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VECTOR BUNDLES ON AN ELLIPTIC CURVE

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Introduction

Let k be an algebraically closed field of characteristic $p \geqq 0$, and let X be an abelian variety over k.

The goal of this paper is to answer the following questions, when $\dim(X) = 1$ and $p \neq 0$, posed by R. Hartshorne:

(1) Is $E^{(p)}$ indecomposable, when E is an indecomposable vector bundle on X?

(2) Is the Frobenius map $F^*: H^1(X, E) \to H^1(X, E^{(p)})$ injective?

We also partly answer the following question posed by D. Mumford:

(3) Classify, or at least say anything about, vector bundles on X when $\dim(X) > 1$.

Let us now summarize our results.

When the Hasse invariant of X is not zero, the answers to (1) and (2) are both affirmative.

When the Hasse invariant of X is zero and E is an indecomposable vector bundle on X of rank r and of degree d, then $E^{(p)}$ is indecomposable, if either $(r, d) = 1$ or $(r, d) \neq 1$ with $r/(r, d)$ divisible by p. Otherwise $E^{(p)}$ decomposes into a direct sum of $\min\{(r, d), p\}$ indecomposable components.

Also when the Hasse invariant of X is zero and E is an indecomposable vector bundle on X, the Frobenius map in (2) is not injective (and in fact the zero map), if and only if $r < p$, $d = 0$ and E has a non-zero section (i.e. in Atiyah's notation $E = E_{r,0}$ with $r < p$). It is surprising that F^* seldom fails to be injective.

When $\dim(X) = 1$, Atiyah [1] classified all the indecomposable vector bundles on X. He also gave the multiplicative structure in case $p = 0$. His construction of indecomposable vector bundles is essentially by succes-

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sive extensions of line bundles. To answer (1) and (2), however, it is very hard to keep track of these extensions after we pull them back by the Frobenius map.

We give here an entirely different way, inspired by Schwarzenberger's results [18] and [19], of constructing indecomposable vector bundles, which is very easy to handle and which gives us a clearer picture, we hope, especially in characteristic $p \neq 0$. This construction, by taking the direct image of line bundles by isogenies, can also be generalized to higher dimension, and thus partially answers the question (3).

When k is the field of complex numbers, Morikawa [10] characterized those indecomposable vector bundles, which we thus get on an abelian variety, or more generally a complex torus, in terms of their factor of automorphy. We shall re-interpret his result at the end of Section 1, after we give our construction.

There are lots of other simple vector bundles on an abelian variety of higher dimension. (cf. Our forthcoming paper, Vector bundles on abelian surfaces, in Inv. Math.)

We also remark here that R. Hartshorne [7] proved the following: *A vector bundle E on an elliptic curve X is ample, if and only if every quotient bundle of E has positive degree.*

He uses Atiyah's multiplicative structure when $p = 0$, and our answer to (2) when $p \neq 0$.

In Section 1 we state the results valid in arbitrary dimension. In Section 2 we apply these results to elliptic curves and solve the problems (1) and (2).

Notation and convention

Throughout this paper we denote by k an algebraically closed field of characteristic $p \geqq 0$.

We use the words vector bundle and locally free sheaf interchangeably. For vector bundles E and E' on a scheme X, we denote by $\mathcal{E}nd_{\mathcal{O}_X}(E)$ and $\mathcal{H}om_{\mathcal{O}_X}(E, E')$ the sheaves of $\mathcal{O}_Y$-endomorphisms and $\mathcal{O}_X$-homomorphisms, while $\mathrm{End}_{\mathcal{O}_X}(E)$ and $\mathrm{Hom}_{\mathcal{O}_X}(E, E')$ mean the sets of global $\mathcal{O}_X$-endomorphisms and $\mathcal{O}_X$-homomorphisms. $\check{E} = \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is the dual vector bundle.

For an abelian variety X, we denote by X^t the dual abelian variety. For an isogeny $\varphi : Y \to X$, φ^t is its dual isogeny $X^t \to Y^t$.

When X is an elliptic curve over k with $p \neq 0$, we denote by Hasse (X) the Hasse invariant of X, that is, Hasse $(X) \neq 0$, if there are p k-valued points of order p on X, while Hasse $(X) = 0$, if there is none besides 0.

Section 1 Vector bundles on an abelian variety

Let $\varphi : Y \to X$ be an isogeny of g-dimensional abelian varieties over k with scheme-theoretic kernel$(\varphi) = G$. Let E be a vector bundle on Y. Since φ is finite and flat, it is obvious that $\varphi_* E$ is a vector bundle on X. We should like to compute $\mathrm{End}_{\mathcal{O}_X}(\varphi_* E)$. First of all it canonically contains $\mathrm{End}_{\mathcal{O}_Y}(E)$.

We have the cartesian diagram

$$\begin{array}{ccc} Y \times G & \xrightarrow{\ \mu\ } & Y \\ {\scriptstyle p_1}\downarrow & \varphi & \downarrow{\scriptstyle \varphi} \\ Y & \longrightarrow & X \end{array}$$

where $\mu : Y \times G \to Y$ is the restriction of the group law $Y \times Y \to Y$, and p_1 is he projection. Since φ is an isogeny, hence affine, we have $\varphi^* \varphi_* E \cong p_{1*} \mu^* E$. Thus by the adjointness of φ^* and φ_*, we get $\mathrm{End}_{\mathcal{O}_X}(\varphi_* E) = \mathrm{Hom}_{\mathcal{O}_Y}(\varphi^* \varphi_* E, E)$ $= \mathrm{Hom}_{\mathcal{O}_Y}(p_{1*} \mu^* E, E) = H^0(Y, [p_{1*} \mu^* E \otimes_{\mathcal{O}_Y} \check{E}]^{\vee})$. Since the canonical line bundle Ω_Y^g is isomorphic to $\mathcal{O}_Y$, this latter is, by Serre duality, dual to $H^g(Y, p_{1*} \mu^* E \otimes_{\mathcal{O}_Y} \check{E})$. By the projection formula, $p_{1*} \mu_* E \otimes_{\mathcal{O}_Y} \check{E} = p_{1*}(\mu^* E \otimes_{\mathcal{O}_{Y \times G}} p_1^* \check{E})$. Moreover p_1 is finite, hence the cohomology group above is isomorphic to $H^g(Y \times G,\ \mu^* E \otimes_{\mathcal{O}_{Y \times G}} p_1^* \check{E})$.

Let L be a line bundle on Y. As in Mumford [13], we denote by $\Lambda(L) : Y \to Y^t$ the homomorphism sending a point y in Y to $T_y^* L \otimes_{\mathcal{O}_Y} L^{-1}$, where $T_y : Y \to Y$ is the translation.

Moreover, suppose C is a vector bundle on Y, such that on $\mu^* C \cong p_1^* C$ on $Y \times G$. Let us apply our previous calculation to $E = L \otimes_{\mathcal{O}_Y} C$. Then $\mu^* E \otimes p_1^* \check{E} = (\mu^* L \otimes p_1^* L^{-1}) \otimes_{\mathcal{O}_{Y \times G}} p_1^* \mathcal{E}nd_{\mathcal{O}_Y}(C)$. By the definition of $\lambda = \Lambda(L) : Y \to Y^t$, we know that

$$\mu^* L \otimes_{\mathcal{O}_{Y \times G}} p_1^* L^{-1} = (1 \times \lambda \circ j)^* \mathcal{P}$$

where $j : G \to Y$ is the injection and $\mathcal{P}$ is the normalized Poincaré line bundle on $Y \times Y^t$, that is, the universal family of line bundles algebraically equivalent to zero parametrized by Y^t, and so normalized that $\mathcal{P}|Y\times\{0\} \cong \mathcal{O}_Y$ and $\mathcal{P}|\{0\}\times Y^t \cong \mathcal{O}_{Y^t}$.

Thus $\mathrm{End}_{\mathcal{O}_X}(\varphi_* E)$ is dual to

$$H^g(Y\times G, (1\times\lambda\circ j)^*[\mathcal{P}\otimes_{\mathcal{O}_{Y\times Y^t}} p_1^*\mathcal{E}nd_{\mathcal{O}_Y}(C)]),$$

which is isomorphic, since G is affine, to

$$H^0(G, R^g p_{2*}(1\times\lambda\circ j)^*[\mathcal{P}\otimes_{\mathcal{O}_{Y\times Y^t}} p_1^*\mathcal{E}nd_{\mathcal{O}_Y}(C)]).$$

By the base change theorem (EGA III §7, Mumford [14]), this is isomorphic to

$$H^0(G, (\lambda\circ j)^* R^g p_{2*}[\mathcal{P}\otimes_{\mathcal{O}_{Y\times Y^t}} p_1^*\mathcal{E}nd_{\mathcal{O}_Y}(C)]).$$

We now state a slight generalization of an important result of Mumford.

DEFINITION. A vector bundle U on a scheme Y is called *unipotent*, if U has a filtration such that the successive quotients are all isomorphic to $\mathcal{O}_Y$.

It is straightforward to see that U is unipotent, if any only if the transition functions can simultaneously be chosen to be upper triangular matrices with 1 along the diagonal.

LEMMA 1.1 Let U be a unipotent vector bundle on a g-dimensional abelian variety over k, and let $\mathcal{P}$ be the normalized Poincaré line bundle on $Y\times Y^t$. Then $R^i p_{2*}[\mathcal{P}\otimes_{\mathcal{O}_{Y\times Y^t}} p_1^* U]$ is the zero sheaf on Y^t, when $i\neq g$. On the other hand, $R^g p_{2*}[\mathcal{P}\otimes_{\mathcal{O}_{Y\times Y^t}} p_1^* U]$ has support at the origin 0 of Y^t, and the stalk there is a rank(U)-dimensional vector space over k. Moreover, the stalk modulo the maximal ideal of $\mathcal{O}_{Y^t,0}$ is isomorphic to $H^g(Y,U)$.

Proof. Mumford has shown this when $U=\mathcal{O}_Y$ in [14]. Since U has a filtration with successive quotients all isomorphic to $\mathcal{O}_Y$, we easily get the first two statements. As for the last, we use the base change theorem.

We now return to our previous situation and suppose further that $\mathcal{E}nd_{\mathcal{O}_Y}(C)$ is unipotent. Examples of vector bundles C with this condition and the previous $\mu^*C\cong p_1^*C$ on $Y\times G$ are the following:

(i) C is a direct sum of vector bundles of the form $L'\otimes E_{r,0}$ with L' algebraically equivalent to zero. (cf. Atiyah [1] and our Section 2).

(ii) k is the field of complex numbers, and C is a vector bundle with a holomorphic (integrable) connection. (cf. Matsushima [9] and Morimoto [11]).

Under this further condition, we know by Lemma 1.1 that

$$R^q p_{2*}[\mathscr{P} \otimes p_1^* \mathscr{E}nd_{\mathscr{O}_Y}(C)]$$

has support at the origin of Y^t, and the stalk there is of dimension rank $(\mathscr{E}nd_{\mathscr{O}_Y}(C)) = [\text{rank}(C)]^2$ over k. Moreover, modulo the maximal ideal of $\mathscr{O}_{Y^t,0}$, this is isomorphic to $H^q(Y, \mathscr{E}nd_{\mathscr{O}_Y}(C))$, which is dual to $H^0(Y, \mathscr{E}nd_{\mathscr{O}_Y}(C)) = \text{End}_{\mathscr{O}_Y}(C)$. On the other hand, we have seen that $\text{End}_{\mathscr{O}_Y}(\varphi_*(L \otimes_{\mathscr{O}_Y} C))$ canonically contains $\text{End}_{\mathscr{O}_Y}(L \otimes_{\mathscr{O}_Y} C) = \text{End}_{\mathscr{O}_Y}(C)$, and that it is dual to

$$H^0(G, (\lambda \circ j)^* R^q p_{2*}[\mathscr{P} \otimes_{\mathscr{O}_{Y \times Y^t}} p_1^* \mathscr{E}nd_{\mathscr{O}_Y}(C)]).$$

Thus counting the dimension we finally get the following:

THEOREM 1.2 Let $\varphi : Y \to X$ be an isogeny of g-dimensional abelian varieties over k, and let L be a line bundle on Y such that the restriction of $\Lambda(L) : Y \to Y^t$ to the (scheme-theoretic) kernel of φ is an isomorphism. Then

(i) $\text{End}_{\mathscr{O}_X}(\varphi_* L) = k$. Especially $\varphi_* L$ is an indecomposable vector bundle on X.

