = e_0$ and $f\sigma^* = \sigma$.*

THEOREM 3. *Let (X, T, Π) be a reducible transformation group, where T is a locally compact, but not compact, connected, locally pathwise connected, X is a locally compact, Hausdorff, connected, locally pathwise connected space and f is a group covering. If every orbit-closure under T in X is compact, then for each $x_0 \in X$, the fundamental group $\pi_1(X, x_0) \neq 0$.*

Proof. Since (X, T, Π) is reducible, there exists a compact group H and a group covering $f: T \to H$ and a continuous map $p: X \to H$ such that the following diagram is commutative:

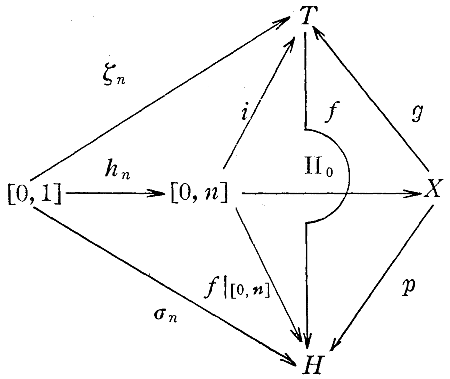
$$\begin{array}{ccccc} X & \times & T & \xrightarrow{\Pi} & X \\ {\scriptstyle p}\downarrow & & {\scriptstyle f}\downarrow & m & \downarrow{\scriptstyle p} \\ H & \times & H & \longrightarrow & H \end{array}$$

where m is the multiplication in H. Let $y_0 \in X$, such that $p(y_0) = e$, where e is the identity of H. Define $\Pi_0 : T \to X$ by $\Pi_0(t) = \Pi(y_0, t)$ for $t \in T$. Then Π_0 is continuous and the following diagram is commutative:

$$\begin{array}{ccc} T & \xrightarrow{\Pi_0} & H \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & H & \end{array}$$

Suppose that $\pi_1(X, y_0) = 0$. We have $p_*(\pi_1(X, y_0)) = 0$ which is obviously contained in $f_*(\pi_1(T, e))$, where e is the identity of T. By the lifting theorem of Homotopy Theory (e.g. see p. 89, [9]), there exists a unique continuous map $g : X \to T$ such that $g(y_0) = e$ and $fg = p$. We shall show that it is impossible.

Since T is a locally compact, connected, but not compact, topological group, by a result of Iwasawa, there is a closed one-parameter subgroup R of T such that it is topologically isomorphic onto the usual additive group of real line (see [10]). Restrict Π_0 to R. For the matter of convenience we denote the multiplication in R by the usual addition. Let $I = [0, 1]$ be the unit interval. Define $h_n : [0,1] \longrightarrow \begin{cases} [0, n], & \text{if } n \text{ is positive} \\ [n, 0], & \text{if } n \text{ is negative} \end{cases}$ by $h_n(t) = nt$, where $[0, n]$ are subsets in R. It is clear that h_n is a homeomorphic onto map. The path $\xi_n = \Pi_0 \circ i \circ h_n : I \to X$ has the property that $\xi_n(t) = \Pi_0(nt)$ where $i = [0, n] \to T$ is the inclusion map (or $i : [n, 0] \to T$ if n is negative). In particular, $\xi_n(0) = y_0$ and $\xi_n(1) = \Pi_0(n) = y_n$ for some y_n in X. The composed map $\sigma_n = p\varepsilon_n : I \to H$ is a path. Then $\sigma_n(0) = e$ in H, where e is the identity in H. By the Lemma 12, there is a unique path $\zeta_n : [0,1] \to T$ such that $\zeta_n(0) = e$, where e is the identity in T and $f\zeta_n = \sigma_n$. Consider $i \circ h_n$, where the map i is defined as above. We have $(i \circ h)(0) = i(0) = e$, and $\sigma_n = p\xi_n = p \circ \Pi_0 \circ i \circ h_n = f \circ i \circ h_n = f \circ (i \circ h_n)$. Hence $f \circ (i \circ h_n) = \sigma_n$. By the uniqueness of ζ_n, (see Lemma 12), we have $\zeta_n = i \circ h_n$.



By the usual construction of $g: X \to T$ (e.g., see p. 90, [9]) we have $g(y_n) = \zeta_n(1) = (i \circ h_n)(1) = n \in R \subset T$, for all integers n and $h(y_0) = 0$, we have $g\Pi_0(n) = g(y_n) = n$ for all n in R. Let $Z = \{0, \pm 1, \pm 2, \cdots\} \subset R \subset T$. Then Z is a discrete subgroup of T. By assumption, we have $cl(\Pi_0(T)) = cl(\Pi(y_0, T))$ is compact, so is its closed subset $cl(\Pi_0(Z))$. Consequently $g(cl(\Pi_0(Z))$ is compact in T. However, we know $g(\Pi_0(Z)) = Z$ and Z is closed in T, it follows that $Z = cl(Z) = cl(g(\Pi_0(Z)))$ is compact, a contradiction! Thus, $\Pi_1(X, y_0) \neq 0$. Since a connected, locally arcwise connected space is arcwise connected, we have $\pi_1(X, x) = \pi_1(X, y_0)$ for every $x \in X$. The theorem is proved.

COROLLARY 1. *Let (X, T, Π) be a reducible transformation group, where T is a locally compact, but not compact, connected, locally pathwise connected, X is a compact, Hausdorff, connected, locally pathwise connected space and f is a group covering. Then for each $x_0 \in X$, the fundamental group* $\Pi_1(X, x_0) \neq 0$.

THEOREM 4. *Let (X, T, Π) be a transformation group where T is a locally compact, but not compact, connected, locally pathwise connected group and X is a compact Hausdorff minimal set under T. Let N be a proper, discrete, normal syndetic subgroup of T having the property (A). Then $\Pi_1(X, x_0) \neq 0$, where $x_0 \in X$.*

Proof. Notice that for each $x \in X$, $\overline{xT} \neq \overline{xN}$, otherwise the subgroup N can not have the property (A). Hence X is not minimal under N.

