AN INTEGRAL FORMULA FOR THE CHERN FORM OF A HERMITIAN BUNDLE

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Introduction

We shall consider a Hermitian n-vector bundle E over a complex manifold X. When X is compact (without boundary), S.S. Chern defined in his paper [3] the Chern classes (the basic characteristic classes of E) $\hat{C}_i(E)$, $i = 1, \cdots, n$, in terms of the basic forms Φ_i on the Grassmann manifold $H(n, N)$ and the classifying map f of X into $H(n, N)$. Moreover he proved ([3], [4]) that if E_k denotes the k-general Stiefel bundle associated with E, the $(n-k+1)$-th Chern class $\hat{C}_{n-k+1}(E)$ coincides with the characteristic class $C(E_k)$ of E_k defined as follows: Let K be a simplicial decomposition of X and $K^{2(n-k)+1}$ the $2(n-k)+1$ – shelton of K. Then there exists a section s of $E_k | K^{2(n-k)+1}$ so that one can define the obstruction cocycle $c(s)$ of s. The cohomology class of $c(s)$ is independent of such a section s. Thus one denotes by $C(E_k)$ the cohomology class of $c(s)$ which is called the characteristic class of E_k. The above fact is well known as the second definition of the Chern classes ([3]).

On the other hand, in case when X is with boundary, R. Bott and S.S. Chern established the so-called Gauss-Bonnet theorem ([1]), which gives an integral formula for the above second definition of the n-th Chern class $\hat{C}_n(E)$, that is, if $C_n(E)$ denotes the n-th Chern form induced by a norm on E (c.f. Prop. 2.1),

$$\int_X C_n(E) = \int_{\partial X} s^* \eta_n + \sum_{j=1}^{l} \text{zero}\,(p_j;\, s),$$

where the p_j are the zero points of a section s of X into E, the zero $(p_j; s)$ denote the zero-numbers of s at p_j, and η_n is the n-th boundary form of E (cf. Def. 3.1).

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The main purpose of this paper is to generalize their theorem to give an integral formula (Theorem 4.1) for the i-th Chern form $C_i(E)\,(1\leq i\leq n)$ induced by a norm on a Hermitian n-vector bundle E over a complex manifold X of a complex dimension m, according to [1] and the obstruction theory [3] and [4].

Roughly speaking, our main theorem 4.1, which is called the generalized realtive Gauss-Bonnet theorem, is as follows: Let E_k be the k-general Stiefel bundle associated with E and $\pi_k^\# E$ the induced bundle of E under the projection π_k of E_k onto X. Suppose there exisit a real $2(m-n+k-1)$-dimensional oriented submanifold A (with smooth boundary ∂A) of X (here $m=\dim_{\boldsymbol{C}} X$), and a smooth section s of $(X-A)$ into E. Then for any interior point q of A we can define the k-th complement obstruction number $obs_k^{\perp}(q, s, A)$ (c.f. Def. 4.2). Let V be a real $2(n-k+1)$-dimensional oriented manifold and D a compact domain with smooth boundary ∂D. Now given a smooth map f of V into X, we obtain the intersection numbers $n(p_i, f, A)$ of the singular chain $f: D\rightarrow X$ and A at the points $p_j\in D\cap f^{-1}(A)\,(i=1,\cdots,l)$.

Then our integral formula is given by

$$\int_D f^*C_{n-k+1}(E)=\int_{\partial D} f^*s^*\eta_{n-k+1}(\pi_k^\# E)$$
$$+\sum_{j=1}^{l} obs_k^{\perp}(f(p_j), s, A)\cdot n(p_j, f, A).$$

As an application of our theorem, we obtain Levine's "The First Main Theorem [7]" concerning holomorphic mappings f from a non-compact complex manifold V into the n-complex projective space $\boldsymbol{P}^n(\boldsymbol{C})$ (c.f. §5).

Finally we note that technics in [2] are used in the proof of Theorem 4.1.

In Section 1 we review the theory of the Chern forms as described in [1]. In Section 2 we refine this theory for the case of complex analytic Hermitian bundles and state the duality formula according to [1]. In Section 3 we define an (n,k)-trivial bundle and its boundary form (c.f. Def. 3.1 and 3.2). Furthermore we study the boundary form $\eta_{n-k+1}(\pi_k^\# E)$ of the (n,k)-trivial bundle $\pi_k^\#(E)$ associated with a Hermitian n-bundle E over a complex manifold X, which plays an important role in our theorem. In Section 4 we define the k-th obstruction number (c.f. Def. 4.1 and 4.2), and prove the generalized relative Gauss-Bonnet theorem.

In preparing this paper, I have received many advices from Dr. N. Tanaka. I would like to express my cordial thanks to him.

§1. The Chern forms

1.1 The Chern forms. Let E be a C^∞-vector bundle of fibre dimension n over a C^∞-manifold X. We denote by $T^* = T^*(X)$ the cotangent bundle of X and by $A(X) = \sum_j A^j(X)$ the graded ring of C^∞-complex valued differential forms on X. More generally we write $A(X; E)$ for the differential forms on X with values in E. Thus if $\Gamma(E)$ denotes the smooth sections of E, then it follows that $A(X; E) = A(X) \underset{A^0(X)}{\otimes} \Gamma(E)$.

DEFINITION 1.1. A *connection* on E is a differential operator $D: \Gamma(E) \longrightarrow \Gamma(T^* \otimes E)$ satisfying the following rule:

$$D(f \cdot s) = df \cdot s + f \cdot Ds \tag{1.1}$$

for $f \in A^0(X)$, $s \in \Gamma(E)$.