(ii) Suppose φ is separable. If C is a vector bundle on Y such that $\mathscr{E}nd_{\mathscr{O}_Y}(C)$ is unipotent and that $T_a^* C \cong C$ for all k-valued points a of $\ker(\varphi)$, then $\text{End}_{\mathscr{O}_X}(\varphi_*(L \otimes_{\mathscr{O}_Y} C))$ is canonically isomorphic to $\text{End}_{\mathscr{O}_Y}(C)$.

Remark. (i) From what we have seen above, it is easy to see that the second statement of the theorem is false in general when $C \neq \mathscr{O}_Y$, and φ is inseparable.

(ii) Even when the restriction $\lambda \circ j$ of $\Lambda(L)$ to $G = \ker(\varphi)$ is not an isomorphism, we can identify the algebra $\text{End}_{\mathscr{O}_X}(\varphi_*(L \otimes_{\mathscr{O}_Y} C))$ in the following way:

As in Mumford [13], let $H(L)$ be the kernel of $\Lambda(L)$ and let $K = K(L) = G \cap H(L)$ be the kernel of $\lambda \circ j$. We denote by $D(K)$ the Cartier dual group scheme of K, and by $A(D(K))$ the affine k-algebra of $D(K)$, that is, $D(K) = \text{Spec}(A(D(K)))$.

By our previous calculation, we see that $\text{End}_{\mathscr{O}_X}(\varphi_*(E))$ is isomorphic, as a k-vector space, to $A(D(K)) \otimes_k \text{End}_{\mathscr{O}_Y}(C)$, where $E = L \otimes_{\mathscr{O}_Y} C$.

We now identify its algebra structure.

We denote by $\mu' : Y\times K \to Y$ the restriction of the group law. By assumption, we have $\mu' E \cong p_1^* E$ on $Y\times K$. Then there is an obstruction $O(E)$ in the group cohomology $H^2_{gr}(K, \mathbf{Aut}_{\mathcal{O}_Y}(E))$, where K acts canonically on the automorphism functor $\mathbf{Aut}_{\mathcal{O}_Y}(E)$. This is the obstruction to the existence of an isomorphism between $\mu' E$ and $p_1^* E$ on $Y\times K$ which satisfies the cocycle condition, i.e. the obstructilon to the descent of E by $Y \to Y/K$. (cf. FGA exposé 190). Then we can show that the algebra structure on $A(D(K))\otimes_k \mathrm{End}_{\mathcal{O}_Y}(C)$ is the ordinary one twisted by multiplying the 2-cocycle $0(E)$.

To see this, let us for simplicity assume that φ is a separable isogeny, i.e. $G = \ker(\varphi)$ is reduced. Then

$$\begin{aligned}\mathrm{End}_{\mathcal{O}_X}(\varphi_* E) &= \mathrm{Hom}_{\mathcal{O}_Y}(p_{1*}\mu^* E, E) = \bigoplus_{a\in G} \mathrm{Hom}_{\mathcal{O}_Y}(T_a^* E, E)\\ &= \bigoplus_{a\in G} \mathrm{End}_{\mathcal{O}_Y}(C)\otimes_k \mathrm{Hom}_{\mathcal{O}_Y}(T_a^* L, L).\end{aligned}$$

K is the subgroup of G of elements a such that $T_a^* L\otimes L^{-1} \cong \mathcal{O}_Y$, that is, those for which $\mathrm{Hom}_{\mathcal{O}_Y}(T_a^* L, L)$ above is not zero. (cf. Mumford [14]. See also Lemma 1.1.). Let us fix isomorphisms $w(a) : T_a^* E \to E$ for a in K. Then

$$\mathrm{End}_{\mathcal{O}_X}(\varphi_* E) = \bigoplus_{a\in K} \mathrm{End}_{\mathcal{O}_Y}(E)\cdot w(a)$$

which is isomorphic, as a k-vector space, to $A(D(K))\otimes_k \mathrm{End}_{\mathcal{O}_Y}(E)$. Note that $A(D(K))$ is now isomorphic to the group algebra of K over k. The algebra structure is defined as follows:

$$w(a)\cdot w(b) = \rho(a, b) w(a+b)$$

where $\rho(a, b) = w(a)\circ T_a^*(w(b))\circ w(a+b)^{-1}$ is in $\mathrm{Aut}_{\mathcal{O}_Y}(E)$ and is a 2-cocycle, which determines $0(E)$ in $H^2_{gr}(K, \mathrm{Aut}_{\mathcal{O}_Y}(E))$.

As one application of this, we see that $\mathrm{End}_{\mathcal{O}_X}(\varphi_*\mathcal{O}_Y)$ is canonically isomorphic, as a k-algebra, to $A(D(G))$. We will later identify $\varphi_*\mathcal{O}_Y$ more explicitly. (cf. Remark after (1.7))

We leave the application of Theorem 1.2 to elliptic curves to Section 2. We later show that when $g=1$, we get all the *stable* vector bundles as in Theorem 1.2 (i). We believe there are lots of other simple vector bundles when $g\neq 1$. Possibly we need direct image of line bundles by a finite ramified coverings. But then we no longer have abelian varieties above. cf. Lang and Serre [8], and Schwarzenberger [18] and [19]. We can also apply our method in Theorem 1.2 (i) to unramified coverings of a non-singular curve. Then we get an algebraic family of simple vector bundles. cf. Our remark after Propositions 1.3 and 2.3 and [15], [20].

PROPOSITION 1.3 Let $\varphi : Y \to X$ be an isogeny of g-dimensional abelian varieties over k, and let L be a line bundle on Y such that $\Lambda(L)$ induces an isomorphism on $G = \ker(\varphi)$. Suppose M is another line bundle on Y algebraically equivalent to L, i.e. $\Lambda(L) = \Lambda(M)$. Then $\varphi_* L \cong \varphi_* M$, if and only if there exists a closed point a in G such that $L \cong T_a{}^* M$, where $T_a : Y \to Y$ is the translation by a. Otherwise $\mathrm{Hom}_{\mathcal{O}_X}(\varphi_* L, \varphi_* M) = 0$.

Proof. In the same way as in the proof of Theorem 1.2, we see that $\mathrm{Hom}_{\mathcal{O}_X}(\varphi_* L, \varphi_* M)$ is dual, as a k-vector space, to

$$H^0(G, (\lambda \circ j)^* R^g p_{2*}[\mathcal{P} \otimes_{\mathcal{O}_{Y \times Y^t}} p_1{}^*(L \otimes_{\mathcal{O}_Y} M^{-1})]).$$

But since L and M are algebraically equivalent, there exists a point b in Y^t such that $\mathcal{P} \otimes_{\mathcal{O}_{Y \times Y^t}} p_1{}^*(L \otimes_{\mathcal{O}_Y} M^{-1}) \cong (1 \times T_b)^* \mathcal{P}$. Thus the cohomology group is equal to $H^0(G, (T_b \circ \lambda \circ j)^* R^g p_{2*}(\mathcal{P}))$. This is 1-dimensional if $T_b \circ \lambda(G)$ contains 0 of Y^t, and otherwise it is zero, by Lemma 1.1. The rest of the proof is easy. Q.E.D.

This Proposition shows that when we vary the line bundle L in its universal algebraic family and make $\varphi_* L$, isomorphic vector bundles on X occur as often as the number of closed points in G.

THEOREM 1.4. Let $\varphi : Y \to X$ be an isogeny of g-dimensional abelian varieties over k, with $\ker(\varphi) = G$. Let L and L' be line bundles on Y and Y^t, respectively. Denote by $\mathcal{P}$ the normalized Poincaré line bundle on $Y \times Y^t$. We also denote $\lambda = \Lambda(L) : Y \to Y^t$, $\lambda' = \Lambda(L') : Y^t \to Y$, $G' = \lambda(G)$, and $\pi : Y^t \to X' = Y^t / G'$. Assume $\lambda' \circ \lambda$ induces the identity map on G. Then the vector bundle $(\varphi \times 1_{Y^t})_* (p_1{}^* L \otimes_{\mathcal{O}_{Y \times Y^t}} p_2{}^* L')$ on $X \times Y^t$ descends via $1_X \times \pi : X \times Y^t \to X \times X'$.

Remark. This Theorem 1.4 says that there exists an algebraic family of vector bundles on X parametrized by an abelian variety X', in which isomorphic ones appear only once.

Proof. For simplicity we denote $Z = Y \times Y^t$, $M = p_1{}^* L \otimes_{\mathcal{O}_Z} \mathcal{P} \otimes_{\mathcal{O}_Z} p_2{}^* L'$, and $E' = (\varphi \times 1)_* M$. First of all $\Lambda(M) : Z = Y \times Y^t \to Z^t = Y^t \times Y$ is the map sending a point (y, v) in Z to $(v + \lambda(y),\ y + \lambda'(v))$. Hence $\Lambda(M)$ sends a point $(a, 0)$ in $G\times\{0\} = \ker(\varphi\times 1)$ to $(\lambda(a), a)$, and thus is an isomorphism on $G\times\{0\}$. Thus applying Theorem 1.2 to M and $\varphi\times 1$, we see that $\mathrm{End}_{\mathcal{O}_{X\times Y^t}}(E)=k$.

Consider the cartesian diagram

$$\begin{array}{ccc} Y\times X' & \xleftarrow{1\times\pi} & Y\times Y^t = Z \\ \downarrow{\scriptstyle \varphi\times 1} & & \downarrow{\scriptstyle \varphi\times 1} \\ X\times X' & \xleftarrow{1\times\pi} & X\times Y^t \end{array}$$

We first show that $(\varphi\times 1)^*E' = (\varphi\times 1)^*(\varphi\times 1)_*M$ is isomorphic on Z to $(1\times\pi)^*(1\times\pi)_*M$.

Let us simply write G and G' the subgroup schemes $G\times\{0\}$ and $\{0\}\times G'$ of Z. Let us also write $\mu, p_1 : Z\times G \rightrightarrows Z$ and $\mu', p_1' : Z\times G' \rightrightarrows Z$, the actions and the projections. Since the diagram

$$\begin{array}{ccc} Z & \xleftarrow{\mu} & Z\times G \\ \downarrow{\scriptstyle \varphi\times 1} & & \downarrow{\scriptstyle p_1} \\ X\times Y^t & \xleftarrow[\varphi\times 1]{} & Z \end{array}$$

is cartesian, we get $(\varphi\times 1)^*(\varphi\times 1)_*M = p_{1*}\mu^*M = p_{1*}(1\times(\lambda, j))^*(p_1{}^*M\otimes_{\mathcal{O}_{Z\times Z^t}}\mathscr{R})$, where $\mathscr{R}$ is the normalized Poincaré line bundle on $Z\times Z^t$, and $(\lambda, j) : G\to Z^t = Y^t\times Y$ sending a to $(\lambda(a), a)$ is equal to the restriction of $\Lambda(M)$ to $G\times\{0\}=G$. Similarly we get $(1\times\pi)^*(1\times\pi)_*M=p_{1*}'[(1\times(j', \lambda'))^*(p_1{}^*M\otimes_{\mathcal{O}_{Z\times Z^t}}\mathscr{R})]$, where $(j', \lambda') : G'\to Z^t$ is the restriction of $\Lambda(M)$ to G'.

By assumption, λ and λ' are inverse to each other on G and G'. Thus the diagram

$$\begin{array}{ccccc} Z & \xleftarrow{p_1} & Z\times G & \xrightarrow{1\times(\lambda, j)} & Z\times Z^t \\ \| & & {\scriptstyle 1\times\lambda}\downarrow\ \uparrow{\scriptstyle 1\times\lambda'} & & \| \\ Z & \xleftarrow{p_1'} & Z\times G' & \xrightarrow{1\times(j', \lambda')} & Z\times Z^t \end{array}$$

commutes. Thus we get $(\varphi\times 1)^*(\varphi\times 1)_*M = (1\times\pi)^*(1\times\pi)_*M$.