From Theorem 2 we know that the transformation group (X, T, Π) is separable and reducible. Since N is discrete and normal, the quotient homomorphism, $f: T \to T^* = T/N$, is a group covering. By Theorem 3, we know that $\Pi_1(X, x_0) \neq 0$ for each $x_0 \in X$.

Corollary 1. Let (X, R, Π) be a transformation group where X is a compact Hausdorff locally arcwise connected, minimal, but not totally minimal set, under the real group R. Then $\Pi_1(X, x_0) \neq 0$, for each $x_0 \in X$.

Proof. Since X is not totally minimal under R, by Lemma 11 there is a proper closed syndetic subgroup Z having the property (A). As we know that every proper closed subgroup of R is discrete and isomorphic to an integer group. Now the corollary is a consequence of Theorem 4.

§ 6. Fibre Bundles

By a fibre bundle we mean an equivalence class of coordinate bundles (see [14]).

Lemma 13. *Let $(X, T, \Pi; H, p, f)$ be reducible and separable transformation group where X is compact. Hausdorff and $f: T \to H$ is a group covering. Then (X, H, p) is a fibre bundle.*

Proof. Since $f: T \to H$ is a group covering, $f^{-1}(e) = N$ is a discrete, normal syndetic, proper subgroup of T. By Lemma 9, we know that X is minimal under T but not minimal under N. From the reducibility we have the following commutative diagram:

$$\begin{array}{ccccc} X & \times & T & \xrightarrow{\Pi} & X \\ p\downarrow & & f\downarrow & m & \downarrow p \\ H & \times & H & \longrightarrow & H \end{array}$$

where m denotes the multiplication of H. Let $x \in X$ such $p(x) = e$ in H. Let $Y = cl(xN)$, we have $p(Y) = e$. For each $h \in H$, there exists a compact neighborhood V of h and a compact neighborhood $\tilde{V}$ of r in T, where $f(r) = h$ such that the restriction of f to $\tilde{V}$, $f_V: \tilde{V} \to V$ is a topological isomorphism onto. The set $\tilde{V}$ is so chosen that any pair of elements in $\{n\tilde{V} \mid n \in N\}$ are disjoint. Define

$$\phi_V : Y \times V \to p^{-1}(V) \text{ by } \phi_V(y, z) = \Pi(y, f_V^{-1}(z)) = \Pi(y, t),$$

where $t \in \tilde{V}$ with $f_V(t) = z$. We shall show ϕ_V is a homeomorphism from $Y \times V$ onto $p^{-1}(V)$. It is obvious that ϕ_V is continuous. We first show that ϕ_V is one-to-one. Let z_1, z_2 in V and y_1, y_2 in Y. If $\phi_V(y_1, z_1) = \phi_V(y_2, z_2)$, then $\Pi(y_1, f_V^{-1}(z_1)) = \Pi(y_2, f_V^{-1}(z_2))$. Since $p(y_1) = p(y_2) = e$, by the commutative diagram above we have $ez_1 = ez_2$ or $z_1 = z_2$. It follows easily that $y_1 = y_2$. We shall show that ϕ_V is onto. Let $x_0 \in p^{-1}(V)$. Then $p(x_0) = h_0 \in V$. Since this transformation group is separable we have $p^{-1}(h_0) = cl(x_0N)$ and $Yf_V^{-1}(h_0) \subset p^{-1}(h_0) = cl(x_0N)$. Let $[f_V^{-1}(h_0)]^{-1} = g$, we have $f(g) = h_0^{-1}$. By the above commutative diagram we have $p(cl(x_0N)g) = p(cl(x_0N))f(g) = h_0h_0^{-1} = e$. It follows that $cl(x_0N)g \subset p^{-1}(e) = Y$ or $cl(x_0N) \subset Y(f_V^{-1}(h_0))$. Hence $Y(f_V^{-1}(h_0)) = cl(x_0N)$ and there exist $y_0 \in Y$ such that $\phi_V(y_0, h_0) = \Pi(y_0, f_V^{-1}(h_0)) = x_0$. This shows that ϕ_V is onto. Hence ϕ_V is a homeomorphism from $Y \times V$ onto $p^{-1}(V)$. Let V_0 be the interior of B. By the preceding argument we have $\phi_{V|V_0} = \phi_{V_0}$ maps $Y \times V_0$ onto $p^{-1}(V_0)$, where $p^{-1}(V_0)$ is open in X as well as in $p^{-1}(V)$ and $Y \times V_0$ is open in $Y \times V$. Hence $\phi_{V_0} : Y \times V_0 \to p^{-1}(V_0)$ is a homeomorphism from $Y \times V_0$ onto $p^{-1}(V_0)$. Let S be a family of those open sets, V_0, which covers H. Let the elements in S be indexed by a set J, i.e., $S = \{V_i \mid i \in J\}$ and denote the corresponding neighborhoods in T by $\tilde{V}_i$ and the corresponding homeomorphisms by f_i and ϕ_i respectively. It is obvious that $\{V_i\}$ are coordinate neighborhoods in H and $\{\phi_i\}$ are their corresponding coordinate functions.

For $h \in V_j$ and $y \in Y$, we have

$$(1) \qquad p\phi_j(y, h) = p(\Pi(y, f_j^{-1}(h)) = p(y) \cdot f(f_j^{-1}(h)) = e \cdot h = h.$$

