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ALGEBRAS OF FINITE COHOMOLOGICAL DIMENSION

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Introduction

The cohomology theory of an associative algebra has been shown to be valuable in the study of the structure of algebras of finite cohomological dimension, especially those of dimension less than or equal to one over a field. M. Harada [9] has shown that every semi-primary hereditary algebra A (for example, A is finitely generated over a field R and has dimension ≤ 1) is isomorphic to a generalized triangular matrix algebra. The concept of a central separable algebra over a commutative ring has been shown to be a useful generalization of the concept of a central simple algebra over a field, where a separable algebra is defined to be an algebra having cohomological dimension zero.

We are able to generalize the structural result of M. Harada to one dimensional algebras over a local hensel ring by first extending a result of Endo and Watanabe [8] on separable algebras, that is, we show that the cohomological dimension of an algebra A finitely generated and projective over R, is determined by the cohomological dimension of the algebras $A/\mathfrak{m}A$ over the residue fields $R/\mathfrak{m}$. As a corollary we show that there are no finitely generated projective commutative algebras of non-zero finite cohomological dimension.

Section 1. Preliminaries:

All rings are assumed to have an identity and all ring homomorphisms carry the identity onto the identity. Throughout this paper, R denotes a commutative ring and A denotes an R-algebra; that is, A is a ring along with a ring homomorphism θ of R into the center of A. N always denotes the Jacobson radical of A. By a finitely generated or projective R-algebra, we mean an algebra which is finitely generated or projective as an R-module.

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If R has a unique maximal ideal, we say R is a local ring. For integers n, we always mean that $0 \leq n < \infty$.

We will repeatedly need the so-called Nakayama's Lemma which states that if R is a commutative ring and M is a finitely generated R-module with the property that $\mathfrak{m}M = M$ for every $\mathfrak{m}$ in the maximal spectrum $\Omega(R)$ of R, then $M = (0)$.

We say that R-dim $A = n$ if and only if the left homological dimension of A as an A^e-module equals n ($l.hd_{A^e}(A) = n$) where $A^e = A \otimes_R A^{op}$ is the enveloping algebra. We let J denote the kernel of the map μ in $\mathrm{Hom}_{A^e}(A^e, A)$ defined by $\mu(a \otimes a') = aa'$. We note that J is finitely generated over A^e if A is finitely generated over R. One can easily see that A^e is isomorphic as an R-module to $J \oplus A$.

Samuel Eilenberg [5, Cor 2 to Th 3, p. 31] has proved the following very interesting and useful characterization of algebras with finite cohomological dimension over a field:

THEOREM 1.1. *Let A be a finitely generated algebra over a field R. Then R-dim $A = n$ if and only if A/N is R-separable and $l.hd_A(N) = n - 1$.*

We will need the following well-known results concerning weak homological dimension of left modules [*cf*, 11, p. 153 ff.]

LEMMA 1.2. (*a*) *If A is an R-algebra and M is a left A-module, then $\sup_{\mathfrak{m}} (w.hd_{A_\mathfrak{m}}(M_\mathfrak{m})) = w.hd_A(M)$. Hence, M is A-flat if and only if $M_\mathfrak{m}$ is $A_\mathfrak{m}$-flat for every $\mathfrak{m}$ in $\Omega(R)$.*

(*b*) *If A is an R-algebra, R a local ring with henselization $\hat{R}$, and M a left A-module, then $w.hd_{\hat{A}}(\hat{M}) = w.hd_A(M)$.*

In particular, Lemma 1.2(b) together with the fact that a module M which is both flat and finitely presented over some ring is also projective [*vid.* 6, p. 64] allows one to see by an easy induction that:

PROPOSITION 1.3. *If A is a finitely generated R-algebra, R a local ring, then R-dim $A = n$ if and only if $\hat{R}$-dim $\hat{A} = n$.*

We close this section with the result of Endo and Watanabe [8, Prop 1.1, p. 234] which we generalize:

THEOREM 1.4. *Let A be a finitely generated R-algebra. The following are equivalent:*

(a) *A is R-separable.*

(b) $A_{\mathfrak{m}}$ *is* $R_{\mathfrak{m}}$*-separable for every* $\mathfrak{m}$ *in* $\Omega(R)$.

(c) $A/\mathfrak{m}A$ *is* $R/\mathfrak{m}$*-separable for every* $\mathfrak{m}$ *in* $\Omega(R)$.