Suppose now that E has a definite connection D. Let $s = \{s_i\}_{1 \leq i \leq n}$ be a frame of E over V, where V is an open subset of X. Then there exist 1-forms θ_{ij} on V which satisfy the following relations:

$$Ds_i = \sum_{j=1}^n \theta_{ij} s_j \qquad i = 1, \cdots, n \tag{1.2}$$

These 1-forms θ_{ij} define a matrix of 1-forms on V, denoted by $\theta(s, D) = \|\theta_{jj}\|$, which is called the *connection matrix* relative to the frame s. From $\theta(s, D)$ we now define a matrix $K(s, D) = \|K_{ij}\|$ of 2-forms on V by $K_{ij} = d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj}$. In matrix notation:

$$K(s, D) = d\theta(s, D) - \theta(s, D) \wedge \theta(s, D). \tag{1.3}$$

$K(s, D)$ is called the *curvature matrix* of D relative to the frame s,

Let us consider any two frames s and s' of $E|V$. Then there exist elements $A_{ij} \in A^0(V)$ such that $s_i' = \sum_j A_{ij} s_j$ and in matrix notation we write simply $s' = As$. Then we have the following transformation law

$$AK(s, D) = K(s, D)A \qquad s' = As. \tag{1.4}$$

From this and the fact that even forms commute with one another, we have

DEFINITION 1.2. The *Chern form* of E relative to D, denoted by $C(E, D)$, is a global form on X defined as follows: Let us cover X by $\{V_\alpha\}$ which admit frames s^α over V_α: Let $\det\{1_n + iK(s, D)/2\pi\}$ denote determinants of matrices $1_n + iK(s^\alpha, D)/2\pi$, where $i = \sqrt{-1}$ and 1_n is the unit matrix. Then we set

$$C(E, D)|V_\alpha = \det\{1_n + iK(s^\alpha, D)/2\pi\}. \tag{1.5}$$

Moreover in terms of the transformation law (1.4), the curvature matrices $K(s^\alpha, D) = \|K_{ij}\|$ determine a definite element $K[E, D] \in A^2(X: \mathrm{Hom}(E, E))$ as follows: Let t be any elemet of $\Gamma(E)$. Then for each open set V_α there exists elements $f_i^\alpha \in A^0(V_\alpha)$ such that $t = \sum_{i=1}^n f_i^\alpha s_i^\alpha$, $s^\alpha = \{s_i^\alpha\}_{1 \leq i \leq n}$. Here we put

$$K[E, D]\cdot t = \textstyle\sum_{i,j=1}^n f_i^\alpha K_{ij}^\alpha \cdot s_j^\alpha \quad \text{on} \quad V_\alpha. \tag{1.6}$$

$K[E, D]$ is called the *curvature element* of E relative to D.

1.2. Reformulation of the Chern forms. We observe that by using the curvature element $K[E, D]$, we can reformulate the Chern form $C(E, D)$ in the following manner.

DEFINITION 1.3. Let M_n denote the vector space of $n \times n$ matrices over $\boldsymbol{C}$. A k-linear function φ on M_n is called *invariant* if for any $B \in GL(n: \boldsymbol{C})$,

$$\varphi(A_1, \cdots, A_k) = \varphi(BA_1B^{-1}, \cdots, BA_kB^{-1}) \text{ for } A_i \in M_n. \tag{1.7}$$

We denote by $I^k(M_n)$ the vector space of all the k-linear invariant functioons.

Now given $\varphi \in I^k(M_n)$ and an open set V of X, we ixtend φ to a k-linear mapping, denoted by φ_V, from $M_n \otimes A(V)$ into $A(V)$ by putting

$$\omega_V(A_1\omega_1, \cdots, A_k\omega_k) = \varphi(A_1, \cdots, A_k)\omega_1 \wedge \cdots \wedge \omega_k$$

for $A_i \in M_n$, $\omega_i \in A(V)$.

On the other hand if $\xi \in A(X: \mathrm{Hom}(E, E))$ and if $s = \{s_i\}$ is a frame of $E|V$, then ξ determines a matrix of forms $\xi(s) = \|\xi(s)_{ij}\| \in M_n \otimes A(V)$ by $\sum_j \xi(s)_{ij} s = \xi \cdot s_i$, and under the substitution $s' = As$. these matrices transform by $\xi(s') = A\xi(s)A^{-1}$. Hence given $\xi_i \in A(X: \mathrm{Hom}(E, E))$ $(i = 1, \cdots, k)$ and $\varphi \in I^k(M_n)$; we can define a form $\varphi(\xi_1, \cdots, \xi_k) \in A(X)$ as follows: Let s be a frame of $E|V$. Then set

$$\varphi(\xi_1, \cdots, \xi_k)|V = \varphi_v(\xi_1(s), \cdots, \xi_k(s)) \tag{1.8}$$

where the $\xi_i(s)$ are matrices of ξ_i relative to s.

For simplicity we put $\varphi(\xi, \cdots, \xi) = \varphi((\xi))$.