By the separability, we have $p^{-1}(h) = \Pi(Y, t) = Y \cdot t$, where $t \in f^{-1}(h)$. Define the map $\phi_{j,h} : Y \to p^{-1}(h)$, by $\phi_{j,h}(y) = \phi_j(y, h)$ for $y \in Y$. It is easy to see that $\phi_{j,h}$ is continuous, one-to-one, onto and because that both Y and $p^{-1}(h)$ are compact, it is a homeomorphism. For each pair i, j in J and each $h \in V_i \cap V_j$, we have $\phi_{i,h}(y) = \Pi(y, f_i^{-1}(h)) = yf_i^{-1}(h)$, $\phi_{j,h}(y) = \Pi(y, f_j^{-1}(h)) = yf_j^{-1}(h)$ where $f_i^{-1}(h) \in \tilde{V}_i$ and $f_j^{-1}(h) \in \tilde{V}_j$. If $V_i \cap V_j \neq \emptyset$. It is easy to see that there exists an $n \in N$ such that $\tilde{V}_i \cap n\tilde{V}_j \neq \emptyset$. We shall show that there is only one such $n \in N$ that has this property. Suppose there is an $m \in N$ wish the same property: $\tilde{V}_i \cap m\tilde{V}_j \neq \emptyset$. Choose $h \in V_i \cap V_j$, then there is $t = nt_1 \in \tilde{V}_i \cap n\tilde{V}_j$ and $t^1 = mt_2 \in \tilde{V}_i \cap m\tilde{V}_j$ such that $f(t) = h$ and $f(t^1) = h$, where t, t^1 in $\tilde{V}_i$ and t_1, t_2 in $\tilde{V}_j$. Since $f_{|\tilde{V}_i} = f_i$ is a homeomorphism from $\tilde{V}_i$ onto V_i, we have $t = t^1$. From $f(m, t_1) = f(t_1) = h$ and $f(nt_2) = f(t_2) = h$, we have $f(t_1) = f(t_2)$. Since $f_{|\tilde{V}_j} = f_j$ is also a homeomorphism, we have $t_1 = t_2$. It follows that $nt_1 = mt_1$ and we have $n = m$. Hence for each pair i and j if $V_i \cap V_j \neq \emptyset$ there is a unique $n \in N$ such that $\tilde{V}_i \cap n\tilde{V}_j \neq \emptyset$. Let $h \in V_i \cap V_j$, choose $t_i \in \tilde{V}_i$ and $t_h \in \tilde{V}_j$ so that

$t = t_i = nt_j$ and $f(t) = h$, where $n \in N$. It follows that $f_i^{-1}(h) = t_i$ and $f_j^{-1}(h) = t_j$. Consider $\phi_{j,h}^{-1} \circ \phi_{i,h} : Y \to Y$, we have

$$\phi_{j,h}^{-1}\phi_{i,h}(y) = \phi_{j,h}^{-1}(\Pi(y, f_i^{-1}(h)) = \phi_{j,h}^{-1}(\Pi(y, t_i)) = \phi_{j,h}^{-1}(\Pi(y, nt_j))$$
$$= \phi_{j,h}^{-1}(\Pi(y \cdot n, t_j) = \phi_{j,h}^{-1}(\Pi(y \cdot n, f_j^{-1}(h)) = y \cdot n = \Pi(y, n)$$

or $\phi_{j,h}^{-1}\phi_{i,h} = n \in N$, on Y. Thus we have

(2) For each pair i and j in J and $h \in V_i \cap V_j$, we have that $\phi_{j,h}^{-1}\phi_{i,h}$ is a homeomorphism from Y onto Y and $\phi_{j,h}^{-1}\phi_{i,h} \in N$ and

(3) Define $g_{ji}(h) = \phi_{j,h}^{-1}\phi_{i,h}$, the map $g_{ji} : V_i \cap V_j \to N$ is trivial, i.e. $g_{ji}(h) = n$ for every $h \in V_i \cap V_j$. Consequently g_{ji} is continuous. Hence it is a coordinate bundle with X as the bundle space, H as the base space, P as the projection from X onto H, Y as the fibre, N as the group of bundle, the open covering on X X, $\{V_i\}$, as coordinate neighborhoods and $\{\phi_i\}$ as coordinate functions.

LEMMA 14. *Let $(X, T, \Pi; H, p, f)$ be a reducible and separable transformation group as we stated in Lemma 13. Then $\{T, f, H, N, N\}$ is the associated principal bundle of the fibre bundle $\{X, p, H, Y, N\}$ as we constructed in Lemma 13.*

Proof. It is a direct consequence of the definition (see [14]) of associated principal bundle and the way we constructed the coordinate neighborhoods $\{V_i \mid i \in J\}$ and coordinated functions $\{\phi_i \mid i \in J\}$ and coordinate transformation $\{g_{ji} \mid j, i \in J\}$ in the proof of Lemma 13.

LEMMA 15. *Let $\{X, p, H, Y, N, \{V_i\}, \{\phi_i\}\}$ be a fibre bundle, where X is a compact Hausdorff space, H is a compact group and N is a discrete group. Let $Y \times N \to N$ be a transformation group. Let $\tilde{p} : T \to H$ be a group covering from T onto H such that $\{T, \tilde{p}, H, N, N\}$ is the principal associated bundle of the given fibre bundle. Then there exists an action σ of the group T on X so that $(X, T, \sigma; H, p, \tilde{p})$ is a reducible transformation group. There is a closed subset Y^1, invariant under N, in X, such that (Y^1, N), in (X, T, σ), is equivalent to the transformation group (Y, N). If furthermore, N is a minimal set under Y, then $(X, T, \sigma; H, p, \tilde{p})$ is separable.*

Proof. Let $\mathscr{B} = \{X, p, H, Y, N\}$. Let $\tilde{\mathscr{B}} = \{T, \tilde{p}, H, N, N\}$. Let $Y \times \mathscr{B} = \{Y \times T, q, T, Y, N\}$, $q(y, \tilde{b}) = \tilde{b}$ where $\tilde{b} \in T$ and $y \in Y$, be the product bundle with group N. Define the principal map:

$$P: Y \times \tilde{\mathscr{B}} \to \mathscr{B}$$

by $P(y, \tilde{b}) = \phi_i(y \cdot \tilde{p}_i(\tilde{b}), x)$, where $x = \tilde{p}(\tilde{b}) \in V_i$ in H and $\tilde{p}_i : \tilde{p}^{-1}(V_i) \to N$ by $\tilde{p}_i(t) = \tilde{\phi}_{i,h}^{-1}(t) \in N$ with $t \in \tilde{p}^{-1}(V_i)$ and $\tilde{p}(t) = h \in H$ (see p. 8 [14]). Then P is well defined and continuous and the following diagram:

$$\begin{array}{ccc} Y \times T & \xrightarrow{p} & X \\ q \downarrow & \tilde{p} & \downarrow p \\ T & \longrightarrow & H \end{array}$$

is commutative. In fact P is a bundle mapping (see p. 38 [14]). From the facts that $\tilde{p}$ is open and ϕ_i is a homeomorphism, it follows that P is open. We shall show that P is onto.