As the first step of the descent, we next show that by $\mu', p_1' : (X\times Y^t)\times G' \rightrightarrows (X\times Y^t)$, μ'^*E' is isomorphic to $p_1'^*E'$. But from the cartesian diagram

$$\begin{array}{ccc} Z & \overset{\mu'}{\underset{p_1'}{\leftleftarrows}} & Z\times G' \\ \downarrow{\scriptstyle \varphi\times 1} & & \downarrow{\scriptstyle \varphi\times 1\times 1} \\ X\times Y^t & \overset{\mu'}{\underset{p_1'}{\leftleftarrows}} & (X\times Y^t)\times G' \end{array}$$

we have $\mu'^*E' = \mu'^*(\varphi\times 1)_*M = (\varphi\times 1\times 1)_*\mu'^*M$
$= (\varphi\times 1\times 1)_*(1\times(j',\lambda'))^*(p_1{}^*M\otimes_{\mathcal{O}_{Z\times Z^t}}\mathscr{P})$
$= (\varphi\times 1\times 1)_*(1\times\lambda')^*(1\times(\lambda, j))^*(p_1^*M\otimes_{\mathcal{O}_{Z\times Z^t}}\mathscr{P})$
$= (\varphi\times 1\times 1)_*(1\times\lambda')^*\mu^*M = (1\times 1\times\lambda')^*(\varphi\times 1\times 1)_*\mu^*M$ from what we have seen above. But since the diagram

$$\begin{array}{ccc} Z & \xleftarrow{\quad\mu\quad} & Z\times G \\ {\scriptstyle\varphi\times 1}\downarrow & & \downarrow{\scriptstyle\varphi\times 1\times 1} \\ X\times Y^t & \xleftarrow{\quad p_1\quad} & (X\times Y^t)\times G \end{array}$$

is cartesian, this latter is equal to
$(1\times 1\times\lambda')^*p_1{}^*(\varphi\times 1)_*M = p_1'^*(\varphi\times 1)_*M = p_1'^*E'$, and we are done.

Thus to show that E' descends via $1_X\times\pi$, we have to check the existence of an isomorphism with cocycle conditions, in other words, the vanishing of the obstruction $0(E')$ in $H^2_{gr}(G'\,\mathbf{Aut}_{\mathcal{O}_{X\times Y^t}}(E'))$. (cf. Remark after Theorem 1.2) But we know that $(\varphi\times 1)^*0(E') = 0((\varphi\times 1)^*E')$ in $H^2_{gr}(G', \mathbf{Aut}_{\mathcal{O}_Z}((\varphi\times 1)^*E')$ vanishes, since we have shown that $(\varphi\times 1)^*E'$ descends via $1_Y\times\pi : Z\to Y\times X'$. Thus it is enough to show that $(\varphi\times 1)^* : H^2_{gr}(G', \mathbf{Aut}_{\mathcal{O}_{X\times Y^t}}(E'))\to H^2_{gr}(G', \mathbf{Aut}_{\mathcal{O}_Z}((\varphi\times 1)^*E'))$ is injective. From what we have seen before, it is easy to show that $\mathbf{End}_{\mathcal{O}_{X\times Y^t}}(E') = G_a$, thus $\mathbf{Aut}_{\mathcal{O}_{X\times Y^t}}(E') = G_m$, with G' acting trivially on these. It is also not hard to see that $\mathbf{End}_{\mathcal{O}_Z}((\varphi\times 1)^*E') = \boldsymbol{A}(G')$, and thus $\mathbf{Ant}_{\mathcal{O}_Z}((\varphi\times 1)^*E') = \boldsymbol{A}(G')^*$ with G' acting trivially on these. Here, for a k-algebra A, we define the ring functor $\boldsymbol{A}$ and the group functor $\boldsymbol{A}^*$ as follows: for a k-prescheme S, $\boldsymbol{A}(S)$ is the ring $A\otimes_k H^0(S, \mathcal{O}_S)$, and $\boldsymbol{A}^*(S)$ is the multiplicative group of invertible elements in it. In our case $A = A(G')$ is a commutative finite k-algebra. Since A_{red} is a direct sum of k as a k-algebra, the surjection $A\to A_{red}$ followed by a projection to one of the factors gives a splitting of the injection $G_m = \boldsymbol{k}^*\to\boldsymbol{A}^*$. Thus we are done.

The choice of E on $X\times X'$ such that $(1_Y\times\pi)^*E\cong E'$ is not unique. By descent theory, they correspond to the elements of

$$H^1_{gr}(G', \mathbf{Aut}_{\mathcal{O}_{X\times Y^t}}(E')) = \mathrm{Hom}_{gr}(G', G_m) = \ker[X'^t(k)\to Y(k)]$$

by Cartier's duality theorem. (cf. e.g. Oda [16], Section 1)
It is easy to see that this corresponds to the fact that E and $E\otimes_{\mathcal{O}_{X\times X'}} p_2{}^*L_0$ give the same vector bundle on $X\times Y^t$ by the pull back by $(1_X\times\pi)$, where L_0 is a line bundle on X' such that $\pi^*L_0\cong\mathcal{O}_{Y^t}$. Q.E.D.

THEOREM 1.5 Let $\varphi : Y \to X$ be an isogeny of abelian varieties over k, and let $\varphi^t : X^t \to Y^t$ be its dual. If $\mathscr{P}$ and $\mathscr{Q}$ are the normalized Poincaré line bundles on $X \times X^t$ and $Y \times Y^t$, respectively, then $(\varphi \times 1_{Y^t})_* \mathscr{Q}$ and $[(1_X \times \varphi^t)_* \mathscr{P}^{-1}]^\vee$ are isomorphic simple vector bundles on $X \times Y^t$.

Proof. Consider the cartesian diagram

$$\begin{array}{ccc} Y \times Y^t & \xleftarrow{1 \times \varphi^t} & Y \times X^t \\ {\scriptstyle \varphi \times 1} \downarrow & & \downarrow {\scriptstyle \varphi \times 1} \\ X \times Y^t & \xleftarrow[1 \times \varphi^t]{} & X \times X^t \end{array}$$

We know that $(1 \times \varphi^t)^* \mathscr{Q} \cong (\varphi \times 1)^* \mathscr{P}$ on $Y \times X^t$ and they are the normalized line bundles corresponding to the divisorial correspondences $\varphi : Y \to X$ and $\varphi^t : X^t \to Y^t$. Let us denote these isomorphic line bundles by M. Thus $\Lambda(M) : Y \times X^t \to (Y \times X^t)^t \cong X \times Y^t$ coincides with $\varphi \times \varphi^t$. Also let us denote $G = \ker(\varphi)$ and $G' = \ker(\varphi^t)$. Thus $H(M) = \ker \Lambda(M) = G \times G'$. We also know (cf. Oda [16], Lemma 1.4) that G and G' are Cartier dual to each other, the non-degenerate pairing $\langle\ ,\ \rangle : G \times G' \to G_m$ being induced by the alternating biadditive pairing e^M on $H(M)$.

$$\begin{array}{ccc} G \times G' & \xrightarrow{\langle\ ,\ \rangle} & G_m \\ {\scriptstyle i \times i'} \downarrow & & \| \\ H(M) \times H(M) & \xrightarrow{e^M} & G_m \end{array}$$

For simplicity we denote $Z = Y \times X^t$. For a k-prescheme S, we denote by Z_S the base extension $Z \times S$. M_S is the pull back of M on Z_S. Then for any S-valued points a and b of G and a' and b' of G', we have an isomorphism

$$\rho(a + a') : M_S \to T^*_{a+a'} M_S$$

satisfying $(T^*_{a+a'} \rho(b + b')) \circ \rho(a + a') = \langle a, b' \rangle \rho(a + b + a' + b')$. The 2-cocycle $\langle a, b' \rangle$ gives the obstruction $0(M)$ in $H^2_{gr}(H(M), G_m)$ and $e^M(a + a', b + b') = \langle a, b' \rangle \langle b, a' \rangle^{-1}$. (See the remark after Theorem 1.2) Moreover, ρ satisfies the cocycle condition on G and G' separately. $\mathscr{P}$ and $\mathscr{Q}$ are the corresponding descent of M to $Y \times Y^t$ and $X \times X^t$, respectively. Therefore our theorem is reduced to the following more general situation:

Let M be a line bundle on an abelian variety Z. Suppose $H(M) = \ker \Lambda(M)$ contains a product $G \times G'$ of finite group schemes with a non-degenerate pairing $\langle\ ,\ \rangle : G \times G' \to G_m$. Suppose further that for any k-prescheme S and any S-valued points a and b of G and a' and b' of G', there is an isomorphism

$$\rho(a + a') : M_S \to T^*_{a+a'} M_S$$

satisying $(T^*_{a+a'}\rho(b + b')) \circ \rho(a + a') = \langle a, b'\rangle \rho(a + b + a' + b')$. Let us denote $\psi : Z \to Y = Z/G'$, $\psi' : Z \to Y' = Z/G$ and $X = Z/G \times G'$ with the projections $\varphi : Y \to X$ and $\varphi' : Y' \to X$. Then since ρ satisfies the cocycle condition on G and G' separately, M descends to line bundles L and L' on Y and Y' respectively. Then

LEMMA 1.6 $\varphi'_* L'$ and $[\varphi_* L^{-1}]^\vee$ are isomorphic vector bundles on X.

Proof. It is easy to see that the diagram

$$\begin{array}{ccc} Y & \xleftarrow{\ \psi\ } & Z \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ X & \xleftarrow[\ \varphi'\ ]{} & Y' \end{array}$$

is cartesian. Let us consider the pull back of $\varphi_* L$ and $\varphi'_* L'$ by $\varphi \circ \psi = \varphi' \circ \psi'$. From the diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\ \psi\ } & Z & \xleftarrow{\ \mu\ } & Z \times G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle p_1} \\ X & \xleftarrow[\ \varphi'\ ]{} & Y' & \xleftarrow[\ \psi'\ ]{} & Z \end{array}$$

we see that $\psi^* \varphi^* (\varphi_* L) = \psi'^* \varphi'^* (\varphi_* L) = A(G) \otimes_k M$. Similarly, we get $\psi'^* \varphi'^* (\varphi'_* L') = A(G') \otimes_k M$.

Let a and a' be S-valued points of G and G' respectively. They define the translations $T_a : G_S \to G_S$ and $T_{a'} : G'_S \to G'_S$, hence algebra automorphisms $\tau(a) = T^*_a : A(G)_S \to A(G)_S$ and $\tau'(a') = T^*_{a'} : A(G')_S \to A(G')_S$. Moreover, the non-degenerate pairing $\langle\ ,\ \rangle : G \times G' \to G_m$ defines an isomorphism $G \cong D(G')$. Thus the automorphism $D(T_{a'}) : D(G')_S \to D(G')_S$ defines a coalgebra automorphism $\sigma(a') = D(\tau'(a')) : A(G)_S \to A(G)_S$. Since $G'(S) = D(G)(S) = \mathrm{Hom}_{S\text{-}gr}(G, G_m)$, it is easy to see that $\sigma(a')$ is equal to the multiplication in $A(G)_S$ of the character defined by a'. We define the coalgebra automorphism $\sigma'(a) =$

$D(\tau(a)) : A(G')_S \to A(G')_S$ in a similar manner. From what we have seen above, we can easily show that

$$\tau(a) \circ \sigma(a') = \langle a, a'\rangle \sigma(a') \circ \tau(a)$$
$$\tau'(a') \circ \sigma'(a) = \langle a, a'\rangle \sigma'(a) \circ \tau'(a').$$

In view of the diagram above, the descent data on $A(G)\otimes_k M$ corresponding to the pull back of $\varphi_* L$ by $\varphi \circ \psi$ is easily seen to be given by

$$\tau(a) \circ \sigma(a') \otimes \rho(a + a') : (A(G)\otimes_k M)_S \to T^*_{a+a'}(A(G)\otimes_k M)_S$$

for S-valued points a and a' of G and G' respectively. Taking into account the skew-symmetry of the Cartier duality for isogenies, we can similarly show that the descent data on $A(G')\otimes_k M$ corresponding to the pull back of $\varphi'_* L'$ by $\varphi' \circ \psi'$ is defined by

$$\begin{array}{c} \tau'(a') \circ \sigma'(-a) \otimes \langle a, a'\rangle \rho(a + a') : (A(G')\otimes_k M)_S \to T^*_{a+a'}(A(G')\otimes_k M)_S. \\ \| \\ \sigma'(-a) \circ \tau'(a') \otimes \rho(a + a') \end{array}$$

The descent data on $D(A(G))\otimes_k M$ corresponding to the pull back of $[\varphi_* L^{-1}]^\vee$ by $\varphi \circ \psi$ is defined by $D(\tau(a) \circ \sigma(-a'))^{-1} \otimes \rho(a + a') = D(\tau(-a)) \circ D(\sigma(a')) \otimes \rho(a + a')$. Since $D(A(G)) = A(D(G))$ is isomorphic via the pairing $\langle\ ,\ \rangle$ to $A(G')$, and since $D(\tau(-a))$ and $D(\sigma(a'))$ correspond to $\sigma'(-a)$ and $\tau'(a')$ respectively, it is easy to see that the isomorphism above $D(A(G)) \cong A(G')$ induces an isomorphism from $D(A(G))\otimes_k M$ to $A(G')\otimes_k M$ which commutes with the descent data corresponding to $[\varphi_* L^{-1}]^\vee$ and $\varphi'_* L'$. Thus by the fundamental theorem of descent theory (FGA, exposé 190) we see that $\varphi'_* L'$ and $[\varphi_* L^{-1}]^\vee$ are isomorphic on X. Q.E.D.