Section 2. Main Theorem and Corollaries

In this section, we prove a generalization of Theorem 1.4 with the unfortunate (but not surprising) additional hypothesis that the algebra be projective.

THEOREM 2.1. *Let* A *be a finitely generated, projective* R*-algebra. The following are equivalent:*

(a) R*-dim* $A = n$

(b) $R_{\mathfrak{m}}$*-dim* $A_{\mathfrak{m}} \leq n$ *for every* $\mathfrak{m}$ *in* $\Omega(R)$ *with equality at some* $\mathfrak{m}$.

(c) $R/\mathfrak{m}$*-dim* $A/\mathfrak{m}A \leq n$ *for every* $\mathfrak{m}$ *in* $\Omega(R)$ *with equality at some* $\mathfrak{m}$.

Remark 2.2. If $n = 1$, then (a) implies (b) implies (c) without the hypothesis that A be R-projective since a minimal projective A^e-resolution of A is given by the R-split exact sequence $0 \to J \to A^e \to A \to 0$.

Proof of the Theorem. We proceed by induction on n with $n = 0$ (Theorem 1.4) as the basis for the induction. Assume the result is true for all $k < n$.

That (a) implies (b) and that (a) implies (c) are clear.

(*b*) *implies* (*a*)*:* Let $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ be an A^e resolution of A with $P_0, \cdots, P_{n-1}$ A^e-projective. Since A^e is finitely generated and projective over R, K_n is finitely A^e-presented and R-projective. Now, for every $\mathfrak{m}$, $(K_n)_{\mathfrak{m}}$ is $A^e_{\mathfrak{m}}$-flat. Hence K_n is A^e-flat; whence K_n is A^e-projective. Thus R-dim $A \leq n$. Equality follows from the inductive hypothesis.

(*c*) *implies* (*a*)*:* It is clear that we may assume R is a local ring with maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m}$-dim [$A/\mathfrak{m}A = n$. By 1.3 we may assume that R is a hensel ring. Let $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ be as in (b) implies (a). Then $K_n/\mathfrak{m}K_n$ is $(A/\mathfrak{m}A)^e$-projective. By an application of the idempotent lifting theorem of G. Azumaya [8, Thm. 24, p. 138], we obtain a finitely generated, projective A^e-module P such that $\bar{f} : P/\mathfrak{m}P \cong K_n/\mathfrak{m}K_n$. Since P is A^e-projective, there exists an f in Hom_{A^e} (P, K_n) which is an epimorphism by Nakayama's Lemma and f is R-split since K_n is R-projective. By an application of Nakayama's Lemma, $P \cong K_n$. Together with the inductive hypothesis, this implies that R-dim $A = n$.

We now apply this theorem to extend some theorems proved for algebras over a field. Auslander [1, Prop. 14 + 15, p. 75] has shown that:

PROPOSITION 2.3. *If A is a finitely generated commutative algebra over a field R, R-dim $A = 0$ or R-dim $A = \infty$.*

THEOREM 2.4. (a) *If A is a finitely generated, commutative R-algebra, then R-dim $A \neq 1$.*

(b) *If A is a finitely generated, projective, commutative R-algebra, then R-dim $A = 0$ or R-dim $A = \infty$.*

Proof. (a) Assume R-dim $A = 1$; then $R/\mathfrak{m}$-dim $A/\mathfrak{m}A = 1$ for some $\mathfrak{m}$ in $\Omega(R)$ by Theorem 2.1 and Remark 2.2. Thus we contradict 2.3. (b) follows in a similar manner.

We also know from Theorem 1.1 that if R is a field, A is finitely generated over R, and R-dim $A < \infty$, then R-dim $A/N = 0$. We can generalize this as follows:

PROPOSITION 2.5. *If A is a finitely generated, projective R-algebra with R a semi-local ring and R-dim $A < \infty$, then R-dim $A/N = 0$.*

Proof. Clearly R-dim $A = n$ if and only if $R/\mathfrak{J}(R)$-dim $A/\mathfrak{J}(R) = n$. So by the application of the Chinese Remainder Theorem, 1.1, and 2.1, the result follows.