For each $x \in X$, there exist $i \in J$ such that $x \in p^{-1}(V_i)$ and $\phi_i : V_i \times Y \to p^{-1}(V_i)$ is a homeomorphism and onto. Consequently there exist $h_i \in V_i$ and $y \in Y$ so that $\phi_i(y, h_i) = x$. Let $t_i \in T$, with $\tilde{p}(t_i) = h_i$. We have $P(y \cdot (\tilde{p}_i(t))^{-1}, t_i) = \phi_i(y \tilde{p}_i(t_i)(\tilde{p}_i(t_i))^{-1}, \tilde{p}(t_i)) = \phi_i(y, h_i) = x$. Hence P is an onto mapping.

Let $P(y_1, \tilde{b}_1) = P(y_2, \tilde{b}_2)$, for $\tilde{b}_1$ and $\tilde{b}_2$ in T and y_1 and y_2 in Y. We shall show that $\tilde{b}_2 = n\tilde{b}_1$ for some $n \in N$ and $y_2 = y_1 n^{-1}$. From $P(y_1, \tilde{b}_1) = P(y_2, \tilde{b}_2)$, we have $\phi_i(y_1 \cdot \tilde{p}_i(\tilde{b}_1), x_1) = \phi_j(y_2 \cdot \tilde{p}_j(\tilde{b}_2), x_2)$ where $x_1 = \tilde{p}(\tilde{b}_1) \in V_i$ and $x_2 = \tilde{p}(\tilde{b}_2) \in V_j$. Since $\phi_i(y_1 \cdot \tilde{p}_i(\tilde{b}_1), x_1) \in p^{-1}(x_1)$ and $\phi_j(y_2 \cdot \tilde{p}_j(\tilde{b}_2), x_2) \in p^{-1}(x_2)$, we have $p^{-1}(x_1) = p^{-1}(x_2)$ in X. It follows that they are the same fibre in X and we have $x_1 = x_2$. From $\tilde{p}(\tilde{b}_1) = \tilde{p}(\tilde{b}_2)$. We conclude that $\tilde{b}_1$ and $\tilde{b}_2$ are in the same coset in T. Consequently, we have $\tilde{b}_2 = n\tilde{b}_1$ for some $n \in N$. Now, we may choose $i = j$. $\phi_i(y_1 \cdot \tilde{p}_i(\tilde{b}_1), x_1) = \phi_i(y_2 \cdot \tilde{p}_i(n\tilde{b}_1), x_1)$, we have $y_1 \cdot \tilde{p}_i(\tilde{b}_1) = y_2 \cdot \tilde{p}_i(n\tilde{b}_1)$ or $y_1 \cdot \phi_{i,x_1}^{-1}(\tilde{b}_1) = y_2 \cdot \tilde{\phi}_{i,x_1}^{-1}(n\tilde{b}_1)$. The mapping $\tilde{\phi}_{i,x} : N \to \tilde{p}^{-1}(x)$ in the covering group T is corresponding to the multiplication of N by an element in T such that $\tilde{\phi}_{i,x_1}(m) = mt$ for every m in N. $y_1 \cdot \tilde{\phi}_{i,x_1}^{-1}(\tilde{b}_1) = y_1 \cdot (\tilde{b}_1)t^{-1}$ and $y_2 \cdot \tilde{\phi}_{i,x}^{-1}(n\tilde{b}_1) = y_2 \cdot (n\tilde{b}_1)t^{-1}$. From $y_2 \cdot (n\tilde{b}_1)t^{-1} = y_1 \cdot (\tilde{b}_1)t^{-1}$, we have $y_2 = y_1 n^{-1}$. Consequently, we also have $P(y, m\tilde{b}) = P(ym, \tilde{b})$ for $\tilde{b} \in T$, $m \in N$ and $y \in Y$.

For every $g \in T$, we define that g acts on X by $(P(y, \tilde{b}))g = P(y, \tilde{b}g)$ for every $P(y, \tilde{b}) \in X$. Notice that $P(Y \times T) = X$. We shall show that it is well defined. Let $P(y_1, \tilde{b}_1) = P(y_2, \tilde{b}_2)$. Then $\tilde{b}_2 = n\tilde{b}_1$ and $y_2 = y_1 n^{-1}$ for some $n \in N$. $(P(y_2, \tilde{b}_2))g = (P(y_1 n^{-1}, n\tilde{b}_1))g = P(y_1 n^{-1}, n\tilde{b}_1 g) = \phi_i(y_1 n^{-1} \cdot \tilde{p}_i(n\tilde{b}_1 g), \tilde{p}(n\tilde{b}_1 g))$, where $\tilde{p}(n\tilde{b}_1 g) = \tilde{p}(\tilde{b}_1 g) \in V_i$, for some i. In the covering group T, there exist some $t\in T$ such that $\tilde{p}_i(z)=\tilde{\phi}_{i,x}^{-1}(z)=(z)t\in N$ for every $z\in\tilde{p}^{-1}(x)$, we have $y_1n^{-1}\tilde{p}_i(n\tilde{b}_1g)=y_1n^{-1}n\tilde{b}_1gt=y_1\tilde{b}_1gt$ where $n\tilde{b}_1gt\in N$ or $\tilde{b}_1gt\in N$ and $(P(y_2,\tilde{b}_2))g$ $=\phi_i(y_1(\tilde{b}_1g)t,\ \tilde{p}(\tilde{b}_1g))=\phi_i(y_1\cdot\tilde{p}_i(\tilde{b}_1g),\ \tilde{p}(\tilde{b}_1g))=P(y_1,\tilde{b}_1g)=(P(y_1,\tilde{b}_1))g$. Since P is open and continuous, the action of g on X is continuous. For g_1, g_2 in T, it is easy to see that $((P(y,\tilde{b}))g_1)g_2=P(y,\tilde{b})g_1g_2$. Consequently, for each $g\in T$, g is a homeomorphism.