Remark. If we apply the duality theorem for the finite morphism $\varphi : Y \to X$ (cf. Hartshorne [6], Chap. III. § 6), we see that

$$[\varphi_* L^{-1}]^\vee = \varphi_*(L \otimes_{\mathcal{O}_Y} \varphi^! \mathcal{O}_X).^{(*)}$$

Consider a finite k-scheme $s : S \to \operatorname{Spec}(k)$ and its base change $p_1 : X \times S \to X$. Since the functor $s \to s^!$ commutes with the flat base change, we see that

(*) (*Added in proof*) We have $\varphi^! \mathcal{O}_X = \mathcal{O}_Y$ for an isogeny $\varphi : Y \to X$ of abelian varieties, thanks to the compatipility of the upper shriek functor under composition. Thus $[\varphi_* L^{-1}]^\vee = \varphi_* L$. Theorem 1.5 and the proof of the results below can thus be much simpler.

$$p_1{}^!\mathcal{O}_X = p_2{}^*s^!\mathcal{O}_{\mathrm{Spec}\,(k)}\,.$$

Now $s^!\mathcal{O}_{\mathrm{Spec}\,(k)} = \mathcal{O}_S$, hence $p_1{}^!\mathcal{O}_X = \mathcal{O}_{X\times S}$, if and only if S is Gorenstein, i.e. the affine ring $A(S) = H^0(S, \mathcal{O}_S)$ is a Gorenstein ring(*). This can be seen as follows. $s^!\mathcal{O}_{\mathrm{Spec}\,(k)}$ is the $\mathcal{O}_S$-module associated to the dual space $D(A(S)) = \mathrm{Hom}_k(A(S), k)$ on which the $A(S)$-module structure is given by $(au)(x) = u(ax)$ for a and x in $A(S)$ and u in $D(A(S))$ (cf. Hartshorne [6]). Hence $s^!\mathcal{O}_{\mathrm{Spec}\,(k)} = \mathcal{O}_S$ if and only if $D(A(S)) = A(S)$ as $A(S)$-modules, hence if and only if $A(S)$ is a Gorenstein ring (cf. H. Bass, On the ubiquity of Gorenstein rings, Math. Zeitschr., 82 (1963), 8–28). $A(S)$ is Gorenstein if and only if each localization of $A(S)$ is Gorenstein. Hence, for example, $s^!\mathcal{O}_{\mathrm{Spec}\,(k)} = \mathcal{O}_S$, if each stalk of $\mathcal{O}_S$ is a complete intersection

$$k[t_1, t_2, \cdots, t_r]/(t_1{}^{s_1}, t_2{}^{s_2}, \cdots, t_r{}^{s_r}).$$

Thus if

(i) S is a finite subscheme of a non-singular curve or a surface over k

or if

(ii) S is a finite group scheme over k (cf. Cartier's structure theorem of local group schemes [4]),

then $s^!\mathcal{O}_{\mathrm{Spec}\,(k)} = \mathcal{O}_S$, hence $p_1{}^!\mathcal{O}_X = \mathcal{O}_{X\times S}$.

COROLLARY 1.7 Let $\varphi : Y \to X$ be an isogeny of abelian varieties over k. Then $\varphi_*\mathcal{O}_Y \cong p_{1*}(\mathscr{P}\,|\,X\times\ker(\varphi^t))$ where $\mathscr{P}$ is the normalized Poincaré line bundle on $X\times X^t$ and $\varphi^t : X^t \to Y^t$ is the dual of φ. Especially if φ^t is separable, then $\varphi_*\mathcal{O}_Y = \oplus L$ where L runs over all the line bundles on X such that $\varphi^* L \cong \mathcal{O}_Y$.

Remark. We have seen in the remark before Proposition 1.3 that $\mathrm{End}_{\mathcal{O}_X}(\varphi_*\mathcal{O}_Y) = A(D(\ker(\varphi)))$. Now we have a sharper result. We also remark that this result generalizes Atiyah's result in [1] p. 451.

Proof. We have seen in Theorem 1.5 that $(\varphi\times 1)_*\mathcal{Q} = [(1\times\varphi^t)_*\mathscr{P}^{-1}]^\vee$. Thus $\varphi_*\mathcal{O}_Y = \varphi_*(\mathcal{Q}\,|\,Y\times\{0\}) = [(\varphi\times 1)_*\mathcal{Q}]\,|\,X\times\{0\} = [(1\times\varphi^t)_*\mathscr{P}^{-1}]^\vee\,|\,X\times\{0\} = [((1\times\varphi^t)_*\mathscr{P}^{-1})\,|\,X\times\{0\}]^\vee = [p_{1*}(\mathscr{P}^{-1}\,|\,X\times\ker(\varphi^t)]^\vee$
since the diagram

(*) The relevance of Goernstein ring here was pointed out to us by Masaki Kashiwabara.

$$\begin{array}{ccc} X\times X^t & \longleftarrow & X\times \ker(\varphi^t) \\ {\scriptstyle 1\times\varphi^t}\downarrow & & \downarrow{\scriptstyle p_1} \\ X\times Y^t & \longleftarrow & X\times\{0\} \end{array}$$

is cartesian. Now apply the duality theorem for the finite morphism p_1. Since $\ker(\varphi^t)$ is a group scheme and hence Gorenstein, we see that

$$\varphi_*\mathcal{O}_Y = p_{1*}([\mathscr{P}^{-1}\,|\,X\times\ker(\varphi^t)]^\vee) = p_{1*}(\mathscr{P}\,|\,X\times\ker(\varphi^t)).$$

If φ^t is separable, $\ker(\varphi^t)$ is reduced, and we easily get the result. Q.E.D.

Let $\varphi: Y\to X$ be an isogeny of g-dimensional abelian varieties over k with $G=\mathrm{Ker}(\varphi)$, and let L and M be line bundles on Y. We are now going to identify the $\mathcal{O}_X$-module $\mathscr{H}om_{\mathcal{O}_X}(\varphi_*L,\varphi_*M)$.

By the adjointness of φ_* and φ^*, and the fact that $\varphi^*\varphi_*L=p_{1*}\mu^*L$ for $\mu, p_1: Y\times G\rightrightarrows Y$, we have

$$\mathscr{H}om_{\mathcal{O}_X}(\varphi_*L,\varphi_*M)=\varphi_*\mathscr{H}om_{\mathcal{O}_Y}(\varphi^*\varphi_*L, M)$$
$$=\varphi_*\mathscr{H}om_{\mathcal{O}_Y}(p_{1*}[\mu^*L\otimes_{\mathcal{O}_{Y\times G}}p_1{}^*M^{-1}],\mathcal{O}_Y)=\varphi_*[p_{1*}(\mu^*L\otimes_{\mathcal{O}_{Y\times G}}p_1{}^*M^{-1})]^\vee.$$

Since G is Gorenstein, this is isomorphic to $\varphi_*p_{1*}[\mu^*L\otimes p_1{}^*M^{-1}]^\vee=\varphi_*p_{1*}(\mu^*L^{-1}\otimes p_1{}^*M)=\varphi_*p_{1*}(1\times\Lambda(L^{-1}))^*[\mathscr{Q}\otimes_{\mathcal{O}_{Y\times Y^t}}p_1{}^*(M\otimes_{\mathcal{O}_Y}L^{-1})]=p_{1*}(1\times\Lambda(L^{-1}))^*(\varphi\times 1)_*[\mathscr{Q}\otimes_{\mathcal{O}_{Y\times Y^t}}p_1{}^*(M\otimes_{\mathcal{O}_Y}L^{-1})]$ by the commutative diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\;p_1\;} & Y\times G & \xrightarrow{\;1\times\Lambda(L^{-1})\;} & Y\times Y^t \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi\times 1} & & \downarrow{\scriptstyle\varphi\times 1} \\ X & \xleftarrow[\;p_1\;]{} & X\times G & \xrightarrow[\;1\times\Lambda(L^{-1})\;]{} & X\times Y^t \end{array}$$

Suppose moreover that L and M are algebraically equivalent. Then there is a k-valued point b in Y^t such that

$$\mathscr{Q}\otimes_{\mathcal{O}_{Y\times Y^t}}p_1{}^*(M\otimes_{\mathcal{O}_Y}L^{-1})\cong(1_Y\times T_b)^*\mathscr{Q}$$

where $T_b: Y^t\to Y^t$ is the translation by b. Hence

$$\begin{aligned}\mathscr{H}om_{\mathcal{O}_X}(\varphi_*L,\varphi_*M)&=p_{1*}(1\times\Lambda(L^{-1}))^*(\varphi\times 1)_*(1\times T_b)^*\mathscr{Q}\\&=p_{1*}(1\times\Lambda(L^{-1}))^*(1\times T_b)^*(\varphi\times 1)_*\mathscr{Q}\\&=p_{1*}(1\times T_b\circ\Lambda(L^{-1}))^*(\varphi\times 1)_*\mathscr{Q}.\end{aligned}$$

Thus by Theorem 1.5, we get

$$\begin{aligned}\mathscr{H}om_{\mathcal{O}_X}(\varphi_*L,\varphi_*M)&=p_{1*}(1\times T_b\circ\Lambda(L^{-1}))^*[(1\times\varphi^t)_*\mathscr{P}^{-1}]^\vee\\&=p_{1*}[(1\times T_b\circ\Lambda(L^{-1}))^*(1\times\varphi^t)_*\mathscr{P}^{-1}]^\vee.\end{aligned}$$

Suppose, moreover, that $\Lambda(L) = -\Lambda(L^{-1})$ induces an isomorphism on G as in Theorem 1.2. Then the sheaf is isomorphic to

$$p_{1*}[(1\times\varphi^t)_*(\mathscr{P}^{-1}|X\times\tilde{G}_b)]^\vee$$

where G_b is the image of G by $T_b\circ\Lambda(L^{-1}) : Y\to Y^t$ and $\tilde{G}_b = (\varphi^t)^{-1}(G_b)$ is the total inverse image by $\varphi^t : X^t\to Y^t$ of the subscheme G_b.

$$\begin{array}{ccccc} X\times X^t & \longleftarrow & X\times\tilde{G}_b & \xrightarrow{p_1} & X \\ {\scriptstyle 1\times\varphi^t}\downarrow & & \downarrow{\scriptstyle 1\times\varphi^t} & & \| \\ X\times Y^t & \longleftarrow & X\times G_b & \xrightarrow{p_1} & X \end{array}$$

Since G_b is a translation of a finite subgroup scheme of Y^t, it is Gorenstein. Applying the duality theorem for $p_1 : X\times G_b\to X$, the sheaf above is isomorphic to $[p_{1*}(1\times\varphi^t)_*(\mathscr{P}^{-1}|X\times\tilde{G}_b)]^\vee = [p_{1*}(\mathscr{P}^{-1}|X\times\tilde{G}_b)]^\vee$. The second p_1 is now the projection $p_1 : X\times\tilde{G}_b\to X$. Since $\tilde{G}_b$ is again a translation of a finite subgroup scheme and hence Gorenstein, this sheaf is isomorphic to

$$p_{1*}[\mathscr{P}^{-1}|X\times\tilde{G}_b]^\vee = p_{1*}[\mathscr{P}|X\times\tilde{G}_b].$$

Thus we get the following:

PROPOSITION 1.8. Let $\varphi : Y\to X$ be an isogeny of abelian varieties over k with $\ker(\varphi) = G$. Let L and M be algebraically equivalent line bundles on Y, with $M\cong L\otimes_{\mathcal{O}_Y}[\mathcal{Q}|Y\times\{b\}]$. If $\Lambda(L)$ induces an isomorphism on G. then $\mathscr{H}om_{\mathcal{O}_X}(\varphi_*L, \varphi_*M) = p_{1*}(\mathscr{P}|X\times\tilde{G}_b)$ where $\mathscr{P}$ is the normalized Poincaré line bundle on $X\times X^t$ and $\tilde{G}_b = (\varphi^t)^{-1}(T_b\circ\Lambda(L^{-1})(G))$ is the total inverse image in X^t by φ^t of the subscheme $T_b\circ\Lambda(L^{-1})(G)$ of Y^t.