One of the distinguishing characteristics of one dimensional algebras over a field is the following [6, Thm. 8, p. 90]:

THEOREM 2.6. *Assume A is a finitely generated algebra over a field R with R-dim $A \leq 1$. Then for every bilateral ideal I of A, R-dim $A/I < \infty$.*

With the usual hypothesis of projectivity, we extend this theorem to the following, by means of Theorem 2.1:

PROPOSITION 2.7. *If A is a finitely generated, projective, faithful algebra over a semi-local ring R, R-dim $A \leq 1$, I a bilateral ideal of A such that $I \cap R = \mathfrak{a}$, and A/I is $R/\mathfrak{a}$-projective, then$R$-dim $A/I < \infty$.*

COROLLARY 2.7.1. *If $I \cap R = \mathfrak{m}$ in $\Omega(R)$ or if $I = \mathfrak{a}A$, the proposition holds.*

We can now prove the following results which show that under reasonably strong hypotheses, an n-dimensional R-algebra A is n-dimensional over it center $Z(A)$ and $Z(A)$ is R-separable and conversely. This mimics the result that a finitely generated R-algebra A is separable iff A is $Z(A)$-separable and $Z(A)$ is R-separable [2, Thm. 2.3, p. 374].

We first give a well-known result in the theory of cohomological dimension [7, Prop. 3, p. 78].

THEOREM 2.8. *If A is a finitely generated, projective, faithful S-algebra and S is a faithful R-algebra, then*

$$R\text{-}dim\ S \leq R\text{-}dim\ A \leq R\text{-}dim\ S + S\text{-}dim\ A$$

LEMMA 2.9. *Let A be a finitely generated, projective, faithful S-algebra and S a faithful R-algebra. If R-dim $S=0$ and R-dim $A=n$, then S-dim $A=n$.*

Proof. Assume that $\underline{X}=0\rightarrow P_n\rightarrow P_{n-1}\rightarrow\cdots\rightarrow A\rightarrow 0$ is a minimal A^e-projective resolution of A. Since S is R-separable, S is S^e-projective; hence $(\ )\otimes_{S^e}$ is exact. By the middle four interchange [10, p. 194], $(A\otimes_R A^{op})\otimes_{S^e}(S\otimes_S S)=A\otimes_S A^{op}$ and, by the pullback lemma [10, Lemma 1.2, p. 140], $\mu: S^e\rightarrow S$ being an epimorphism implies that $A\otimes_{S^e}S\cong A\otimes_S S\cong A$. Therefore, $\underline{X}\otimes_{S^e}S$ is an $A\otimes_S A^{op}$-projective resolution of A. So S-dim $A\leq n$. Assume S-dim $A=k<n$. Then R-dim $A=k<n$ by 2.8. Whence equality holds.

THEOREM 2.10. *Let A be a finitely generated, projective, faithful S-algebra and S a finitely generated, projective, faithful R-algebra. R-dim $A=n$ if and only if R-dim $S=0$ and S-dim $A=n$.*

Proof. If R-dim $A=n$, then R-dim $S\leq n$ by 2.8. Hence R-dim $S=0$ by 2.4(b). Now 2.9 shows that S-dim $A=n$.

If, on the other hand, R-dim $S=0$ and S-dim $A=n$, then R-dim $A\leq n$ by 2.8. Assume R-dim $A=k<n$, then S-dim $A=k<n$, by 2.9. Contradiction.

We note that the necessity of the hypothesis of finite generation is guaranteed by results of Rosenberg and Zelinsky [12].

Section 3. Structural results

DEFINITION 3.1. Let $A_1,\cdots,A_n$ be algebras over a fixed ring R and let M_{ij} be left A_i-and right A_j-bimodules with $M_{ij}=(0)$ for $i>j$, and $M_{ii}=A_i$. Assume that R commutes with the M_{ij}. We consider a family of left A_i-and right A_j-bihomomorphisms which satisfy the following properties:

$$\phi^t_{ij}: M_{it}\otimes_{A_t}M_{tj}\rightarrow M_{ij}$$

$$\phi^t_{it}: M_{it}\otimes_{A_t}A_t\cong M_{it}$$

$$\phi^i_{it}: A_i\otimes_{A_i}M_{it}\cong M_{it}$$

and commutative diagrams:

$$\begin{array}{ccc} M_{ij}\otimes_{A_j}M_{jt}\otimes_{A_t}M_{tk} & \xrightarrow{id_{ij}\otimes\phi^t_{jk}} & M_{ij}\otimes_{A_j}M_{jk} \\ \Big\downarrow \phi^j_{it}\otimes id_{tk} & & \Big\downarrow \phi^j_{ik} \\ M_{it}\otimes_{A_t}M_{tk} & \xrightarrow{\phi^t_{ik}} & M_{ik} \end{array}$$

