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COMPLEX-HARMONIC MEIER'S THEOREM

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1. Fatou's theorem is true for a bounded complex-valued harmonic function in the disk D: $|z| < 1$. One asks naturally: "Is Meier's topological analogue of Fatou's theorem (simply, "*MF* theorem"; [14, p. 330, Theorem 6], cf. [10, p. 154, Theorem 8.9]) true for a bounded complex-valued harmonic function in D?" We shall give the affirmative answer to this question. Furthermore, the horocyclic *MF* theorem [2, p. 14, Theorem 5] in the complex-harmonic form will be proved in parallel.

For recent various discussions on Plessner's and Meier's theorems we consult [1∼7, 11, 12, 15∼18].

2. In the rest of this note we denote by $\delta(\zeta_0, \rho)$ the open disk $|z - \zeta_0| < \rho$ in the z-plane.

LEMMA 1. *Let a function $g(\zeta)$ be complex-valued and harmonic (simply, "complex-harmonic") in $\delta(\zeta_0, \rho)$ and $|g(\zeta)| < 1$ for $\zeta \in \delta(\zeta_0, \rho)$. Then we have*

$$|g(\zeta) - g(\zeta_0)| \leqq (8/\pi) \text{ arc } \tan(|\zeta - \zeta_0|/\rho) \tag{1}$$

for $\zeta \in \delta(\zeta_0, \rho)$ (Schwarz's lemma).

Proof. Let $w = (\zeta - \zeta_0)/\rho$ and consider the function

$$G(w) = \{g(\rho w + \zeta_0) - g(\zeta_0)\}/2$$

in D: $|w| < 1$. Then $G(0) = 0$ and $|G(w)| < 1$ in D, so that we may apply the ready Schwarz lemma [13, p. 101, Lemma] to the complex-harmonic G in D. The inequality [13, p. 101, (3)]

$$G(w)| \leqq (4/\pi) \text{ arc } \tan|w|$$

for $w \in D$ proves (1). Q.E.D.

The reader should know the definition of cluster set, chordal cluster set and angular cluster set [10, pp. 1, 72 and 73].

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LEMMA 2. *Let a function* $f(z)$ *be complex-harmonic in* D *with* $|f(z)|<1$ *for* $z\in D$. *Assume that*

(2) $$C_X(f, 1) \neq C_D(f, 1),$$

where X *is a chord of the unit circle passing through the point* $z=1$. *Then there exists an angle* Δ *at* $z=1$ *(i.e., the interior of a triangle lying in* D *except for one vertex* $z=1$*) such that*

(3) $$C_\Delta(f, 1) \neq C_D(f, 1).$$

Proof. Choose a point $P\in C_D(f, 1)-C_X(f, 1)$ and let

$$0<2\varepsilon<\text{dis}\ \{P,\ C_X(f, 1)\}.$$

By (2) such a point P does exist and further we can find a rectilinear segment $X_1\subset X$ terminating at $z=1$ such that

(4) $$\overline{f(X_1)}\cap\delta(P, \varepsilon)=\phi \text{ (empty)}$$

by the very definition of $C_X(f, 1)$. Let φ be the directed angle, $|\varphi|<\pi/2$, made by X and the radius of D at $z=1$ and suppose without loss of generality that $0\leqq\varphi<\pi/2$. Set

$$\gamma(z_0)=|1-z_0|\sin(\pi/4-\varphi/2),\quad z_0\in X_1$$

and choose a constant μ such that

(5) $$0<\mu<\tan(\pi\varepsilon/16).$$

Then $\tan(\pi\varepsilon/16)<1<\pi/2$ because of $\varepsilon<1$ and for any point $z\in\delta(z_0, \mu\gamma(z_0))$ $(z_0\in X_1)$ we have

(6) $$|f(z)-f(z_0)|\leqq(8/\pi)\ \text{arc}\ \tan\{\mu\gamma(z_0)/\gamma(z_0)\}<\varepsilon/2$$

by (1) of Lemma 1 and (5) if z_0 is so near to $z=1$ that $\delta(z_0, \gamma(z_0))\subset D$. Now, as $X_1\ni z_0\to 1$, the disks $\delta(z_0, \mu\gamma(z_0))$ sweep an angle Δ at $z=1$, so that by (4) and (6) we have

(7) $$\overline{f(\Delta)}\cap\delta(P, \varepsilon/4)=\phi.$$

Now that (7) means $P\notin\overline{f(\Delta)}$ we have

$$P\in C_D(f, 1)-C_\Delta(f, 1),$$

which proves (3). Q.E.D.

For the terminology, "right horocycle", "right horocyclic cluster set", "right horocyclic angle", etc. we refer to [2, pp. 4–6].