Define $\sigma: X\times T\rightarrow X$ by $\sigma(P(y,\tilde{b}),g)=(P(y,\tilde{b}))g$ for $g\in T$ and $P(y,\tilde{b})\in X$. We shall show that σ is continuous. Let V be an open neighborhood of $(P(y,\tilde{b}))g$ in X. Let $\tilde{p}(\tilde{b})=x$ in H with $x\in V_i$ for some i. Then $P(y,\tilde{b})=$ $\phi_i(y\cdot\tilde{p}_i^{-1}(\tilde{b}),\ x)\in p^{-1}(V_i)$ and $P(y,\tilde{b})g\in V\ (p^{-1}(V_i))g$ which is open in X. Since p is continuous from $Y\times T$ onto X and $P(y,\tilde{b})g=P(y,\tilde{b}g)$, there exist an open neighborhood W of $\tilde{b}g$ in T and an open neighborhood U of y in Y such that $P(U,W)\subset V\cap(p^{-1}(V_i))g$. In the group T, there exist an open neighborhood W_1 of g and an open neighborhood W_2 of $\tilde{b}$ in T such that $\tilde{p}(W_2)\subset V_i$ and $W_2W_1\subset W$, we have $P(U,W_2)W_1=P(U,W_2\cdot W_1)\subset P(U,W)\subset$ $V\cap(p^{-1}(V_i))g$. Since $P(U,W_2)=\phi_i(U\cdot\tilde{p}_i(W_2),\ \tilde{p}(W_2)),\ \tilde{p}(W_2)),\ \tilde{p}(W_2)$ is open in H, $U\cdot\tilde{p}_i(W_2)$ is open in Y and ϕ_i is a homeomorphism from $Y\times V_i$ onto $p^{-1}(V_i)$, we have that $\phi_i(U\cdot\tilde{p}_i(W_2),\ \tilde{p}(W_2))$ is open in $p^{-1}(V_i)$, therefore $P(U,W_2)$ is open in X. It follows that $P(U,W_2))W_1\subset V$, where $P(U,W_2)$ is a neighborhood of $P(y,\tilde{b})$ and W_1 is a neighborhood of g. This shows that σ is continuous. It is easy to see that $\sigma(\sigma(x,t_1),t_2)=\sigma(x,t_1t_2)$ and $\sigma(x,e)=x$ where $x\in X$, t_1, $t_2\in T$ and e is the identity of T. Hence (X,T,σ) is a transformation group.

We shall show that $(X,T,\sigma;H,p,\tilde{p})$ is a reducible transformation group. It is sufficient to show that the following diagram is commutative:

$$\begin{array}{ccccc} X & \times & T & \xrightarrow{\ \sigma\ } & X \\ p\big\downarrow & \tilde{p} & \big\downarrow & m & \big\downarrow p \\ H & \times & H & \longrightarrow & H \end{array}$$

We know that $P: Y\times T\rightarrow X$ is onto. Let $t\in T$ and $P(y,\tilde{b})\in X$ for $y\in Y$ and $\tilde{b}\in T$. We show that $p\sigma(P(y,\tilde{b}),t)=p(P(y,\tilde{b}))\cdot\tilde{p}(t)$. $\sigma(P(y,\tilde{b}),t)=P(y,\tilde{b}t)=$ $\phi_i(y\cdot\tilde{p}_i(\tilde{b}t),\ \tilde{p}(\tilde{b}t))$, where $\tilde{p}(\tilde{b}t)\in V_i$ in H for some i. $p(\sigma(P(y,\tilde{b}),t))=\tilde{p}(\tilde{b}t)$ and $p(P(y,\tilde{b}))\cdot\tilde{p}(t)=p(\phi_j(y\cdot\tilde{p}_j(\tilde{b}),\ \tilde{p}(\tilde{b}))\cdot\tilde{p}(t)=\tilde{p}(\tilde{b})\cdot\tilde{p}(t)=\tilde{p}(\tilde{b}t)$, where $\tilde{p}(\tilde{b})\in V_j$ for some j. Hence it is reducible.

Let $P(Y,e)=Y^1$ where e is the identity in T. It is not hard to see that

Y^1 is invariant under N. For $n \in N$, $y \in Y$, we have $P(y \cdot n, e)\ P(y, n \cdot e) = P(y, n) = (P(y, e))n$. Hence (Y^1, N), as in (X, T, σ), is equivalent to the transformation group (Y, N).

Furthermore, if Y is minimal under N, we shall show it is reducible. It is enough to show that for $P(y_1, \tilde{b}_1) \notin cl((P(y, \tilde{b}), N))$, $p(P(y_1, \tilde{b}_1)) \neq p(cl(\sigma(P(y, \tilde{b}), N))$. $p(P(y_1, \tilde{b}_1)) = p(\phi_i(y_1 \cdot \tilde{p}_i(\tilde{b}_1), \tilde{p}(\tilde{b}_1)) = \tilde{p}(\tilde{b}_1)$ where $\tilde{p}(\tilde{b}_1) \in V_i$ in H for some i. $\sigma(P(y, \tilde{b}), N) = P(y, \tilde{b}N) = P(y, N\tilde{b}) = P(y \cdot N, \tilde{b}) = \phi_j(y \cdot N \cdot \tilde{p}_j(\tilde{b}), \tilde{p}(\tilde{b})) = \phi_j(y \cdot N, \tilde{p}(\tilde{b}))$, where $\tilde{p}(\tilde{b}) \in V_j$ in H for some j and $\tilde{p}_j(\tilde{b}) \in N$. $cl(\phi_j(y \cdot N, \tilde{p}(\tilde{b})) = cl(\phi_{j, \tilde{p}(\tilde{b})}(y \cdot N))$, $\phi_{j, \tilde{p}(\tilde{b})}(cl(y \cdot N)) = \phi_{j, \tilde{p}(\tilde{b})}(Y) = \phi_j(Y, \tilde{p}(\tilde{b})) = P(Y, \tilde{b})$. Thus $cl(\sigma(P(y, \tilde{b}), N)) = \phi_j(Y, \tilde{p}(\tilde{b})) = P(Y, \tilde{b})$ and $p(cl(\sigma(P(y, \tilde{b}), N)) = p(\phi_j(Y, \tilde{p}(\tilde{b})) = \tilde{p}(\tilde{b})$. From $P(y_1, \tilde{b}_1) \notin P(Y, \tilde{b})$, we have $\phi_i(y_1 \tilde{p}_i(\tilde{b}_1), \tilde{p}(\tilde{b}_1)) \notin \phi_j(Y, \tilde{p}(\tilde{b}))$, it follows that $\tilde{p}(\tilde{b}_1) \neq \tilde{p}(\tilde{b})$. By Lemma 9, we know X is minimal under T. By Lemmas 14 and 15, we have