COROLLARY 1.9 Let $\varphi : Y\to X$ be an isogeny of abelian varieties over k with $\ker(\varphi) = G$. Let L be a line bundle on Y such that $\Lambda(L)$ induces an isomorphism on G. Then

$$\mathscr{E}nd_{\mathcal{O}_X}(\varphi_*L) = p_{1*}(\mathscr{P}|X\times\tilde{G})$$

where $\tilde{G}$ is the total inverse image by $\varphi^t : X^t\to Y^t$ of the subgroup scheme $\Lambda(L)(G)$ of Y^t. Especially if both φ and φ^t are separable,

$$\mathscr{E}nd_{\mathcal{O}_X}(\varphi_*L) = \oplus L'$$

where L' runs over all the line bundles L' on X such that

$$\varphi^*L' \cong T_a^*L\otimes_{\mathcal{O}_Y}L^{-1}$$

for some point a in $\ker(\varphi)$.

Remark. We shall show later that it is a generalization valid in all characteristic and dimension of Atiyah's key Lemma 22, p. 439 in [1].

PROPOSITION 1.10 In the notation of Theorem 1.4, let E be a universal vector bundle on $X\times X'$ such that $(1_X\times\pi)^*E=(\varphi\times 1_{Y^t})_*M$. Then

$$R^g p_{23*}\mathscr{H}om_{\mathcal{O}_{X\times X'\times X'}}(p_{12}{}^*E, p_{13}{}^*E)\cong \Delta_*\mathcal{O}_{X'},$$

where $p_{12}, p_{13}: X\times X'\times X'\to X\times X'$ and $p_{23}: X\times X'\times X'\to X'\times X'$ are the projections and $\Delta: X'\to X'\times X'$ is the diagonal map.

Proof. Considering the stalk at each point of $X'\times X'$, we conclude, by Proposition 1.3, that the sheaf on the left hand side has support on the diagonal $\Delta(X')$ and that at a point on the diagonal its stalk modulo the maximal ideal is one dimensional. Moreover, the canonical injection $\mathcal{O}_{X'}\to p_{2*}\mathscr{E}nd_{\mathcal{O}_{X\times X'}}(E)$ dualizes to give a surjection $R^g p_{2*}\mathscr{E}nd_{\mathcal{O}_{X\times X'}}(E)\to\mathcal{O}_{X'}$. Thus from what we have seen above and the flat base change theorem we get a canonical surjection

$$R^g p_{23*}\mathscr{H}om_{\mathcal{O}_{X\times X'\times X'}}(p_{12}{}^*E, p_{13}{}^*E)\to \Delta_*\mathcal{O}_{X'}.$$

To show that this is an isomorphism, it is enough to show that its pull back by the faithfully flat morphism $\pi\times\pi: Y^t\times Y^t\to X'\times X'$ is an isomorphism. By the flat base change theorem and the fact that $(1\times\pi)^*E=(\varphi\times 1)_*M$, we get

$$(\pi\times\pi)^*R^g p_{23*}\mathscr{H}om_{\mathcal{O}_{X\times X'\times X'}}(p_{12}{}^*E, p_{13}{}^*E)$$
$$=R^g p_{23*}\mathscr{H}om_{\mathcal{O}_{X\times Y^t\times Y^t}}((\varphi\times 1\times 1)_*p_{12}{}^*M, (\varphi\times 1\times 1)_*p_{13}{}^*M)$$

where p_{12} and p_{13} on the right hand side are now the projections for $Y\times Y^t\times Y^t$.

Thus we are in the situation before Proposition 1.8, and see that this latter is isomorphic to

$$R^g p_{23*}(\varphi\times 1\times 1)_*p_{123*}[(\mu^*p_{12}{}^*M)^{-1}\otimes p_{13}{}^*M]$$
$$=p_{12*}R^g p_{234*}[(\mu^*p_{12}{}^*M)^{-1}\otimes p_{13}{}^*M]$$

where $\mu:(Y\times Y^t\times Y^t)\times G\to Y\times Y^t\times Y^t$ is induced by the action of G on Y, and various projections are for $Y\times Y^t\times Y^t\times G$.

Since $M=p_1{}^*L\otimes\mathscr{P}\otimes_2 p^*L'$, it is not hard to see that $(\mu^*p_{12}{}^*M)^{-1}\otimes p_{13}{}^*M$

$=[(1\times\nu\times 1)^*(p_{12}{}^*\mathscr{Q}\otimes p_{13}{}^*(1\times\lambda)^*\mathscr{Q}^{-1}]\otimes p_{234}{}^*[p_1{}^*L'^{-1}\otimes p_2{}^*L'\otimes p_{13}{}^*(1\times j)^*\widetilde{\mathscr{Q}}^{-1}]$ where $\nu: Y^t\times Y^t\to Y^t$ sends a point (u,v) to $v-u$, and $\widetilde{\mathscr{Q}}$ is the normalized Poincaré line bundle on $Y^t\times Y$. Thus we get

$$(\pi\times\pi)^*R^g p_{234*}\mathscr{H}om_{\mathscr{O}_{X\times X'\times X'}}(p_{12}{}^*E, p_{13}{}^*E)$$

$$=p_{12*}[(p_1{}^*L'^{-1}\otimes p_2{}^*L'\otimes p_{13}{}^*(1\times j)^*\widetilde{\mathscr{Q}}^{-1})\otimes R^g p_{234*}(1\times\nu\times 1)^*(p_{12}{}^*\mathscr{Q}\otimes p_{13}{}^*(1\times\lambda)^*\mathscr{Q}^{-1})].$$

But by the flat base change it is not hard to show that

$$R^g p_{234*}(1\times\nu\times 1)^*[p_{12}{}^*\mathscr{Q}\otimes p_{13}{}^*(1\times\lambda)^*\mathscr{Q}^{-1}]=(p_2-p_1-\lambda\circ p_3)^*R^g p_{2*}\mathscr{Q},$$

where $p_2-p_1-\lambda\circ p_3: Y^t\times Y^t\times G\to Y^t$ is the obvious map. By Lemma 1.1 $R^g p_{2*}\mathscr{Q}$ has support at 0 with one dimensional stalk there. Thus this latter is isomorphic to $i_*\mathscr{O}_{Y^t\times G}$ where $i: Y^t\times G\to Y^t\times Y^t\times G$ sends (u,a) to $(u, u+\lambda(a), a)$. Hence

$$\begin{aligned}&(\pi\times\pi)^*R^g p_{23*}\mathscr{H}om_{\mathscr{O}_{X\times X'\times X'}}(p_{12}{}^*E, p_{13}{}^*E)\\ &=p_{12}{}^*[p_1{}^*L'^{-1}\otimes p_2{}^*L'\otimes p_{13}{}^*(1\times j)^*\widetilde{\mathscr{Q}}^{-1})\otimes i_*\mathscr{O}_{Y^t\times G}]\\ &=(p_{12}\circ j)_*\mathscr{O}_{Y^t\times G}=(\pi\times\pi)^*\Delta_*\mathscr{O}_{X'}.\end{aligned}$$

Moreover, keeping track of the isomorphisms, we see that this final isomorphism is the $(\pi\times\pi)^*$ of the original canonical surjection. Thus we are done. Q.E.D.

Remark. When $Y=X$, $\varphi=1_X$, $L=\mathscr{O}_X$ and $L'=\mathscr{O}_{X^t}$, we have $X'=X^t$ and $E=\mathscr{P}$, and this result is a slight modification of Lemma 1.1 for $i=g$.

COROLLARY 1.11 In the notation of Proposition 1.10, suppose S is a Gorenstein finite subscheme of X'. Let $E|X\times S$ be the restriction of E on $X\times X$ to $X\times S$. Then

$$\mathrm{End}_{\mathscr{O}_X}(p_{1*}[E|X\times S])=A(S)$$

where $p_1: X\times S\to X$ is the projection.

Proof. For simplicity, we denote $E'=E|X\times S$. Via scalar multiplication, $A(S)$ is canonically contained in $\mathrm{End}_{\mathscr{O}_X}(p_{1*}E')$. Thus it is enough to show that the dimension of these, as k-vector spaces, coincide.

By adjointness of p_{1*} and $p_1{}^*$, and the cartesian diagram

$$\begin{array}{ccc} X\times S\times S & \xrightarrow{p_{13}} & X\times S \\ \downarrow p_{12} & p_1 & \downarrow p_1 \\ X\times S & \longrightarrow & X \end{array}$$

$$\begin{aligned}\mathrm{End}_{\mathcal{O}_X}(p_{1*}E') &= \mathrm{Hom}_{\mathcal{O}_{X\times S}}(p_{12*}p_{13}{}^*E', E') = H^0(X\times S, [p_{12*}(p_{13}{}^*E'\otimes p_{12}{}^*\check{E}')]^\vee) \\ &= H^0(X, p_{1*}[p_{12}{}^*(p_{13}{}^*E'\otimes p_{12}{}^*\check{E}')]^\vee).\end{aligned}$$

Since S is Gorenstein, we can apply the duality theorem for the finite morphism p_1 and see that this is equal to

$$H^0(X, [p_{1*}p_{12*}(p_{13}{}^*E'\otimes p_{12}{}^*\check{E}')]^\vee) = H^0(X, [p_{1*}(p_{13}{}^*E'\otimes p_1{}^*\check{E}')]^\vee),$$

which is dual, by Serre duality, to

$$\begin{aligned}H^g(X, p_{1*}(p_{13}{}^*E'\otimes p_{12}{}^*\check{E}')) &= H^g(X\times S\times S, \mathcal{H}om_{\mathcal{O}_{X\times S\times S}}(p_{12}{}^*E', p_{13}{}^*E')) \\ &= H^0(S\times S, R^g p_{23*}\mathcal{H}om_{\mathcal{O}_{X\times S\times S}}(p_{12}{}^*E', p_{13}{}^*E')).\end{aligned}$$

By the base change theorem, the sheaf inside is equal to

$$(i\times i)^*R^g p_{23*}\mathcal{H}om_{\mathcal{O}_{X\times X'\times X'}}(p_{12}{}^*E, p_{13}{}^*E).$$

where $i : S\to X'$ is the injection. By Proposition 1.10 this is equal to $(i\times i)^*\Delta_*\mathcal{O}_{X'} = \Delta_*\mathcal{O}_S$. Thus $\mathrm{End}_{\mathcal{O}_X}(p_{1*}E')$ is dual, as a k-vector space, to $H^0(S\times S, \Delta_*\mathcal{O}_S) = A(S)$.

COROLLARY 1.12 Let X be an abelian variety over k and let $\mathcal{P}$ be the normalized Poincaré line bundle on $X\times X^t$. If S is a Gorenstein finite subscheme of X^t, then

$$\mathrm{End}_{\mathcal{O}_X}(p_{1*}[\mathcal{P}\,|X\times S]) = A(S)$$

where $\mathcal{P}\,|X\times S$ is the restriction of $\mathcal{P}$ to $X\times S$.

Proof. As we remarked before Corollary 1.11, $\mathcal{P}$ is a special case of E in Corollary 1.11.

Remark. When k is the field of complex numbers, Morikawa [10] characterized those simple vector bundles on a complex torus X which we get as in Theorem 1.2 (i) as follows:

Let $\tilde{X}$ be the universal covering space of X, that is, a g-dimensional vector space over k. Let Γ be the fundamental group, which can be identified as the subgroup of periods of $\tilde{X}$. Then a vector bundle E of rank r on X corresponds to a cohomology class of a 1-cocycle (a factor of automorphy or a matric multiplier) $h(\alpha, z)$ in $H^1_{gr}(\Gamma, GL_r(\mathscr{H}))$, where $\mathscr{H}$ is the ring of all holomorphic functions on $\tilde{X}$, and $h(\alpha, z)$ in $GL_r(\mathscr{H})$ for α in Γ and z in $\tilde{X}$ satisfies

$$h(\alpha+\beta, z) = h(\beta, z+\alpha)\cdot h(\alpha, z).$$

Then E is of the form in Theorem 1.2 (i), if and only if the corresponding 1-cocycle is cohomologous to one of the form

$$h(\alpha, z) = \exp(B(\alpha, z))\cdot C(\alpha)$$

where $B(\alpha, z)$ is a bilinear form, k-linear in z, $C(\alpha)$ is a constant matrix in $GL_r(k)$, and the linear envelope of $C(\alpha)$ with α running over Γ is the full matrix ring $M_r(k)$. (See also Gunning [5])

We can re-interpret this result as follows:
There exists an isogeny $\varphi: Y \to X$ and a line bundle L on Y such that $\Lambda(L)$ induces an isomorphism on $\ker(\varphi)$ and that $E \cong \varphi_* L$, if and only if the canonical inclusion $\mathscr{O}_X \to \mathscr{E}nd_{\mathscr{O}_X}(E)$ induces an isomorphism

$$H^j(X, \mathscr{O}_X) \underset{\to}{\sim} H^j(X, \mathscr{E}nd_{\mathscr{O}_X}(E))$$

for $j=0$ and 1 (resp. all j).