Now let D be a connection on E and let $C(E,D)$ and $K[E,D]$ denote the Chern form and the curvature element of E relative to D respectively. Then we want to construct k-linear invariant functions $b_k^n \in I^k(M_n)$ $(k=1,\cdots,n)$ such that

$$C(E,D) = 1 + \textstyle\sum_{k=1}^n b_k^n(\kappa K[E,D])) \qquad \kappa = i/2\pi.$$

For this purpose let L be a k-tuples $(i_1,\cdots,i_k)$ of integers from $\{1,\cdots,n\}$ such that $i_1 < \cdots < i_k$. Then we define linear mappings L_l on M_n $(l = 1,\cdots,k)$ as follows: For any $A = \|a_{ij}\| \in M_n$, we put

$$L_l(A) = \begin{pmatrix} a_{i_1 i_l} \\ \vdots \\ a_{i_k i_l} \end{pmatrix} \qquad l = 1,\cdots,k.$$

If $A_\alpha = \|a^\alpha_{ij}\| \in M_n$, $(a = 1,\cdots,k)$, then $\det\{L_1(A_1),\cdots,L_k(A_k)\}$ denotes the determinant of the matrix $\|a^\alpha_{i_\beta i_\gamma}\|_{1\le\beta,\gamma\le k}$. With this notation k-linear functions b_k^n are defined as follows: For any $A_\alpha \in M_n$ $(a = 1,\cdots,k)$,

$$b_k^n(A_1,\cdots,A_k) = \textstyle\sum_{\sigma,L} \dfrac{1}{k!} \det\{L_1(A_{\sigma(1)}),\cdots,L_k(A_{\sigma(k)})\}, \tag{1.9}$$

where the summation is extended over all permutations σ of $\{1,\cdots,k\}$ and all k-tuples $L = (i_1,\cdots,i_k)$ of integers from $\{1,\cdots,n\}$ such that $i_1 < \cdots < i_k$.

It is clear from definition that the b_k^n are symmetric, that is, for any permutation σ of $\{1,\cdots,k\}$,

$$b_k^n(A_1,\cdots,A_k) = b_k^n(A_{\sigma(1)},\cdots,A_{\sigma(k)}) \quad A_i \in M_n.$$

Therefore in a case of $A_1 = \cdots = A_k = A$, it follows that

$$b_k^n((A)) = \textstyle\sum_L \det\{L_1(A),\cdots,L_k(A)\} \tag{1.10}$$

Hence we find that

$$\det(1_n + A) = 1 + \textstyle\sum_{k=1}^n b_k^n((A)) \quad A \in M_n, \tag{1.11}$$

where 1_n is the unit matrix of M_n.

LEMMA 1.1. *The k-linear function b_k^n is invariant, i.e., $b_k^n \in I^k(M_n)$.*

Proof. Let $\lambda_1,\cdots,\lambda_k$ be indeterminates and let $A_1,\cdots,A_k$ be any fixed elements of M_n. Then it follows from (1.10) and (1.11) that

(1.12) $$\det(1_n+\sum_{\alpha=1}^k \lambda_\alpha A_\alpha)=1+\sum_{r=1}^n\Big[\sum_{L=(i_1,\cdots,i_k)}\sum_{j_1,\cdots,j_r=1}^k \lambda_{j_1},\ldots,\lambda_{j_r}$$
$$\det\{L_1(A_{j_1})\cdots L_r(A_{j_r})\}\Big]$$

Since both sides of (1.2) are considered smooth functions of k variables $\lambda_1,\cdots,\lambda_k$, we operate $\partial^k/\partial\lambda_1\cdots\partial\lambda_k$ on each side of (1.12) at the origin $(\lambda_1,\cdots,\lambda_k)=(0,\cdots,0)=0$. Then from $\frac{\partial^k}{\partial\lambda_1\cdots\partial\lambda_k}\Big|_0(\lambda_{j1}\cdots\lambda_{jr})$

$$=\begin{cases}1 \text{ if } r=k \text{ and } \{j_1,\cdots,j_r\}=\{1,\cdots,k\}\\ 0 \text{ otherwise,}\end{cases}$$

(1.13) $$\frac{\partial^k}{\partial\lambda_1\cdots\partial\lambda_k}\Big|_0\det(1_n+\sum_{\alpha=1}^k\lambda_\alpha A_\alpha)=\sum_{\sigma,L=(i_1,\cdots,i_k)}\det\{L_1(A_{\sigma(1)}),\cdots L_k(A_{\sigma(k)}$$

Thus it follows from (1.9) and (1.13) that

(1.14) $$b_k^n(A_1,\cdots,A_k)=\frac{1}{k!}\frac{\partial^k}{\partial\lambda_1\cdots\partial\lambda_k}\Big|_0\det(1_n+\sum_{\alpha=1}^k\lambda_\alpha A_\alpha).$$

It is clear from (1.14) that b_k^n is invariant. Q.E.D.

Now let $C(E,D)$ and $K[E,D]$ be as before. Then in views of Lemma 1.1 and (1.11), we find that the b_k^n are invariant and satisfy the next relation:

(1.15) $$C(E,D)=1+\sum_{k=1}^n b_k^n((\kappa K[E,D])).$$

Notice that $b_k^n((\kappa K[E,D]))$ becomes a global form of degree $2k$ on X because of $K[E,D]\in A^2(X\colon \mathrm{Hom}(E,E))$. Here we have

Definition 1.4. Let $K[E,D]$ be the curvature element of E relative to D. Let b_k^n denote the k-linear invariant function defined by (1.9). Then the $2k$-form $b_k^n((\kappa K[E,D]))$ is called the *k th Chern form* of E relative to D, denoted by $C_k(E,D)$.