LEMMA 3. *Let a function* $f(z)$ *be complex-harmonic in* D *with* $|f(z)|<1$ *for* $z\in D$. *Assume that*

$$C_{h(1)}(f, 1) \neq C_D(f, 1), \tag{8}$$

where $h(1) = h_r^+(1)$ *is a right horocycle at* $z=1$. *Then there exists a right horocyclic angle* $H(1) = H^+_{r_1,r_2,r_3}(1)$ *at* $z=1$ *such that*

$$C_{H(1)}(f, 1) \neq C_D(f, 1). \tag{9}$$

Proof. We use a different method from Bagemihl's [2, p. 14, Lemma 3]. By (8) we can find a point $P\in C_D(f, 1) - C_{h(1)}(f, 1)$ and we then set

$$0 < 2\varepsilon < \text{dis}\,\{P, C_{h(1)}(f, 1)\}.$$

By the definition of $C_{h(1)}(f, 1)$ we obtain a subarc α of $h(1)$ terminating at $z=1$ such that

$$\overline{f(\alpha)} \cap \delta(P, \varepsilon) = \phi. \tag{10}$$

We consider next the map

$$z = \chi(\zeta) = (\zeta-1)/(\zeta+1)$$

from the half plane $\text{Re}\zeta > 0$ onto D. The initial point of $h(1)$ lies on the real axis, which we denote by x, $|x|<1$. Then the image L_x of $h(1)$ by the map χ^{-1} is the half line

$$L_x = \{\zeta;\ \text{Re}\zeta = (1+x)/(1-x) \text{ and } \text{Im}\zeta \leqq 0\}.$$

Let β be the image of α by χ^{-1} and let

$$0 < \mu < \tan(\pi\varepsilon/16). \tag{11}$$

Let $0<\rho<(1+x)/(1-x)$ and consider the composed function $F(\zeta) = f\circ\chi(\zeta)$ in the disk $\delta(\zeta_0, \rho)$, where $\zeta_0\in\beta$. Then for $\zeta\in\delta(\zeta_0, \mu\rho)\subset\delta(\zeta_0, \rho)$, we have

$$|F(\zeta) - F(\zeta_0)| \leqq (8/\pi)\ \text{arc}\ \tan(\mu\rho/\rho) < \varepsilon/2 \tag{12}$$

by (1) of Lemma 1 combined with (11). Now, as $\beta\ni\zeta_0\to\infty$ (i.e., $\alpha\ni\chi(\zeta_0)\to 1$) the disks $\delta(\zeta_0, \mu\rho)$ sweep a strip of width $2\mu\rho$ whose image by χ contains a right horocyclic angle $H(1) = H^+_{r_1,r_2,r_3}(1)$ at $z=1$. By (10) and (12) we have

$$\overline{f(H(1))} \cap \delta(P, \varepsilon/4) = \phi,$$

so that we have (9). Q.E.D.

Remark. Lemma 3 is true if the word "right" is replaced by "left" where it is.

3. A point $e^{i\theta}$ of the circle is a Meier point (horocyclic Meier point, resp.) of a complex-harmonic function $f(z)$ in D if $C_D(f, e^{i\theta})$ is a proper subset of the Riemann sphere and if every chordal cluster set (every right or left horocyclic cluster set, resp.) of f at $e^{i\theta}$ coincides with $C_D(f, e^{i\theta})$ [10, p. 153], [2, p. 6].

By means of Lemmas 2 and 3, and Collingwood's maximality theorem ([8, p. 1241, Theorem 4], [9, p. 8, Theorem 4]; [10, p. 80, Theorem 4.10]) or its ready generalization from (Stolz) angles to horocyclic angles we have the following two theorems.

THEOREM 1. *Let a function $f(z)$ be bounded, complex-valued and harmonic in the disk $|z|<1$. Then all points of the circle Γ: $|z|=1$ are, except perhaps for a set of first Baire category on Γ* [10, p. 75], *Meier points of f.*

THEOREM 2. *Let a function $f(z)$ be bounded, complex-valued and harmonic in the disk $|z|<1$. Then all points of the circle Γ: $|z|=1$ are, except perhaps for a set of first Baire category on Γ, horocyclic Meier points of f.*

4. As a concluding remark we note that further generalizations of Theorems 1 and 2 are possible (cf. [15]). Let Ω and Ω' be domains in the z-plane and in the ζ-plane respectively. A complex-valued function $f(z)$ in Ω is called K-quasi-conformal harmonic (simply, "*KQCH*") in Ω provided that $f(z)$ is of the composed form $f(z) = g \circ Q(z)$, where $\zeta = Q(z)$ is a K-quasi-conformal homeomorphism ($K \geqq 1$) from Ω onto Ω' and $g(\zeta)$ is complex-harmonic in Ω'*⁾. The key lemma for the proof of *MF* or horocyclic *MF* theorem of *KQCH* functions in D is, of course, an analogue of the Schwarz lemma:

LEMMA 1^{bis}. *Let a function $f(z)$ be KQCH and $|f(z)|<1$ in the disk $\delta(z_0, q)$. Then for $z \in \delta(z_0, q)$ we have*

$$|f(z)-f(z_0)| \leqq (8/\pi) \text{ arc } \tan(4q^{-1/K}|z-z_0|^{1/K}). \tag{13}$$

Proof. We may consider $f = g \circ T$, where T is a K-quasi-conformal self-homeomorphism of $\delta(z_0, q)$ with the additional property that $z_0 = T(z_0)$, and g is complex-harmonic in $\delta(z_0, q)$. Furthermore, we know about T that [15, p. 323, line 2 from below]

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*⁾ A domain Ω' may depend upon f.

$$|T(z) - T(z_o)| \leqq 4q^{1-(1/K)}|z - z_o|^{1/K}, \quad z \in \delta(z_o, q),$$

an inequality due to A. Mori, so that combining this with Lemma 1 of section 2 we obtain (13).

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