The necessity follows immediately from (cf. Corollary 1.9)

$$\mathscr{E}nd_{\mathscr{O}_X}(\varphi_* L) = \oplus L'$$

where L' runs over all the line bundles on X such that $\varphi^* L' \cong T_a^* L \otimes_{\mathscr{O}_X} L^{-1}$ with a in $\ker(\varphi)$, and the calculation of the cohomology groups of a line bundle on an abelian variety in Mumford [14]. (See also Lemma 1.1). The sufficiency follows from Morikawa's characterization and the following: First of all $h(\alpha, z)^{-1} dh(\alpha, z)$ determines a fundamental class in $H^1(X, \Omega^1_X \otimes \mathscr{E}nd_{\mathscr{O}_X}(E))$ (cf. Atiyah [2]), which, by assumption, is isomorphic to $H^1(X, \Omega^1_X)$, since Ω^1_X is trivial. Hence we may assume $h(\alpha, z)$ is of the form $\exp(B(\alpha, z))\cdot C(\alpha)$. Moreover, since $\mathrm{End}_{\mathscr{O}_X}(E) = H^0(X, \mathscr{E}nd_{\mathscr{O}_X}(E)) = H^0(X, \mathscr{O}_X) = k$, the linear envelope of $C(\alpha)$ is the full matrix ring.

We also remark that $H^1(X, \mathscr{E}nd_{\mathscr{O}_X}(E))$ measures the infinitesimal deformation of E on X, that is, it is isomorphic to the tangent space at E of the moduli of vector bundles on X. Our characterization above says that the vector bundle of the form $\varphi_* L$ as above, moves essentially in a g-dimensional family.

Section 2 Vector bundles on an elliptic curve.

In this section, we let X be an abelian variety of dimension 1 over k of characteristic p, i.e. an elliptic curve with a base point.

Atiyah [1] classified all the vector bundles on X. Among other things, he proved the following (Theorems 7 & 10):

(i) Let $\mathscr{E}_X(r, d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and of degree d. If we fix one E in $\mathscr{E}_X(r, d)$, then every other vector bundles is of the form $E \otimes L$ with L in $Pic^\circ(X) = \mathscr{E}_X(1, 0)$. Moreover, $E \otimes L_1 \cong E \otimes L_2$ if and only if $L_1^{\otimes r'} = L_2^{\otimes r'}$ where $r' = r/(r, d)$.

(ii) In $\mathscr{E}_X(r, 0)$ there is a unique element $E_{r,0}$ such that $H^0(X, E_{r,0}) \neq 0$ (in fact it is one dimensional). We fix this notation hereafter.

(iii) (Riemann-Roch) Let $h^i(E)$ be the dimension of $H^i(X, E)$. Then for E in $\mathscr{E}_X(r, d)$, we have

$$h^0(E) = d \text{ and } h^1(E) = 0 \text{ when } d \text{ is positive.}$$

$$h^0(E) = 0 \text{ and } h^1(E) = |d| \text{ when } d \text{ is negative.}$$

$$h^0(E) = h^1(E) = 0 \text{ when } d = 0 \text{ and } E \neq E_{r,0}$$

$$h^0(E) = h^1(E) = 1 \text{ when } E = E_{r,0}$$

(iv) Suppose $p = 0$. For E in $\mathscr{E}_X(r, d)$ with $(r, d) = 1$, $\mathrm{End}_{\mathscr{O}_X}(E) = k$. (In fact such E is "stable" hence simple regardless of p. cf. Raynaud [17] when $p = 0$. In general due to Takemoto.)

(v) Suppose $p=0$. For E in $\mathscr{E}_X(r, d)$ with $(r, d)=1$, $E \otimes E_{h,0}$ is in $\mathscr{E}_X(rh, dh)$.

For these results, the key is his Lemma 7 to the effect that for E in $\mathscr{E}_X(r, d)$ with $(r, d) = 1$ and $p \nmid r$, $\mathscr{E}nd_{\mathscr{O}_X}(E) = \oplus L$, where L runs over all the line bundles on X with $L^{\otimes r} \cong \mathscr{O}_X$.

(vi) When $p = 0$, $E_{r,0}$ is isomorphic to the $(r-1)$-st symmetric power $S^{r-1}(E_{2,0})$.

(vii) When k is the field of complex numbers, a vector buncle has a holomorphic integrable connection if and only if it is a direct sum of those in $\mathscr{E}_X(r, 0)$ for various r. Matsushima [9] and Morimoto [11] generalized this result to complex tori.

We now apply our results in section 1.

PROPOSITION 2.1 Let $\varphi: Y \to X$ be an isogeny of degree r and let L be a line bundle of degree d on Y with $(r, d) = 1$. Then $\varphi_* L$ is in $\mathscr{E}_X(r, d)$. For E in $\mathscr{E}_X(r, d)$, we have $\mathrm{End}_{\mathscr{O}_X}(E) = k$.

Proof. Since Y is also an elliptic curve, we have $Y^t = Y$ and $\Lambda(L) = d_Y$. Hence $\Lambda(L)$ induces as isomorphism on $\ker(\varphi)$ of order r if and only if $(r, d) = 1$. Apply Theorem 1.2 (i). $\varphi_* L$ has rank r and degree d by Riemann-Roch theorem for the finite morphism φ.

PROPOSITION 2.2 Let E be an element of $\mathscr{E}_X(r, d)$ with $(r, d) = 1$. Then

$$\mathscr{E}nd_{\mathscr{O}_X}(E) = p_{1*}(\mathscr{P} \,|\, X \times {}_r X)$$

where $\mathscr{P}$ is the normalized Poincaré line bundle on $X \times X^t = X \times X$ and ${}_r X$ is the (scheme-theoretic) kernel of $r_X : X \to X$.

Proof. Since elements in $\mathscr{E}_X(r, d)$ differ only by tensor products by line bundles of degree 0, it is enough to show that for an isogeny $\varphi: Y \to X$ of degree r and for a line bundle L on Y of degree d, $\mathscr{E}nd_{\mathscr{O}_X}(\varphi_* L)$ is of the form in Proposition 2.2.
By Corollary 1.9 we get

$$\mathscr{E}nd_{\mathscr{O}_X}(\varphi_* L) = p_{1*}(\mathscr{P} \,|\, X \times \tilde{G})$$

where $\tilde{G} = (\varphi^t)^{-1}(\Lambda(L)(\ker(\varphi)))$. Identifying, as before, X^t and Y^t with X and Y respectively (via Λ of a line bundle of degree 1), we can easily show that $\Lambda(L) = d_Y$, $\varphi \circ \varphi^t = r_X$ and $\varphi^t \circ \varphi = r_Y$. Since $(r, d) = 1$, we have $\Lambda(L)(\ker(\varphi)) = \ker(\varphi)$. Thus $\tilde{G} = (\varphi^t)^{-1}(\ker(\varphi)) = \ker(\varphi \circ \varphi^t) = \ker(r_X) = {}_r X$.

PROPOSITION 2.3 Given r and d with $(r, d) = 1$. There exists a (simple) vector bundle $E = E(r, d)$ on $X \times X$, such that among the vector bundles $E \,|\, X \times \{a\}$ for a moving over the k-valued points of X, each element in $\mathscr{E}_X(r, d)$ appears once and only once. Moreover,

$$R^1 p_{23*} \mathscr{H}om_{\mathscr{O}_{X \times X \times X}}(p_{12}{}^* E, p_{13}{}^* E) = \Delta_* \mathscr{O}_X$$

where $p_{12}, p_{23}, p_{13} : X \times X \times X \to X \times X$ are the projections and $\Delta : X \to X \times X$ is the diagonal map.

Remark. This is a sharpening of Atiyah's classification of $\mathscr{E}_X(r, d)$. He has shown that it is set-theoretically isomorphic to X. We now have an

algebraic family parametrized by X. This family is "universal" in the sense that for any scheme S over k, the set of S-equivalence classes (cf. *FGA*, 190–24) of vector bundles on $X\times S$ with each fiber in $\mathscr{E}(r,d)$ is isomorphic to the set of morphisms $S\to X'=X$, via the pull back of $E(r,d)$.
We shall see later that this algebraic family is indispensable to handling the problem (1), when the Hasse invariant of X is zero.

We also remark that when $r=1$, we can take $E(r,d)=p_1{}^*L\otimes_{\mathcal{O}_{X\times X}}\mathscr{P}$, where $\mathscr{P}$ is the Poincare line bundle on $X\times X$ and L is a line bundle of degree d on X.

Proof. Let $\varphi:Y\to X$ be an isogeny of degree r. Let L and L' be line bundles on Y of degree d and d' respectively such that $d\cdot d'\equiv 1$ (mod. r). Such d' exists since $(r,d)=1$. Since $\lambda=\Lambda(L)=d_Y$ and $\lambda'=\Lambda(L')=d'_Y$, the conditions of Theorem 1.4 are satisfied. Thus we have $\pi:Y=Y^t\to X'=Y^t/\lambda(G)$. But $\lambda(G)=G$. It is easy to show that $X'=X$ and $\pi=\varphi$. Thus we are done by Theorem 1.4 and Proposition 1.10.

PROPOSITION 2.4. Let S be a finite subscheme of X. Then for the vector bundle $E=E(r,d)$ on $X\times X$ with $(r,d)=1$ defined in Proposition 2.3, we have

$$\mathrm{End}_{\mathcal{O}_X}(p_{1*}[E\,|\,X\times S])=A(S).$$

Especially when S is an artinian local subscheme of X, the vector bundle $p_{1*}[E\,|\,X\times S]$ is in $\mathscr{E}_X(rh,dh)$ with $h=\dim_k A(S)$.

Proof. Any finite subscheme of X is Gorenstein, since X is of dimension 1. Thus the Proposition follows from Corollary 1.11.

COROLLARY 2.5 Let E be an element of $\mathscr{E}_X(rh,dh)$ with $(r,d)=1$. Then

$$\mathrm{End}_{\mathcal{O}_X}(E)=k[t]/(t^t).$$

Remark. We shall show in Proposition 2.13 that when there is a *separable* isogeny of degree r, we do not need Proposition 2.4 to prove Corollary 2.5.

COROLLARY 2.6 Let S be the $(h-1)$-st order neighborhood of the origin 0 of X, i.e. $S=\mathrm{Spec}\,(\mathcal{O}_{X,0}/m^h_{X,0})$. Then $p_{1*}(\mathscr{P}\,|\,X\times S)=E_{h,0}$, where $\mathscr{P}$ is the normalized Poincaré line bundle on $X\times X$. Moreover, $\mathrm{End}_{\mathcal{O}_X}(E_{h,0})=k[t]/(t^h)$.

Proof. We know by Proposition 2.4 and the Remark before that, that this bundle is in $\mathscr{E}_X(h,0)$. It is enough to show that this has a non-zero global section. But since S is Gorenstein, $H^0(X, p_{1*}(\mathscr{P}|X\times S))$ is dual, as a k-vector space, to

$$H^1(X,[p_{1*}(\mathscr{P}|X\times S)]^\vee) = H^1(X, p_{1*}(\mathscr{P}^{-1}|X\times S))$$
$$= H^1(X\times S, \mathscr{P}^{-1}|X\times S) = H^0(S, R^1p_{2*}(\mathscr{P}^{-1}|S)).$$

By the base change theorem, this is equal to $H^0(S,(R^1p_{2*}\mathscr{P}^{-1})|S) = k$, since $R^1p_{2*}\mathscr{P}^{-1}$ is also concentrated at 0 and has 1-dimensional stalk there. (This result is contained in Mumford's result mentioned in Lemma 1.1).

Remark. We have shown in Proposition 2.1 that elements in $\mathscr{E}_X(r,d)$ with $(r,d)=1$ can be obtained by the direct image of a line bundle by an isogeny $\varphi: Y\to X$. Corollary 2.6 and in fact Theorem 1.5 says that we get other bundles when we allow Y to be a *non-reduced* covering of X.