With this notation the relation (1.15) becomes

(1.15)′ $$C(E,D)=1+\sum_{k=1}^n C_k(E,D),\quad C_k(E,D)=b_k^n((\kappa K[E,D])).$$

Moreover, applying the next proposition to the invariant functions b_k^n, it follows that

(1.16) $$dC_k(E,D)=0 \qquad k=1,\cdots,n$$

so that

$$dC(E, D) = 0 \tag{1.17}$$

PROPOSITION 1.2. [1]. *Let E be a C^∞-vector bundle of fibre dimension n over a C^∞-manifold X with a connection D. Let $K[E, D]$ be the curvature element. Given any $\varphi \in I^k(M_n)$, then we obtain*

$$d\varphi((K[E, D])) = 0. \tag{1.18}$$

Next we introduce notations used in the later sections, For $\varphi \in I^k(M_n)$ we abbreviate $\sum_{\alpha=1}^{k} \varphi(A, \cdots, B, \cdots, A)$ to $\varphi'((A: B))$. We put for any $A, B \in M_n$

$$\widetilde{\det}((A)) = 1 + \sum_{k=1}^{n} b_k^n((A)) \text{ and } \widetilde{\det}'((A: B)) = \sum_{k=1}^{n} b_k^{n\prime}((A: B)).$$

Then it follows that

$$\widetilde{\det}'((A: B)) = \frac{\partial}{\partial \lambda}\Big|_0 \det(1_n + A + \lambda B), \tag{1.19}$$

$$\widetilde{\det}((\kappa K[E, D])) = C(E, D). \tag{1.20}$$

In order to prove (1.19) it is sufficient to notice that $\det(1_n + A + \lambda B) = 1 + \sum_{k=1}^{n} b_k^n((A + \lambda B))$. (1.20) is trivial.

REMARK. A connection D on E is extended uniquely to an antiderivation of the $A(X)$ module $A(X: E)$, so as to satisfy the law:

$$D(\theta \cdot s) = d\theta \cdot s + (-1)^p \theta \cdot Ds \qquad \theta \in A^p(X), \ s \in \Gamma(E). \tag{1.21}$$

Then from the definition (1.6) of the curvature element $K[E, D]$, we find that

$$D^2 s = K[E, D] \cdot s \qquad \text{for any } s \in \Gamma[E]. \tag{1.22}$$

§2. The duality formula

2.1. The canonical connection of a Hermitian bundle. Let E be a holomorphic vector bundle over a complex manifold X. Then a norm N on E is a real-valued function $N: E \longrightarrow \boldsymbol{R}$ such that the restriction of N to any fibre is a Hermitian norm on that fibre. Thus for each $x \in X$, a positive definite Hermitian form, denoted by $\langle u, v \rangle_N$, or simply $\langle u, v \rangle$, is defined by putting for any $u, v \in E_x$,

$$\langle u, v \rangle_N = \frac{1}{2}\{N(u+v) - N(u) - N(v)\} + i\frac{1}{2}\{N(u+iv) - N(u) - N(v)\}.$$

Moreover this Hermitian form $\langle\ ,\ \rangle_N$ is extended as follows: For any sections s and s', we define $\langle s, s'\rangle$ as the function $\langle s, s'\rangle(x) = \langle s(x), s'(x)\rangle$ and we set in general $\langle\theta\cdot s, \theta'\cdot s'\rangle = \theta\wedge\bar{\theta}'\langle s, s'\rangle$ $\theta, \theta'\in A(X)$. A holomorphic vector bundle with a norm is called a hermitian vector bundle. Let E be a Hermitian vector bundle. Then we will find from the following Proposition 2.1 that E has a canonical connection induced by a norm on E. It is our aim to study the Chern form of E relative to this canonical connection.

Now let X be a complex manifold. The complex valued differential froms $A(X)$ split into a direct sum $\sum A^{p,q}(X)$ where $A^{p,q}(X)$ is generated over $A^0(X)$ by forms of the type $df_1\wedge\cdots\wedge df_p\wedge d\bar{f}_{p+1}\wedge\cdots\wedge d\bar{f}_{p+q}$, the f_i being local holomorphic functions on X. Therefore d spilts into $d'+d''$ where

$$d'\colon A^{p,q}\longrightarrow A^{p+1,q} \text{ and } d''\colon A^{p,q}\longrightarrow A^{p,q+1}.$$

If E is a vector bundle over X, then $A(X\colon E)$ split into the direct sum $\sum A^{p,q}(X; E) = \sum A^{p,q}(X)\otimes\Gamma(E)$ according to the decomposition of $A(X)$. Hence any connection D on E is decomposed into $D'+D''$:

$$D'\colon \Gamma(E)\longrightarrow A^{1,0}(X\colon E) \text{ and } D''\colon \Gamma(E)\longrightarrow A^{0,1}(X\colon E).$$

With these preliminaries we obtain

PROPOSITION 2.1. [1]. Let N be a norm on a Hermitian vector bundle E. Then N induces a canonical connection $D=D(N)$ on E which is characterized by the two conditions:

(2.1) D preserves the norm N, i.e., for any $s, s'\in\Gamma(E)$

$$d\langle s, s'\rangle = \langle Ds, s'\rangle + \langle s, D'\rangle.$$

(2.2) If s is a holomorphic section of $E|V$, then $D''s=0$ on V.

This proposition shows that if $s=\{s_i\}$ is a holomorphic frame of $E|V$ and if $N(s)$ denotes the matrix of functions $N(s)=\|\langle s_i, s_j\rangle\|$, then the connection matrix $\theta(s,N)$ of $D(N)$ relative to the frame s is given by

$$\theta(s,N) = d'N(s)\cdot N(s)^{-1} \quad\text{on}\quad V, \tag{2.3}$$

and the curvature matrix $K(s,N)$ is expressed as follows:

$$K(s,N) = d''\theta(s,N), \quad \text{whence } K(s,N) \text{ is of type } (1,1) \text{ and } d''K(s,N)=0. \tag{2.4}$$

It follows from (2.4) and Definition 1.4 that the kth Chern forms $C_k(E, D(N))$ are of type (k, k).