THEOREM 5. *Let $(X, T, \Pi; H, p, f)$ be reducible and separable transformation group where X is a compact Hausdorff space, and $f: T \to H$ is a group covering then $\{X, p, H, Y, N\}$ is a fibre bundle and $\{T, f, H, N, N\}$ is its principal associated bundle. Conversely, if $\{X, p, H, Y, N\}$ is a fibre bundle, where X is a compact Hausdorff space, H is a compact group, Y is a minimal set under N, and N is discrete group, let $f: T \to H$ be a group covering from T onto H such that $f^{-1}(e) = N$ and $\{T, f, H, N, N\}$ is the principal associated bundle of the given fibre bundle, then there exists an action σ of the group T, on X so that $(X, T, \Pi; H, p, f)$ is a reducible and separable transformation group and X is minimal under T.*

§ 7. Extension of transformation groups and universal minimal sets.

THEOREM 6. *Let $\tilde{p}: T \to H$ be a group covering, where H is a compact group and T is connected. Let $\tilde{p}^{-1}(e) = N$. Let Y be a compact Hausdorff space such that (Y, N, Π) is a transformation group. Then there is a compact Hausdorff space X such that (X, T, σ) is a trnasformation group and there is a proper closed subset Y^1 in X, which is invariant under N, so that (Y^1, N, σ) is equivalent to (Y, N, Π). Furthermore if Y is minimal under N, then X is minimal under T. If Y is universal minimal under N, then X is universal minimal under N. Conversely, if (X, T, σ) is a universal minimal set under T, where T is a connected group, X is a compact Hausdorff space and there is a group covering f from T onto a compact group H, then for each $x \in X$ (Y_x, N, σ) must be a universal minimal set under N, where $N = \ker(f)$ and $Y_x = cl(\sigma(x, N))$.*

Proof. It is easy to see that $\{T, \tilde{p}, H, N, N\}$ is a principal bundle. Let, in this principal bundle, $\{V_i \mid i \in J\}$ be a set coordinate neighborhoods in H. Since H is compact the index set J must be finite. Let $\{g_{i_j} \mid i, j \in J\}$ be the set of coordinate transformations in this principal bundle. Then there exists a unique fibre bundle, (see p. 14 [14]), $\mathscr{B} = \{X, p, H, Y, N\}$ with projection p, base space H, fibre Y, group N, coordinate neighborhoods $\{V_i \mid i \in J\}$ and coordinate transformations $\{g_{ij} \mid i, j \in J\}$. Since H and Y are compact Hausdorff spaces and J is a finite set, X must be compact Hausdorff. It is easy to see that $\{T, f, H, N, N\}$ is the principal associated bundle of $\mathscr{B} = \{X, p, H, Y, N\}$. By Lemma 15, there exists an action σ of the group T on X such that (X, T, σ) is a transformation group and a closed subset Y^1 in X such that (Y^1, N, σ) is equivalent to the transformation group (Y, N, Π). Again by Lemma 15, if Y is minimal under N, then X is minimal under T.

Let now Y be universal minimal under N. Since (X, T, σ) is minimal, there exists a compact Hausdorff, universal minimal set (X^1, T, Π^1) (see [3]) and a continuous map f from X^1 onto X such that the following diagram:

$$\begin{array}{ccccc} X^1 & \times & T & \xrightarrow{\ \Pi\ } & X^1 \\ f\big\downarrow & & i\big\downarrow & \sigma & f\big\downarrow \\ X & \times & T & \longrightarrow & X \end{array}$$

is commutative. We know that from the proof of Lemma 15, $P(Y, T) = X$, we shall show that for each $t \in T$ $Y_t = P(Y, t) = p^{-1}(h)$, where $h = p(t)$, is invariant under N and (Y_t, N, σ) is equivalent to (Y, N, Π). For $z = P(y, t) \in Y_t$ and $n \in N$ where $y \in Y$ and $t \in T$, we have $z \cdot n = P(y, t)n = P(y, tn)$. Since T is connected and N is a discrete normal subgroup of T, N must be in the center of T. Hence, we have $tn = nt$ for $t \in T$ and $n \in N$. It follows that $z \cdot n = P(y, tn) = P(y, nt) = P(y \cdot n, t) \in Y_t$, this show that Y_t is invariant under N. For a fixed $t \in T$, define $f_t : Y \to Y_t$ by $f_t(y) = P(y, t)$. It is easy to see that f_t is a homeomorphism from Y onto Y_t. In fact, $f_t(y) = P(y, t) = \phi_i(y \cdot \tilde{p}_i(t), \tilde{p}(t)) = \phi_{i,h}(y \cdot \tilde{p}_i(t))$ where $h = \tilde{p}(t) \in V_i$ for some $i \in J$ and $\tilde{p}_i(t) = \phi_{i,h}^{-1}(t) \in N$. Consider the following diagram

$$\begin{array}{ccccc} Y & \times & N & \xrightarrow{\ \Pi\ } & Y \\ f_t\big\downarrow & & i\big\downarrow & \sigma & f_t\big\downarrow \\ Y_t & \times & N & \longrightarrow & Y_t \end{array}$$

For $y \in Y$ and $n \in N$, we have $f_t \Pi(y, n) = P(y \cdot n, t) = (P(y, t)) \cdot n = \sigma(f_t(y), n)$. Thus the diagram is commutative, (Y, N, Π) and (Y_t, N, σ) are equivalent, and (Y_t, N, σ) is universal minimal set under N for each $t \in T$.