COROLLARY 2.7 Let E be an element of $\mathscr{E}_X(r,d)$ with $(r,d)=1$. Suppose $r=qr'$, with $(r',p)=1$ and q a power of p. Then

$\mathscr{E}nd_{\mathscr{O}_X}(E)\cong\oplus[L\otimes E_{q,0}]$ if Hasse$(X)\neq 0$, where L runs over all the line bundles on X with $L^{\otimes r}\cong\mathscr{O}_X$. (There are r'^2q of those). If Hasse$(X)=0$, $\mathscr{E}nd_{\mathscr{O}_X}(E)=\oplus[L\otimes E_{q^2,0}]$, where L runs over all the line bundles on X with $L^{\otimes r'}=\mathscr{O}_X$.

Remark. This generalizes Atiyah's key Lemma 7 in [1].

Proof. As we have seen in Proposition 2.2, the left hand side is isomorphic to $p_{1*}(\mathscr{P}|X\times_r X)$. But the subgroup scheme $_rX$ is isomorphic to the product $_qX\times_{r'}X$, and $_{r'}X$ is reduced. If Hasse$(X)=0$, $_qX$ is local. If Hasse$(X)\neq 0$, $_qX$ is isomorphic to the procudt $\mu_q\times Z/(q)$, where μ_q is the kernel of $q: G_m\to G_m$ and is local. Corollary follows immediately from Corollary 2.6.

Let $R=\mathscr{O}_{X,0}$ be the local ring of X at the origin 0, and let $\mathfrak{m}=\mathfrak{m}_{X,0}=tR$ be its maximal ideal with a generator t. We denote by $\hat{R}$ the completion of R with respect to $\mathfrak{m}$. $\hat{R}$ is isomorphic to the formal power series ring $k[[t]]$. The group law $\mu: X\times X\to X$ induces a map $\mu^*: \hat{R}\to\hat{R}\hat{\otimes}\hat{R}$, which gives a one-parameter formal group $\hat{X}$, the local part of the p-divisible group $X(p)$. (For the detail see e.g. Oda [16].) When $p=0, \hat{X}$ is isomorphic to the additive group $\hat{G}_a$. When $p\neq 0$ and Hasse$(X)\neq 0$. X is isomorphic to $\hat{G}_m = G_{1,0}$, while $\hat{X}$ is isomorphic to the group $G_{1,1}$, when $p \neq 0$ and Hasse$(X) = 0$.

PROPOSITION 2.8 $E_{h,0} \otimes_{\mathcal{O}_X} E_{h',0}$ decomposes into the direct sum of $E_{h(i),0}$ $(i = 1, 2, \cdots, s)$, where $R/\mathfrak{m}^h \otimes_k R/\mathfrak{m}^{h'}$ decomposes, as an R-module via the group law μ^*, into the direct sum of cyclic modules of length $h(i)$ $(i = 1, 2, \cdots, s)$.

Remark. When $p = 0$, Atiyah ([1] Theorem 8) found $h(i)$. For $h' \geqq h$,

$$h(i) = (h' - h) + (2i - 1), \quad (i = 1, 2, \cdots, h).$$

We can show that when $h' = p^e \geqq h$, $h(i) = p^e$ $(i = 1, 2, \cdots, h)$.

Proof. Let $S = \mathrm{Spec}(R/\mathfrak{m}^h)$ and $S' = \mathrm{Spec}(R/\mathfrak{m}^{h'})$ be the $(h-1)$-st and the $(h'-1)$-st order neighborhood of 0 in X. We have shown in Corollary 2.6 that $E_{h,0} = p_{1*}(\mathscr{P} | X \times S)$ and $E_{h',0} = p_{1*}(\mathscr{P} | X \times S')$. Then

$$E_{h,0} \otimes_{\mathcal{O}_X} E_{h',0} = p_{1*}(p_{12}{}^* \mathscr{P} \otimes_{X \times X \times X} p_{13}{}^* \mathscr{P} \,|\, X \times S \times S')$$

But $p_{12}{}^* \mathscr{P} \otimes_{\mathcal{O}_{X \times X \times X}} p_{13}{}^* \mathscr{P} = (1 \times \mu)^* \mathscr{P}$. The rest follows immediately from this.

COROLLARY 2.9 $\mathscr{E}nd_{\mathcal{O}_X}(E_{h,0})$ is a unipotent vector bundle on X.

We denote by $F: X \to X^{(p)} \cong X$ the Frobenius morphism and by $V: X^{(p)} \to X$ the "Verschiebung" morphism. We know (for example Oda [16] Section 2) that $F^t = V$, and that

(a) When Hasse$(X) \neq 0$, V is separable and coincides with the quotient map of X by the unique reduced subgroup of order p.

(b) When Hasse$(X) = 0$, V coincides with F.

After Hartshorne, we denote F^*E by $E^{(p)}$ for a vector bundle E on X.

PROPOSITION 2.10

When Hasse$(X) \neq 0$, $E_{h,0}^{(p)} \cong E_{h,0}$.

When Hasse$(X) = 0$, we have

$$E_{h,0}^{(p)} \cong \mathcal{O}_X{}^h \quad \text{for} \quad 1 \leqq h \leqq p$$

$$E_{h,0}^{(p)} \cong \bigoplus E_{[(h-i)/p]+1,0} \quad (i = 1, 2, \cdots, p), \quad \text{for } p < h,$$

where [ ] is the Gauss symbol.

Proof. Let $S=\operatorname{Spec}(R/\mathfrak{m}^h)$ be the $(h-1)$-st order neighborhood of 0 in X. Then $E_{h,0}=p_1(\mathscr{P}\,|\,X\times S)$. Thus $E_{h,0}^{(p)}=p_{1*}((F\times 1)^*\mathscr{P}\,|\,X\times S)$. But as we have seen in the proof of Theorem 1.5, $(F\times 1)^*\mathscr{P}=(1\times F^t)^*\mathscr{P}=(1\times V)^*\mathscr{P}$. When Hasse$(X)\neq 0$, $V:X\to X$ is separable, hence locally isomorphic at 0. Therefore $E_{h,0}^{(p)}\cong E_{h,0}$. When Hasse$(X)=0, V$ coincides with F. Thus $E_{h,0}^{(p)}$ decomposes into the direct sum of $E_{h(i),0}$, while $R/\mathfrak{m}^h$, as an R-module via the p-th power map $F^*:R\to R$ and the projection $R\to R/\mathfrak{m}^h$, decomposes into the direct sum of cyclic modules of length $h(i)$. It suffices to compute the decomposition of $k[t]/(t^h)$ as a $k[t^p]$-module, which is easy.

COROLLARY 2.11 *If $\varphi:Y\to X$ is an isogeny such that $\varphi^t:X\to Y$ is separable, then $\varphi^*E_{h,0}$ is isomorphic to $E_{h,0}$ of Y.*

Proof. The proof is similar to that of Proposition 2.10, in view of the fact that φ^t is locally isomorphic.

PROPOSITION 2.12 *Let G be a finite subgroup scheme of X and let μ_G, $p_1:X\times G\to X$ be the action and the projection. Then*

$$\mu_G{}^*E_{h,0}\cong p_1{}^*E_{h,0}$$

on $X\times G$.

Remark. When k is the field of complex numbers, this means that $E_{h,0}$ has a holomorphic integrable connection.

Proof. $E_{h,0}=p_{1*}(\mathscr{P}\,|\,X\times S)$, for the $(h-1)$-st order neighborhood S of 0. But $(\mu\times 1)^*\mathscr{P}=p_{13}{}^*\mathscr{P}\otimes_{\mathcal{O}_{X\times X\times X}}p_{23}{}^*\mathscr{P}$ on $X\times X\times X$. The ristriction of $p_{23}{}^*\mathscr{P}$ to $X\times G\times S$ is trivial, since $G\times S$ is finite. Hence we are done.

PROPOSITION 2.13 *Let $\varphi:Y\to X$ be a separable isogeny of degree r. For a line bundle L of degree d on Y with $(r,d)=1$, $\varphi_*(L\otimes_{\mathcal{O}_Y}E_{h,0})$ is in $\mathscr{E}_X(rh,dh)$, and $\operatorname{End}_{\mathcal{O}_X}(\varphi_*(L\otimes_{\mathcal{O}_Y}E_{h,0}))=\operatorname{End}_{\mathcal{O}_Y}(E_{h,0})=k[t]/(t^h)$. If moreover φ^t is separable, $\varphi_*L\otimes_{\mathcal{O}_X}E_{h,0}$ is in $\mathscr{E}_X(rh,dh)$.*

Proof. By Corollary 2.9 and Proposition 2.12, $C=E_{h,0}$ satisfies all the conditions of Theorem 1.2 (ii). The rank and the degree are as in the Proposition by the Riemann-Roch theorem for the finite morphism φ. For the last statement use Corollary 2.11 and the projection formula.

Remark. Because of this Proposition and Proposition 2.1 when $h=1$, we can construct, more easily than Proposition 2.4, an element in $\mathscr{E}_X(rh, dh)$ with $(r, d)=1$, if either $h=1$, or $h\neq 1$ and there is a *separable* isogeny of degree r. The reason is that $\pi: Y^t \to X'$ coincides with φ and hence locally isomorphic. Thus we can take the local family near 0 of $(\varphi\times 1)_* M$ instead of that of its descent by $(1\times\pi)$ and then project onto X. If Hasse$(X)\neq 0$, r can be arbitrary. If Hasse$(X)=0$, however, r should not be divisible by p, when $h\neq 1$. Proposition 2.4 is essential to construct and study E in $\mathscr{E}_X(rh, dh)$, when $h\neq 1$ and r is divisible by p. The last statement of Proposition 2.3 is the restatement of Atiyah's result (v) quoted at the beginning of Section 2.

As we have remarked in (vi) at the beginning of Siection 2, Atyah showed that for $p=0$, $E_{h,0}$ is isomorphic to $S^{h-1}(E_{2,0})$. This fact can be interpreted as follows:

Let u be a non-zero element in $H^1(X, \mathscr{O}_X)$. u defines a principal G_a-bundle over X. Let $J=J_h$ be the $h\times h$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & & 0 \\ & 0 & 1 & & \\ & & 0 & & \\ & & & & 1 \\ & & & & \\ 0 & & & & 0 \end{pmatrix}$$

Then $J^h=0$. With this J we have a representation $G_a \to GL_h$ by sending x to $\exp(xJ)$. Take the principal GL_h-bundle $\exp(uJ)$ over X, which is obtained from the G_a-bundle u via this representation. The vector bundle associated to this is easily seen to be $E_{h,0}$. This corresponds to the fact that $\hat{X}=\hat{G}_a$.

We now examine the case when $p\neq 0$. Let W be the ordinary Witt scheme over k and W' be its Cartier dual. (For the detail see Cartier [3] and [4]. See also Oda [16].)

Let $\langle\ ,\ \rangle$ be the dual pairing $W\times W' \to G_m$ defined by

$$\langle u, x\rangle = \exp\Big(-\sum_{m\geq 0} (u\cdot x)^{(m)}/p^m\Big)$$

where $(u\cdot x)^{(m)}$ is the m-th phantom component of the product Witt vector $u\cdot x$. The matrix $J=J_h$ is nilpotent, and hence defines an element

$$\{J\} = (J, 0, 0, \cdots\cdots)$$

of W'.

On the other hand, $H^1(X, W(\mathscr{O}_X))$ coincides with the Dieudonné module of the Serre dual of $\hat{X}$. (See Oda [16], Proposition 4.3). When Hasse$(X)\neq 0$, this is a free $W(k)$-module of rank 1 with a base u, such that $Fu = u$ and $Vu = pu$. When Hasse$(X) = 0$, this is a free $W(k)$-module of rank 2, with an element u and $Fu = Vu$ forming a base.

In either case, take this u and get $\langle u, \{J\}\rangle = \exp\left(-\sum_{m\geq 0} u^{(m)} J^{p^m}/p^m\right)$ in $H^1(X, GL_h(\mathscr{O}_X))$. It is not hard to show that this element determines the principal GL_h-bundle associated to $E_{h,0}$.

We can prove Propositions 2.8, 2.10 and 2.12 using this construction of $E_{h,0}$.

Let us now begin to answer our question (1).

PROPOSITION 2.14 Let E be an element in $\mathscr{E}_X(p^i h, dh)$ with $(p, d) = 1$. Then $E^{(p^i)}$ is indecomposable and is of the form $L\otimes_{\mathscr{O}_X} E_{p^i h, 0}$ with a line bundle L on X of degree d.