Suppose now that E is a line bundle. Then a holomorphic frame is a nonvanishing holomorphic section s of $E|V$, so that, relative to s,

$$\theta(s, N) = d' \log N(s) \text{ and } K[E, D(N)] \cdot s = d''d' \log N(s).$$

Thus if E admits a global nonvanishing holomorphic sections s, then

$$C_1(E, D(N)) = \frac{i}{2\pi} d''d' \log N(s). \tag{2.5}$$

(Note that the invariant function b_1^1 defining $C_1(E, D(N))$ becomes the identity mapping of $M_1 = C$.)

2.2. Homotopy lemma. We state the homotopy lemma on which the duality formula is based.

DEFINITION 2.1. A connection D on a holomorphic bundle E over X, is called *of type* $(1, 1)$ if

(i) For any holomorphic section s of $E|V$, $D''s = 0$

(ii) The curvature matrix $K(s, D)$ relative to a holomorphic frame s over V, are of type $(1, 1)$, i.e., $K[E, D] \in A^{1,1}(X: \text{Hom}(E, E))$.

It is obvious from (2.4) that a cannonical connection $D(N)$ is of type (1.1).

DEFINITION 2.2. A family of connections Dt of type $(1, 1)$ will be called *bounded by* $L_t \in A^0(X: \text{Hom}(E, E))$ if for any frame s,

$$dD_t(s)/dt = d'L_t(s) + \{L_t(s) \cdot \theta(s, D_t) - \theta(s, D_t)L_t(s)\}.$$

Then we obtain the following homotopy lemma.

PROPOSITION 2.2. [1]. *Let D_t be a smooth family of comnections of type* $(1, 1)$ *on a holonorphic vector bundle E. Suppose that D_t is bounded by $L_t \in A$* $(X: \text{Hom}(E, E))$. *Then for any $\varphi \in I^k(M_n)$,* $n = \dim E$,

$$\begin{aligned} &\varphi((K[E, D_b])) - \varphi((K[E, D_a])) \\ &= d''d' \int_a^b \varphi'((K[E, D_t]: L_t))dt \end{aligned} \tag{2.6}$$

2.3. The duality formula. Now let us consider an exact sequence of holomorphic vector bundles:

$$(2.7) \qquad 0 \longrightarrow E_I \longrightarrow E \longrightarrow E_{II} \longrightarrow 0$$

over a complex manifold X. We twrite ξ for the homomorphism from E onto E_{II} defining (2.7). Let N be a norm on E. Then the norm N on E iinduces norms N_I on E_I and N_{II} on E_{II} as follows: Let $E_I^{\perp}$ be the orthocomplement of E_I, i.e., if for each $x \in X$, we put $(E_I^{\perp})_x = \{a \in E_x : \langle a, b\rangle_N = 0,$ for all $b \in E_x\}$, then $E_I^{\perp} = \cup_{x \in X}(E_I^{\perp})_x$.

Hence $E_I^{\perp}$ becomes the C^∞-vector bundle over X. The restriction of ξ to $E_I^{\perp}$ is the C^∞-isomorphism of $E_I^{\perp}$ and E_{II}. Let $\hat{\xi}$ denote the inverse mapping of $\xi | E_I^{\perp}$. Then the norm N_{II} on E_{II} is defined by

$$N_{II}(a') = N(\hat{\xi} \cdot a') \qquad \text{for any } a' \in E_{II}.$$

On the other hand, the norm N_I on E_I is the restriction of N to E_I.

To the exact sequence (2.7), there correspond the canonical connections $D(N) = D(\text{on } E)$, $D(N_i)$ (on E_i) and the Chern forms $C(E) = C(E, D((N))$, $C(E_i, D(N_i))$.

Now let $P_i (i = I,\ II)$ be the orthogonal projections

$$(2.8) \qquad P_I : E \longrightarrow E_I \quad \text{and} \quad P_{II} : E \longrightarrow E_I^{\perp}.$$

Since $P_i (i = I,\ II)$ are elements of $\Gamma(\text{Hom}(E, E))$, these are interpreted as degree zero operator, that is, $P_i(\theta \cdot s) = \theta \cdot P_i \cdot s$, $\theta \in A(X)$, $s \in \Gamma(E)$. Then the connection $D = D(N)$ is decomposed into four parts

$$(2.9) \qquad D = \textstyle\sum_{i,j} P_i D P_j \qquad j, i = I,\ II.$$

With these preliminaries we obtain

LEMMA 2.3, [1]. *In the decomposition*

(i) $P_i D P_i$ $(i \neq j)$ *are degree zero operators of type* (1.0) *and* (0,1) *respectively:*

$$(2.10) \qquad P_{II} D'' P_I = 0, \ P_I D' P_{II} = 0.$$

(ii) $P_i D P_i$ *induces the connection* $D(N_i)$ *on* $E_i \cdot i = I,\ II$.