Set $T_n(A_i;\ M_{ij}/R) = \left\{ \begin{pmatrix} a_{11} & m_{12} & \cdots m_{1n} \\ & a_{22} & \cdots m_{2n} \\ 0 & & \vdots \\ & & a_{nn} \end{pmatrix} :\ a_{ii} \text{ in } A_i;\ m_{ij} \text{ in } M_{ij} \right\}$

$T_n(A_i;\ M_{ij}/R)$ is an R-algebra under the operations:

(a) $(m_{ij}) \pm (m'_{ij}) = (m_{ij} \pm m'_{ij})$

(b) $(m_{ij})\cdot(m'_{ij}) = (\sum_t \phi^t_{ij}(m_{it} \otimes m'_{tj}))$

(c) $r(m_{ij}) = (rm_{ij}) = (m_{ij}r) = (m_{ij})r$

We call $T_n(A_i;\ M_{ij}/R)$ a generalized triangular matrix algebra over R.

The definition is similar to that given by M. Harada in [9, p. 465–466]. From [9, Cor. 2, p. 469] of Harada and Theorems 1.1 and 2.1, we have the following

PROPOSITOIN 3.2. *If A is a block triangular matrix algebra, that is, $A = T_n(A_i;\ M_{ij}/R)$ where A_i is an $(r_i \times r_i)$-matrix with coefficients in R and M_{ij} is an $(r_i \times r_j)$-matrix with coefficients in R, then R-dim $A = 1$.*

We now show that every finitely generated algebra A over a local hensel ring R such that R-dim $A \leq 1$ is a generalized triangular matrix algebra.

LEMMA 3.3. *Let A be a finitely generated algebra over a local hensel ring R with R-dim $A \leq 1$. Then there exists a complete set of mutually orthogonal idempotents $e_1, \cdots, e_n$ such that $e_iNe_j \subseteq \mathfrak{m}A$ for $i \geq j$.*

Proof. The idempotents may be lifted from a complete set of orthogonal idempotents in $A/\mathfrak{m}A$ by the idempotent lifting property of hensel rings [3, Thm. 24, p. 138]. The idempotents $\bar{e}_1, \cdots, \bar{e}_n$ in $A/\mathfrak{m}A$ satisfy the property that $\bar{e}_i(N/\mathfrak{m}A)\bar{e}_j = 0$ by [9, Lemma 7, p. 471] for $i \geq j$.

PROPOSITION 3.4. *Let A be a finitely generated algebra over a local hensel ring R such that A/N is a direct sum of division algebras D_i over $R/\mathfrak{m}$. R-dim $A \leq 1$ implies that A is a generalized triangular matrix algebra.*

Proof. By [9, Lemma 7 and Thm. 9, p. 471], there exist idempotents $\bar{e}_1, \cdots, \bar{e}_n$ in $A/\mathfrak{m}A$ which satisfy the properties:

(a) $\bar{e}_i(N/\mathfrak{m})\bar{e}_j = (0)$ for $i \geq j$

(b) $\bar{e}_i(A/\mathfrak{m}A)\bar{e}_i = D_i$

(c) $\bar{e}_i(A/\mathfrak{m}A)\bar{e}_j = 0$ for $i > j$,

since $R/\mathfrak{m}$-dim $A/mA \leq 1$ by Theorem 2.1 and Remark 2.2.

Applying Lemma 3.3, we obtain

(a′) $A = \bigoplus_R \sum_{i,j} e_iAe_j$

(b′) $e_iAe_j = 0$ for $i > j$ (apply Nakayama's lemma)

and (c′) $\phi_{it}^j : e_iAe_j \otimes_{e_jAe_j} e_jAe_t \to e_iAe_t$, defined by multiplication, are $e_iAe_i - e_jAe_j$ bimodule homomorphisms. Setting $M_{ij} = e_iAe_j$ and $A_i = e_iAe_i$, we have that $A \cong T_n(A_i;\ M_{ij}/R)$ where $a \to (e_iae_j)$.