Proof. It is enough to show this for one E, since all the other elements differ from E by tensor product of a line bundle of degree 0. Hence by Proposition 2.4 it is enough to assume $E = p_{1*}[E(p^i, d) | X\times S]$, where $E(p^i, d)$ is the "universal" vector bundle on $X\times X$ and S is the $(h-1)$-st order neighborhood of a point of X. Recall the construction of $E(p^i, d)$ in Theorem 1.4 and Proposition 2.3. There, $Y = X$, $\varphi = F^i$. It follows that $X = X'$ and $\pi = \varphi = F^i$. We work with the diagram

$$\begin{array}{ccccccc} X & \xleftarrow{\;p_1\;} & X\times S & \longrightarrow & X\times X & \xleftarrow{\;1\times\varphi\;} & X\times X \\ \varphi\big\downarrow & & \varphi\times 1\big\downarrow & & \big\downarrow\varphi\times 1 & & \big\downarrow\varphi\times 1 \\ X & \xleftarrow[\;p_1\;]{} & X\times S & \longrightarrow & X\times X & \xleftarrow[\;1\times\varphi\;]{} & X\times X \end{array}$$

We have $E^{(p^i)} = \varphi^* p_{1*}[E(p^i, d) | X\times S] = p_{1*}(\varphi\times 1)_*[E(p^i, d) | X\times S]$
$\qquad = p_{1*}[(\varphi\times 1)^* E(p^i, d) | X\times S]$.

But we know that $(\varphi\times 1)^* E(p^i, d) = (1\times\varphi)_* M$, where $M = p_1{}^* L\otimes\mathscr{P}\otimes p_2{}^* L'$ with L and L' line bundles on X of degree d and d' respectively.

Thus $E^{(p^i)} = p_{1*}[(1\times\varphi)_* M | X\times S] = p_{1*}(1\times\varphi)_*(M | X\times\varphi^{-1}(S))$
$\qquad = p_{1*}(M | X\times\varphi^{-1}(S)) = p_{1*}(p_1{}^* L\otimes\mathscr{P} | X\times\varphi^{-1}(S))$
$\qquad = L\otimes_{\mathscr{O}_X} p_{1*}(\mathscr{P} | X\times\varphi^{-1}(S))$.

But since S is the $(h-1)$-st order neighborhood of a point on X, $\varphi^{-1}(S) = (F^i)^{-1}(S)$ is the $(p^i h - 1)$-st order neighborhood of the same point. Thus we are done.

Remark. We can prove this Proposition more easily, when Hasse$(X) \neq 0$. See the remark after Theorem 2.16.

COROLLARY 2.15 Suppose $(p^i r', d) = 1$ and $(r', p) = 1$. For an isogeny $\varphi : Y \to X$ of degree r' and an element E' in $\mathscr{E}_Y(p^i h, dh)$, $\varphi_*(E')$ is in $\mathscr{E}_X(p^i r' h, dh)$. Moreover, for a vector bundle E in $\mathscr{E}_X(p^i r' h, dh)$, $E^{(p^i)}$ is indecomposable and is of the form $E'' \otimes_{\mathcal{O}_X} E_{p^i h, 0}$ with E'' in $\mathscr{E}_X(r', d)$.

Proof. Since φ is of degree r' with $(r', p) = 1$, it is separable and the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad F^i \quad} & Y \\ {\scriptstyle\varphi}\downarrow & F^i & \downarrow{\scriptstyle\varphi} \\ X & \xrightarrow{\qquad\quad} & X \end{array}$$

is cartesian. (For the proof we use the fact that the inductive limit of all the finite subgroup schemes of X^t is the dual of the "true fundamental group" of X. cf. *SGA* 1960/1961, exposé XI.)

Hence $(F^i)^*\varphi_*(E') = \varphi_*(F^i)^*(E')$. By Proposition 2.14, $(F^i)^*(E')$ is of the form $L \otimes_{\mathcal{O}_Y} E_{p^i h, 0}$ with L a line bundle of degree d on Y. Thus by Proposition 2.13 $(F^i)^*\varphi_*(E') = \varphi_*(L \otimes E_{p^i h, 0})$ is contained in $\mathscr{E}_X(r'(p^i h), d(p^i h))$. Thus $\varphi_*(E')$ itself should be contained in $\mathscr{E}_X(r' p^i h, dh)$. As for the second statement of the Corollary, it is enough to prove for only one E, and we have done so above, in view of the second statement of Proposition 2.13.

THEOREM 2.16 Let $(r, d) = 1$, and let E be an element of $\mathscr{E}_X(rh, dh)$. When Hasse$(X) \neq 0$, $E^{(p)}$ is indecomposable. When Hasse$(X) = 0$, $E^{(p)}$ is indecomposable, if and only if either $h = 1$, or $h \neq 1$ and r is divisible by p. Otherwise, $E^{(p)}$ decomposes into $\min\{p, h\}$ components in the following manner: If $1 \neq h \leqq p$, $E^{(p)} = E'^h$ where E' is an element in $\mathscr{E}_X(r, dp)$ with $(r, p) = 1$. If $p \leqq h$, $E^{(p)} = \oplus E_j$ $(j = 1, 2, \cdots, p)$, where E_j is an element of $\mathscr{E}_X(rh(j), dph(j))$, with $h(j) = [(h-j)/p] + 1$ and $(r, p) = 1$.

Proof. Write $r = p^i r'$ with $(r', p) = 1$.
If $i \neq 0$, we have seen in Corollary 2.15 that $E^{(p^i)} = [E^{(p)}]^{(p^{i-1})}$ is indecomposable. Thus $E^{(p)}$ itself should be indecomposable. If $i = 0$, we may, by

Proposition 2.13. assume E to be of the form $\varphi_*(L\otimes_{\mathcal{O}_Y}E_{h,0})$ for an isogeny $\varphi: Y\to X$ of degree r' and a line bundle L of degree d on Y. The diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;F\;} & Y \\ {\scriptstyle\varphi}\downarrow & F & \downarrow{\scriptstyle\varphi} \\ X & \longrightarrow & X \end{array}$$

is cartesian. (cf. Proof of Corollary 2.15). Hence

$$F^*\varphi_*(L\otimes E_{h,0}) = \varphi_* F^*(L\otimes E_{h,0}) = \varphi_*(L^{\otimes p}\otimes E^{(p)}_{h,0}).$$

Suppose $\mathrm{Hasse}(X)\neq 0$. Then by Proposition 2.10, $E^{(p)}_{h,0}\cong E_{h,0}$. Thus we have $E^{(p)}\cong\varphi_*(L^{\otimes p}\otimes_{\mathcal{O}_Y}E_{h,0})$ and we can apply Proposition 2.13, since φ is separable and $\deg(L^{\otimes p}) = dp$ is prime to $\deg(\varphi)$.

Suppose $\mathrm{Hasse}(X) = 0$. Then by Proposition 2.10, we know that $E^{(p)}_{h,0}\cong\mathcal{O}_X^h$ when $h\leqq p$, while for $p\leqq h$, $E^{(p)}_{h,0} = \oplus E_{h(j),0}$ $(j = 1, 2, \cdots, p)$, where $h(j) = [(h-j)/p]+1$. Thus applying Proposition 2.13, we get the required result.

Remark. The cases (a) $\mathrm{Hasse}(X)\neq 0$ and $i\neq 0$, and (b) $\mathrm{Hasse}(X) = 0$, $i\neq 0$ and $h = 1$ can be proved more easily without using Corollary 2.15. In both cases $E = \varphi_*(L\otimes E_{h,0})$ is also an element of $\mathcal{E}_X(p^i r' h, dh)$, where $\varphi: Y\to X$ is an isogeny (in case (a), separable) of degree $p^i r'$ and L is a line bundle of degree d on Y. In case (a) we can again show that $E^{(p)} = \varphi_*(L^{\otimes p}\otimes E_{h,0})$. Since $\ker(\varphi)\cap\ker(\Lambda(L^{\otimes p}))\cong \boldsymbol{Z}/(p)$ and $D(\boldsymbol{Z}/(p)) = \mu_p$, $\mathrm{End}_{\mathcal{O}_X}(E^{(p)})$ is the algebra $A(\mu_p)\otimes_k \mathrm{End}_{\mathcal{O}_Y}(E_{h,0})$ twisted by $0(L^{\otimes p}\otimes E_{h,0})$, where $0(L^{\otimes p}\otimes E_{h,0})$ is an element of $H^2_{gr}(\boldsymbol{Z}/(p), \mathbf{Aut}_{\mathcal{O}_Y}(E_{h,0}))$. (See the remark after Theorem 1.2.) It is not hard to show that this twisted algebra is isomorphic to $k[t]/(t^{p^h})$. In case (b), $E = \varphi_*(L)$ is an element of $\mathcal{E}_X(p^i r', d)$. In this case the diagram we had above is no longer cartesian. We decompose φ into a composite $\psi\circ F^i$, with ψ separable of degree r'. Using Corollary 2.6 and Corollary 2.13, we can show that $E^{(p^i)}$, hence $E^{(p)}$, is indecomposable.

We now answer question (2).

THEOREM 2.17 Let E be an element of $\mathcal{E}_X(r, d)$. Then the Frobenius map

$$F^*: H^1(X, E)\to H^1(X, E^{(p)})$$

is injective, unless $\text{Hasse}(X)=0$ and $E=E_{r,0}$ with $r<p$. In the latter case F^* is the zero map.

Remark. $H^1(X,E)=0$, unless either $d<0$ or $E=E_{r,0}$ (cf. (iii) at the beginning of Section 2). We only have to consider these cases. $E=E_{1,0}=\mathcal{O}_X$ is the crucial case, which distinguishes whether $\text{Hasse}(X)$ is zero or not. It is surprising that even when $\text{Hasse}(X)=0$, F^* seldom fails to be injective.

Proof. By projection formula, we have $F_*F^*E=E\otimes_{\mathcal{O}_X}F_*\mathcal{O}_X$. F^* in the Theorem coincides with the map

$$j_*: H^1(X,E)\to H^1(X,E\otimes_{\mathcal{O}_X}F_*\mathcal{O}_X)$$

induced by the injection $j:\mathcal{O}_X\to F_*\mathcal{O}_X$.
We have seen in Corollary 1.7 that $F_*\mathcal{O}_X=p_{1*}(\mathcal{P}\,|X\times\ker(F^t))$. Also $F^t=V$.

Hence when $\text{Hasse}(X)\neq 0$, $F_*\mathcal{P}_X=\oplus L$ where L runs over all the line bundles on X with $L^{\otimes p}\cong\mathcal{O}_X$. Especially j splits, hence $1\otimes j: E\to E\otimes_{\mathcal{O}_X}F_*\mathcal{O}_X$ also splits. Thus j_* is injective.

Suppose now that $\text{Hasse}(X)=0$. Then $\ker(F^t)=\ker(F)$ is the $(p-1)$-st order neighborhood of 0, hence $F_*\mathcal{O}_X=E_{p,0}$ by Corollary 2.6. $E_{p,0}$ is unipotent, i.e. has a filtration

$$E_{p,0}=E_p\supset E_{p-1}\supset\cdots\cdots\supset E_1\supset E_0=0$$

whose successive quotients E_{i+1}/E_i are isomorphic to $\mathcal{O}_X$. The image of $j:\mathcal{O}_X\to F_*\mathcal{O}_X$ coincides with the last member E_1 of the filtration.

We only have to consider the cases (a) $d<0$, and (b) $E=E_{r,0}$, since otherwise $H^1(X,E)=0$.

Case (a): $E\otimes F_*\mathcal{O}_X$ inherits a filtration $E\otimes E_i$, and the image of $1\otimes j$ is $E\otimes E_1$. From the exact sequence

$$0\to E\otimes E_i\to E\otimes E_{i+1}\to E\to 0$$

we get an exact sequence

$$H^0(E)\to H^1(E\otimes E_i)\to H^1(E\otimes E_{i+1})\to H^1(E)\to 0.$$

But $H^0(E)=0$, since $d<0$. Hence by induction on i, we can easily show that $H^1(E\otimes E_1)\to H^1(E\otimes E_p)$ is injective.

It only remains to treat the case (b) $E=E_{r,0}$. $H^1(X,E_{r,0})$ is one dimensional and generated by the characteristic class u of the extension

$$0 \to E_{r,0} \to E_{r+1,0} \to \mathcal{O}_X \to 0$$

it is enough to show that $F^*u \neq 0$ when $r \geqq p$, and $F^*u = 0$ when $r < p$. F^*u is the characteristic class of the pull back

$$0 \to E_{r,0}^{(p)} \to E_{r+1,0}^{(p)} \to \mathcal{O}_X \to 0$$

As we have seen in the proof of Proposition 2.10, this extension behaves in exactly the same manner as the extension of $k[t^p]$-modules

$$0 \to (t)/(t^{r+1}) \to k[t]/(t^{r+1}) \to k \to 0$$

which is easily seen to be non-trivial when $r \geqq p$, and trivial when $r < p$.

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