Proof. The first statement is already proved in [1]. We shall prove only (ii). Let ξ, $\hat{\xi}$ be as above. Then ξ and $\hat{\xi}$ are considered as degree zero operators. Therefore it is clear that $\xi D \hat{\xi}$ defines a connection on E_{II}. We show that $\xi D \hat{\xi}$ is the canonical connection $D(N_{II})$. In order to prove this, it is sufficient to check the conditions (2.1) and (2.2) in Proposition

2.1. At first, (2.1) follows directly from the definition of N_{II} and the fact that D preserves the inner product $\langle\ ,\ \rangle_N$:

Let t, t' be sections of E_{II}. Then it follows that

$$d\langle t, t'\rangle_{N_{II}} = d\langle\hat{\xi}t, \hat{\xi}t'\rangle = \langle D\hat{\xi}t, \hat{\xi}t'\rangle_N + \langle\hat{\xi}t, D\hat{\xi}t'\rangle_N$$
$$= \langle\xi D\hat{\xi}t, t'\rangle_{N_{II}} + \langle t, \xi D\hat{\xi}t'\rangle_{N_{II}}.$$

For (2.2), let s be a holomorphic section of $E|V$. Then, D satisfying the condition (2.2), it follows that $D''s = 0$ on V. Hence from (2.9) we have

$$0 = D''s = (P_I D'' P_{II} + P_I D'' P_I)\cdot s + P_{II}D'P_{II}s + P_{II}D''P_I s.$$

Thus we find from (2.10) that if s is a holomorphic section of $E|V$, then

$$P_{II}D''P_{II}s = 0 \quad \text{on} \quad V. \tag{2.11}$$

Now let t be a holomorphic section of $E_{II}|V$. Then for each $x\in V$, there exist a neighborhood $V(x)\subset V$ of x and a holomorphic section s of $E|V(x)$ such that $\xi\cdot s = t$ on $V(x)$. On the other hand, it is clear that $(\xi D\hat{\xi})'' = \xi D''\hat{\xi}$, $\xi = \xi P_{II}$ and $\hat{\xi}\xi = P_{II}$. Therefore we have

$$(\xi D\hat{\xi})''\cdot t = \xi D''\hat{\xi}\cdot t = \xi D''\hat{\xi}\cdot\xi s = \xi P_{II}D''P_{II}s.$$

From (2.11) it follows that $(\xi D\hat{\xi})''t = 0$ on $V(x)$. Thus we have proved that $(\xi D\hat{\xi})''t = 0$ on V. Therefore $\xi D\hat{\xi}$ is the canonical connection $D(N_{II})$.

Hence if we identify $E_I^\perp$ and E_{II} under the isomorphism $\hat{\xi}$, then we can zalso identify $P_{II}DP_{II}$ and $\xi D\hat{\xi}$. Therefore, as we have proved, $P_{II}DP_{II}$ is regarded as the connection $D(N_{II})$ on E_{II}. Similarly it is proved that $P_I DP_I$ induces the connection $D(N_I)$ on E_I. Q.E.D.

Now a family D_t which we need for the duality theorem is given by

$$D_t = D + (e^t - 1)P_{II}DP_I \qquad \text{for all } t\in\boldsymbol{R}. \tag{2.12}$$

From (i) in Lemma 2.3 and the fact that D is the connection of type (1.1), D_t is a connection of type (1,1) for every $t\in\boldsymbol{R}$. We have further

LEMMA 2.4, [1]. *The family D_t defined by* (2.12) *is "bounded" by the element* $P_I\in\Gamma(\mathrm{Hom}(E, E))$.

Using the identifications $P_i DP_i = D(N_i)$ $(i = I,\ II)$, we obtain the following decompositions of $K[E, D_t]$ according to P_i $(i = I,\ II)$, [1]: Let $P_i K[E, D_t]P_j$ be denoted by $K_{ji}[E, D_t]$. Then we have

$$K_{II}[E, D_t] = K[E_I, D(N_I)] + e^t\square_I \tag{2.13}$$

$$K_{II\ II}[E, D_t] = K[E_{II}, D(N_{II})] + e^t\square_{II} \tag{2.14}$$

$$K_{I\ II}[E, D_t] = e^t K_{I\ II}[E, D],\ \ K_{II\ I}[E, D_t] = K_{II\ I}[E, D] \tag{2.15}$$

where $\square_I = P_I D P_{II} D P_I$ and $\square_{II} = P_{II} D P_I D P_{II}$.

Notice that $\xi K[E, D]\hat{\xi} \in A^{1,1}(X; \mathrm{Hom}(E_{II}, E_{II}))$ is identified with $K_{II,II}[E, D]$ under the isomorphism $\hat{\xi}: E_{II} \longrightarrow E_I^{\perp}$. Under this identification, $\square_{II}$ is also considered as the element of $A^2(X: \mathrm{Hom}(E_{II}, E_{II}))$, that is, from (2.14),

$$\square_{II} = K_{II\ II}[E, D] - K[E_{II}, D(N_{II})] \in A^2(X: \mathrm{Hom}(E_{II}, E_{II})).$$

We are now in a position to state the duality theorem. Let us suppose that $\dim E = n$ and let $b_k^n \in I^k(M_n)$. $(k = 1, \cdots, n)$ and let $\widetilde{\det}$ be as defined in § 1. Then from Lemma 2.4 we can apply Proposition 2.2 to D_t, P_I and $\widetilde{\det}$. Here it follows that

$$C(E, D) - C(E, D_t) = d''d' \int_t^0 \widetilde{\det}'((\kappa K[E, D_t]; \kappa P_I)). \tag{2.16}$$