Remark 3.5. In the above proposition, e_iAe_i are R-separable by 1.1 and if A is R-projective, so also are the e_iAe_j by (a′).

We now proceed as did Harada [9, p. 472] and define an induced generalized triangular matrix algebra.

Let $A = T_n(A_i;\ M_{ij}/R)$ be a generalized triangular matrix algebra over R. Let $\mathfrak{M}_{ij} = \underbrace{\begin{pmatrix} M_{ij} \cdots M_{ij} \\ \cdots\cdots \\ M_{ij} \quad M_{ij} \end{pmatrix}}_{s_j} \Big\} s_i = M_{ij}(s_i \times s_j)$. Each $\mathfrak{M}_{ij}$ can naturally be made into a left $(A_i)s_i$ and a right $(A_j)s_j$-bimodule, where $(A)s$ denotes the $(s \times s)$-matrices with entries from A.

Define $\phi_{ik}^j : ((X_{t,p})) \otimes_{(A_j)s_j} ((Y_{r,q})) \to ((\sum_p \phi_{ik}^j(X_{t,p} \otimes Y_{p,q}))$ to be the induced bilinear mappings. Then $B = T_n((A_i)_{s_i};\ \mathfrak{M}_{ij}/R)$ is called the generalized triangular matrix algebra induced from A.

THEOREM 3.6. *Let A be a finitely generated algebra over a local hensel ring R. If R-dim $A \leq 1$, then A is isomorphic to a generalized triangular matrix algebra induced from eAe, where e is the sum of a single representative from each isomorphism class of primitive orthogonal idempotents.*

Proof. Suppose $e = \sum_i e_{i1}$, where the first index denotes the idempotent class. Then eAe/eNe is isomorphic to a direct sum of division algebras. Whence $eAe = T_n(A_i;\ M_{ij}/R)$ by [9, Cor. 1, p. 465].

By the isomorphism of the idempotents $e_{ij}Ae_{kt} = e_iAe_k = M_{ik}$, for all j and for all t. Hence if we set s_i equal to the number of idempotents in the class of e_i, and set $\mathfrak{M}_{ij} = M_{ij}(s_i \times s_j)$, one can verify by straightforward computation that A is isomorphic to $T_n((A_i)_{s_i};\ \mathfrak{M}_{ij}/R)$.

Section 4. Miscellaneous Results and Conjectures

We have left the question of under what conditions a generalized triangular matrix algebra over a local hensel ring has dimension less than or equal to one. It remains unknown whether every one dimensional algebra, even over a local ring, is a generalized triangular matrix algebra. It is unknown, even in the case $n = 1$, whether the hypothesis that A is R-projective is necessary. This is particularly interesting in view of the fact that one may show the following not necessarily projective algebras have dimension 1 by constructing a dual basis for J, and using Theorem 1.4.

We construct such R-algebras in the following way. Set $\bar{R}$ equal to a non-zero homomorphic image of R.

$$\text{Let } A = \left\{ \left(\begin{array}{c|c} s_1 & s_2 \\ \hline 0 & \bar{s} \end{array} \right) : s_i \text{ in } \bar{R},\ \bar{s} \text{ in } R \right\}$$

$$B = \left\{ \left(\begin{array}{c|c|c} s_1 & s_2 & s_3 \\ \hline 0 & s_4 & s_5 \\ \hline 0 & 0 & \bar{s} \end{array} \right) : s_i \text{ in } \bar{R},\ \bar{s} \text{ in } R \right\}$$

$$C = \left\{ \left(\begin{array}{cc|c} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ \hline 0 & 0 & \bar{s} \end{array} \right) : s_i \text{ in } \bar{R},\ \bar{s} \text{ in } R \right\}$$

$$D = \left\{ \left(\begin{array}{c|cc} s_1 & s_2 & s_3 \\ \hline 0 & \bar{s}_1 & \bar{s}_2 \\ 0 & \bar{s}_3 & \bar{s}_4 \end{array} \right) : s_i \text{ in } \bar{R},\ \bar{s}_j \text{ in } R \right\}$$

Hence it is not unreasonable to conjecture that any block triangular matrix algebra with entries from R in the lower left block and entries from $\bar{R}$ in all the other non-zero blocks has R dimension one. This lends support to the possibility that every one dimensional algebra is generalized triangular and to the possibility that for $n = 1$, Theorem 2.1 is true without A being projective.

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