Since X' is minimal under T, we have, for each $x' \in X'$, x' is almost periodic under T. Since N is discrete and H is compact, T must be locally compact. From the fact that N is a syndetic closed subgroup of T, this implies that for each $x' \in X'$, x' is almost periodic under N (see [8]). Thus $cl(\Pi'(x', N)) = cl(x' \cdot N)$ is minimal under N. For each $x' \in X$, there exist $y \in Y$ and $t \in T$ such that $f(x') = P(y, t)$. We know that $f(cl(\Pi'(x', N)) \supset cl(f(\Pi'(x', N))$. However, $cl(f(\Pi'(x', N)) = cl(\sigma(P(y, t), N)) = cl(P(y, tN)) = cl(P(y, Nt)) = cl(P(yN, t) = cl(\phi_{i,h}(y \cdot N \cdot \tilde{p}_i(t))) = cl(\phi_{i,h}(y \cdot N))$, where $h = \tilde{p}(t) \in V_i$, for some $i \in J$, in H, and $\tilde{p}_i(t) \in N$. Because $\phi_{i,h}$ is a homeomorphism from Y onto $p^{-1}(h)$, we have $cl(\phi_{i,h}(y \cdot N)) = \phi_{i,h}(cl(y \cdot N)) = \phi_{i,h}(Y) = \phi_i(Y, h) = P(Y, t) = Y_t$. Hence $f(cl(\Pi'(x', N)) \supset Y_t$. Since both $f(cl(\Pi'(x', N))$ and Y_t are minimal under N in (X, T, σ), we have $f(cl(\Pi'(x', N)) = Y_t$ and $f(cl(\Pi'(x, N))$, for each $x' \in X'$, is a universal minimal set under N. It follows that, on $cl(\Pi'(x', N))$, f is an one-to-one map for all $x' \in X'$. We shall show that f is an one-to-one map on X'.

Let x_1 and x_2 in X' with the property that $f(x_1) = f(x_2)$. We shall show that $x_2 \in cl(\Pi'(x_1, N))$. Notice that $cl(\Pi'(x_1, N))$ is a universal minimal set under N. Since N is syndetic in T, there exists a compact subset K in T such that $T = N \cdot K$ and $X' = cl(\Pi'(x_1, T)) = cl(x_1 \cdot T) = cl(x_1 \cdot N) \cdot K$. It follows that there is $k \in K$ and $x' \in cl(x_1 \cdot N)$ such that $x_2 = x' \cdot k = \Pi'(x', k)$. There exist y_1 and y' in Y and t_1 and t' in T such that $f(x_1) = P(y_1, t_1)$ and $f(x') = P(y', t)$. From the fact that $f(cl(\Pi'(x_1, N)) = P(Y, t_1) = Y_{t_1}$ and $f(x_2) = f(\Pi'(x', k)) = \sigma(f(x'), k) = \sigma(P(y', t'), k) = P(y', t'k)$, we have $P(y', t'k) \in P(Y, t_1)$ and $t'k \in Nt_1$. Since $x' \in cl(\Pi'(x_1, N))$ we have $f(x') = P(y', t') \in f(cl(\Pi'(x_1, N)) = P(Y, t_1)$ and $t' \in Nt_1$. Consequently $(t')^{-1}Nt_1 = t_1^{-1}Nt_1 = N$ and $k \in N$. Let $k = n$ for some $n \in N$, we have $x_2 = \Pi'(x', n) \in cl(\Pi(x_1, N))$. It follows that f is an one-to-one map from X' onto X. Since X' is compact Hausdorff, f is a homeomorphism and (X, T, σ) must be a universal minimal set under T.

Conversely, assume that (X, T, σ) is a universal minimal set under T. Let $x_1 \in X$ and $Y_1 = cl(\sigma(x_1, N))$. We know that (Y_1, N, σ) is a minimal set. There exists a universal minimal set (Y, N, Π) and a continuous map ξ from Y onto Y_1 such that the following diagram is commutative:

$$\begin{array}{ccccc} Y & \times & N & \xrightarrow{\Pi} & Y \\ \xi\downarrow & & i\downarrow & \sigma & \xi\downarrow \\ Y_1 & \times & N & \longrightarrow & Y_1 \end{array}$$

By the first part proof of this theorem there exists a transformation group (X', T, Π') such that $\mathscr{B} = \{X', p, H, Y, N\}$ is a fibre bundle, (X', T, Π) is a universal minimal set under T, and for each $x' \in X'$ $cl(\Pi'(x', N))$ is a universal minimal set under N and equivalent to (Y, N, Π). Since universal minimal set is unique up to equivalence class, there is a homeomorphism ϕ from X' onto X such that the following diagram

$$\begin{array}{ccccc} X' & \times & T & \xrightarrow{\Pi'} & X' \\ \phi\downarrow & & i\downarrow & \sigma & \phi\downarrow \\ X & \times & T & \longrightarrow & X \end{array}$$

is commutative. Let $\phi^{-1}(x_1) = x'$ then $\phi(cl(\Pi'(x', N)) = cl(\phi(\Pi'(x', N))) = cl(\sigma(x_1, N)) = Y_1$ and (Y_1, N, σ) must be also universal minimal under N. The theorem is proved.