In the case of $\dim E_I = 1$, we calculate (2.16). Let us take a frame $u = \{u_i\}_{1 \leq i \leq n}$ of E over an open set V of X such that u_1 and $v = \{u_i\}_{2 \leq i \leq n}$, respectively, form frames of $E_I|V$ and $E_I^{\perp}|V$. Then $v = \{u_i\}_{2 \leq i \leq n}$ is considered as the frame of $E_{II}|V$. As, relative to the frame u, $P_I(u) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)$ we find from (1.19), (2.13), (2.14) and (2.15) that $\det'((\kappa K[E, D_t]: \kappa P_I))|_V = .$
$\dfrac{\partial}{\partial \lambda}\Big|_{\lambda=0} \{1_n + \kappa K[E, D_t](u) + \lambda \kappa P_I(u)\} = \dfrac{\partial}{\partial \lambda}\Big|_{\lambda=0} \det$

$$\left(\begin{array}{c|c} 1 + \kappa K[E_I, D(N_I)](u_1) + \kappa e^t \square_I(u_1) + \lambda\kappa & \kappa e^t K_{I\ II}[E, D](u) \\ \hline \kappa K_{II\ I}[E, D](u) & 1_{n-1} + \kappa K[E_{II}, D(N_{II})](v) + \kappa e^t \square_{II}(v) \end{array}\right)$$

$$\begin{aligned}
&= \kappa \det\{1_{n-1} + \kappa K[E_{II}, D(N_{II})](v) + e^t \kappa \square_{II}(v)\} \\
&= \kappa\{1 + \textstyle\sum_{k=1}^{n-1} b_k^{n-1} v((\kappa K[E_{II}, D(N_{II})](v) + \kappa e^t \square_{II}(v))\} \\
&= \kappa\{1 + \textstyle\sum_{k=1}^{n-1} b_k^{n-1}((\kappa K[E_{II}, D(N_{II})] + \kappa e^t \square_{II}))\}|V,
\end{aligned}$$

so that, $\widetilde{\det}'((\kappa K[E, D_t]: \kappa P_I)) = \kappa\{1 + \sum_{k=1}^{n-1} b_k^{n-1}((\kappa K[E_{II}, D(N_{II})] + e^t \kappa \square_{II}))$ on X. For simplicity put $b_\alpha^{n-1}((A: (l)B)) = b_\alpha^{n-1}(\overbrace{A, \cdots, A}^{\alpha - l}, \overbrace{B, \cdots, B}^{l})$ $A, B \in M_{n-1}$ and set $b_0^{n-1}((A)) = 1$, $A \in M_{n-1}$. Then in terms of the symmetry of b_α^{n-1} and $K[E_{II}, D(N_{II})], \square_{II} \in A^2(X: \mathrm{Hom}[E_{II}, E_{II}))$, it follows that $b_\alpha^{n-1}((\kappa K[E_{II}] + e^t \kappa \square_{II})) = \sum_{l=1}^{\alpha} \binom{\alpha}{l} e^{lt} b_\alpha^{n-1}((\kappa K[E_{II}]: (l)\kappa \square_{II}))$ where $K[E_{II}] = K[E_{II}, D(N_{II})]$ and $\binom{\alpha}{0} = 1$ for $l = 0$. Therefore it follows that

$\widetilde{\det}'((\kappa K[E.D_t]\colon \kappa P_I))$

$= \kappa\sum_{\alpha=0}^{n-1} b_\alpha^{n-1}((\kappa K[E_{II}])) + \kappa\sum_{\alpha=0}^{n-1}\sum_{l=1}^{\alpha}\binom{\alpha}{l}e^{lt}b_\alpha^{n-1}((\kappa K[E_{II}]\colon (l)\kappa\,\square_{II}))$. Hence as $d''d'(\sum_{\alpha=0}^{n-1} b_\alpha^{n-1}((\kappa K[E_{II}])) = d''d'C_\alpha(E_{II}) = 0$, we have

$$\lim_{t=-\infty} d''d'\int_t^0 \widetilde{\det}'((\kappa K[E, D_t]\colon \kappa P_I))$$

$$= \kappa\sum_{\alpha=1}^{n-1}\sum_{l=1}^{\alpha-1}\frac{1}{l}\binom{\alpha}{l}\, b_\alpha^{n-1}((\kappa K[E_{II}, D(N_{II})]\colon (l)\kappa\square_{II})).$$

On the other hand, it is obvious that

$$\lim_{t=-\infty} C(E, D_t) = C(E_I)\cdot C(E_{II}).$$

Thus we obtain from (2.16) the duality formula for the case of $\dim E_I = 1$:

$$C(E) - C(E_I)\cdot C(E_{II}) \tag{2.17}$$

$$= \kappa d''d'\sum_{\alpha=1}^{n-1}\sum_{l=1}^{\alpha}\frac{1}{l}\binom{\alpha}{l} b_\alpha^{n-1}((\kappa K[E_{II}, D(N_{II})]\colon (l)\kappa\square_{II})).$$

Here we put, in general,

$$C_0(E) = 1 \quad \text{and} \quad C_\alpha(E) = 0 \quad \text{if} \quad \alpha > \dim E.$$

Then using $b_\alpha^{n-1}((\kappa K[E_{II}, D(N_{II})]\colon (l)\kappa\square_{II}))\in A^{2\alpha}(X)$, we obtain from (2.17) the following

PROPOSITION 2,5. *Let* $0\longrightarrow E_I\longrightarrow E\longrightarrow E_{II}\longrightarrow 0$ *be an exact sequence of holomorphic vector bundles over a complex manifold* X, *and let* $C(E)$, *and* $C(E_i)$ $i = I$, II *be the Chem forms induced by a norm* N *on* E. *Suppose now* $\dim E = n$. *Then if* $\operatorname{dem} E_I = 1$, *we obtain*

$$C_{n-k+1}(E) - C_1(E_I)\cdot C_{n-k}(E_{II}) - C_{n-k+1}(E_{II}) \tag{2.18}$$

$$= \kappa\, d''d'\sum_{l=1}^{n-k}\frac{1}{l}\binom{n-k}{l} b_{n-k}^{n-1}((\kappa K[E_{II}, D(N_{II})]\colon (l)\kappa\square_{II})),$$

$$k = 1, \cdots, n,$$

where $\square_{II} = P_{II}K[E, D(N)]P_{II} - K[E_{II}, D(N_{II})]\in A^2(X\colon \mathrm{Hom}\,(E_{II}, E_{II}))$.