Remark. In the classical topological dynamics, it is known (e.g. see [8]) that for any discrete flow (Y, Z, Π) where Y is a compact Hausdorff space and Z is the integer group with the discrete topology, there is an extension to a continuous flow. Let $I = [0, 1]$, the closed unit interval. Consider $Y \times I$. For y, y' in Y, t, t' in I, we say (y, t) and (y, t') are *equivalent* if $t = 0$ and $t' = 1$. Let L be the closed equivalence relation induced by this equivalence relation in $Y \times I$. Let $X = (Y \times I)/L$. Let $(y, t)L \in X$, $r \in R$ and $n \in Z$. Define $\Pi' : X \times R \to X$ by $\Pi'((y, t)L, r) = (yn, r_1)L$ where $t + r = r_1 + n$ with $r_1 \in [0, 1]$. It is known that (X, R, Π') is a transformation group. We call (X, R, Π') the *P-extension* of (Y, Z, Π). It is clear that this extension is a special case of the extension in Theorem 6, with $N = Z$ and $T = R$. Hence we have the following corollary:

COROLLARY: *If (X, R, Π) is a compact Hausdorff universal minimal set under R, then there exists an integer subgroup X of R, $Y = cl(xZ)$ is a universal minimal set under Z. Conversely, if (Y, Z, Π) is a compact Hausdorff universal minimal set and (X, R, Π') is the P-extension of (Y, Z, Π), then (X, R, Π') is the universal minimal set under R.*

§ 8. The Whitney Sum

Let $(X, T, \Pi; X^*, H, \Pi^*, p, f)$ and $(X', T, \Pi'; X^*, H, \Pi^*, p', f)$ be two semi-reducible transformation group. Let X'' be the subset of $X \times X'$ consisting of all pairs (x, x') such that $p(x) = p'(x')$. It is obvious that X'' is closed in $X \times X'$. Define $p: X'' \to X$ by $p_1(x, x') = x$, $p_2: X'' \to X'$ by $p_2(x, x') = x'$, $p'': X'' \to X^*$ by $p''(x, x') = pp_1(x, x') = p' p_2(x, x')$, $\Pi'': X'' \times T \to X''$ by $\Pi''((x, x'), t) = (\Pi(x, t), \Pi'(x', t)) = (xt, x't)$. We shall show that $p(xt) = p'(x't)$.

By the semi-reducibility, we have

$$P(xt) = p(\Pi(x, t)) = \Pi^*(p(x), f(t)) = \Pi^*(p'(x'), f(t)) = p'(\Pi'(x', t)) = p'(x't).$$

It is easy to verify that (X'', T, Π'') is a transformation group. We shall show that the following diagram is commutative,

$$\begin{array}{ccccc} X'' & \times & T & \xrightarrow{\Pi} & X'' \\ {\scriptstyle p''}\downarrow & & {\scriptstyle f}\downarrow & \Pi^* & \downarrow{\scriptstyle p''} \\ X^* & \times & H & \longrightarrow & X^* \end{array}$$

Let $(x, x') \in X''$, $t \in T$. We have

$$\Pi''((x, x'), t)) = (xt, x't), \quad p''\Pi''((x, x'), t) = p''(xt, x't) = p(xt)$$

$$\Pi^*(p''(x, x'), f(t)) = \Pi^*(p(x), f(t)) = p(\Pi(x, t)) = p(xt).$$

Hence it is semi-reducible.

DEFINITION 6. We call this transformation group (X'', T, Π'') is the *Whitney sum* of (X, T, Π) and (X', T, Π'). This definition is same as the Whitney sum of two fibre bundles (e.g. see [11]).

LEMMA 16. *Let (X, T, Π) and (X', T, Π'') be two transformation group. Then if both are semi-reducible, so is its Whitney sum.*

THEOREM 7. *Let $(X, T, \Pi; H, p, f)$ and $(X, T, \Pi'; H, p', f)$ be two reducible, and separable transformation groups where $f: T \to H$ is a group covering and X and X' are compact Hausdorff spaces. Then its Whitney sum is separable if and only if the cartesian product Y'' of fibres Y and Y' is minimal under N, where $N = f^{-1}(e)$.*

Proof. From Theorem 5 (X, p, H) and (X', p', H) are fibre bundle, it follows that their Whitney sum (X'', H, p') is also a fibre bundle with H as its base space, $Y'' = Y \times Y'$ as its fibre, N as its group and N acts on Y''

induced from actions of N on Y and Y'. The theorem is now a consequence of Lemma 15 and Lemma 9. However, we give a direct proof here. Let Y'' be minimal under N. Then for each $h \in H$, $((p'')^{-1}(h), N, \Pi'')$ is minimal under N. We know that both Y and Y' are minimal sets under N, for $(y, y') \in Y''$, where $y \in Y$ and $y' \in Y'$, we have $cl(\Pi''((y, y'), N)) = cl(\Pi(y, N) \times cl(\Pi(y', N))$. Consequently, if $(z, z') \notin cl(\Pi''(y, y')N)$ in X'', then both $z \notin cl(\Pi(y, N))$ and $z' \notin cl(\Pi'(y', N))$. Say $z \notin cl(\Pi(y, N))$ then $p(z) \neq p(y)$. From $p(z) = p'(z')$ and $p(y) = p'(y')$, we have $p'(y') \neq p'(z')$ and $z' \notin cl(\Pi'(y, N))$. It follows that $p''(z, z') = p(z) \neq p(y) = p''(y, y')$. This show that (X'', T, Π'') is separable. Conversely, let (X', T, Π'') be separable, then if $(z, z') \notin cl(\Pi''((y, y'), N))$ we have $p''(z, z') \neq p''(y, y')$ or $p(z) \neq p(y)$ and $p'(z') \neq p'(y')$. It follows that $z \notin cl(\Pi(y, N))$ and $z' \notin cl(\Pi'(y', N))$. We have $(z, z') \notin cl(\Pi(y, N)) \times cl(\Pi'(y', N))$ and $cl(\Pi''((y, y'), N)) \supset cl(\Pi(y, N)) \times cl(\Pi'(y', N))$. It is obvious that $cl(\Pi''((y, y'), N)) \subset cl(\Pi(y, N)) \times cl(\Pi'(y', N))$. Hence $cl(\Pi''(y, y'), N)) = cl(\Pi(y, N)) \times cl(\Pi'(y', N))$. Since both $cl(\Pi(y, N))$ and $cl(\Pi'(y', N))$ are minimal, this implies that $cl(\Pi(y, N)) = Y$ and $cl(\Pi'(y', N)) = Y'$ for all $y \in Y$ and $y' \in Y'$, we have $cl(\Pi''((y, y'), N)) = Y \times Y' = Y''$ for all $(y, y') \in Y''$ and it must be minimal. The theorem is proved.

Remark. This theorem shows that the cartesian product of two minimal, but not totally minimal, real flows is not minimal, if they have a same integer subgroup satisfying the property (A), because, by Theorem 7, at most its Whitney sum is minimal.

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