Here we require explicit representations of $K[E_{II}, D(N_{II})]$ and $\square_{II}$.

LEMMA 2.6. *Notations being as above, let* $u = \{u_i\}_{1\le i\le n}$ *be a frame of* $E|V$ *such that* u_1 *and* $v = \{u_i\}_{2\le i\le n}$, *respectively, are frames of* $E_I|V$ *and* $E_I^\perp|V$. *Then, relative to the frame* v,

$$(2.19)\qquad K[E_{II}, D(N_{II})](v) = \|d\theta_{ij} - \textstyle\sum_{k=2}^{n}\theta_{ik}\wedge\theta_{kj}\|_{2\leq i,j\leq n}$$

$$(2.20)\qquad \square_{II}(v) = \| - \theta_{i1}\wedge\theta_{1j}\|_{2\leq i,j\leq n}\,.$$

Proof. It is trivial from assumptions that

$$P_{II}DP_{II}\cdot u_i = \textstyle\sum_{j=2}^{n}\theta_{ij}\cdot u_j \qquad i = 2,\cdots,n.$$

Therefore it follows from (1.22) and $P_{II}DP_{II} = D(N_{II})$ that

$$\begin{aligned} K[E_{II}, D(N_{II})]\cdot u_i &= (P_{II}DP_{II})^2\cdot u_i \\ &= \textstyle\sum_{j=2}^{n}(d\theta_{ij} - \sum_{k=2}^{n}\theta_{ik}\wedge\theta_{kj})u_j, \quad 2\leq i\leq n. \end{aligned}$$

Thus (2.19) is proved. On the other hand, it follows that; for each integer i $(2\leq i\leq n)$,

$$\begin{aligned} P_{II}K[E, D]P_{II}\cdot u_i &= P_{II}D^2P_{II}u_i = P_{II}D^2u_i \\ &= \textstyle\sum_{j=2}^{n}(d\theta_{ij} - \sum_{k=1}^{n}\theta_{ik}\wedge\theta_{kj})u_j \end{aligned}$$

Then, relative to the frame v,

$$P_{II}K[E, D]P_{II}(v) = \|d\theta_{ij} - \textstyle\sum_{k=1}^{n}\theta_{ik}\wedge\theta_{kj}\|_{2\leq i,j\leq n}.$$

Therefore (2.20) follows immediately:

$$\begin{aligned} \square_{II}(v) &= P_{II}K[E, D]P_{II}(v) - K[E_{II}, D(N_{II})](v). \\ &= \| - \theta_{i1}\wedge\theta_{1j}\|_{2\leq i,j\leq n}. \end{aligned}$$

Q.E.D.

Using these relations (2.19) and (2.20), we shall apply Proposition 2.5 to the case when E is the product bundle $X\times \boldsymbol{C}^n$ over X. Let (,) be the inner product of $\boldsymbol{C}^n$ defined as follows: Let $e_1,\cdots,e_n$ be the natural basis of $\boldsymbol{C}^n$ and let $z^1,\cdots,z^n$ denote the complex coordinates corresponding to this basis. Then put

$$(2.21)\qquad (u, v) = \textstyle\sum_{i=1}^{n}\bar{z}^i(u)\bar{z}^i(v) \qquad u, v\in \boldsymbol{C}^n.$$

We take a norm N_0 on the product bundle E to be one induced by the inner product (,) of $\boldsymbol{C}^n$. Then we have

COROLLARY 2.7. *Let* $0\longrightarrow E_I\longrightarrow E\longrightarrow E_{II}\longrightarrow 0$ *be as in Proposition 2.5. Suppose that* E *is the product bundle* $X\times \boldsymbol{C}^n$ *over* X *and that* $\dim E_I = 1$. *Then it follows that*

$$(2.22)\qquad C_k(E_{II}) = (-C_1(E_I))^k \qquad 1\leq k.$$

Proof. Let $s=\{s_i\}_{1\leq i\leq n}$ be a global holomorphic frame of E defined by
$$s_i(x)=(x, e_i) \qquad x\in X,\ i=1,\cdots,n$$

Further let E_I denote the orthocomplement to E_I and let us take a frame $u=\{u_i\}_{1\leq i\leq n}$ of $E|V$ as defined in Lemma 2.6. Then there exist elements $a_{ij}\in A(V)$ such that $v_i=\sum_{j=1}^n a_{ij}\cdot s_j$ $i=1,\cdots,n$. Let A be the matrix of functions $\|a_{ij}\|$, and let put $A^{-1}=\|b_{ij}\|$. Then from $D(N_0)\cdot s_i=0$ $(i=1,\cdots,n)$ we have

$$D(N_0)\cdot u_i=\textstyle\sum_{k=1}^n(\sum_{k=1}^n da_{ik}b_{kj})\cdot u_j.$$

Therefore if we put $\omega_{ij}=\sum_{k=1}^n da_{ik}b_{kj}$ $(i,j=1,\cdots,n)$, it follows that, relative to the frame u,

$$\theta(u, D(N_0))=\|\omega_{ij}\|_{1\leq i,j\leq n}.$$

Thus if N_{oII} denotes a norm on E_{II} induced by N_0, we find from (2.19) and (2.20) that, relative to the frame $v=\{u_i\}_{2\leq i\leq n}$,

$$K[E_{II}, D[N_{oII})](v)=\|d\omega_{ij}-\textstyle\sum_{k=2}^n\omega_{ik}\wedge\omega_{kj}\| \tag{2.23}$$

$$\Box_{II}(v)=\|-\omega_{i1}\wedge\omega_{1j}\|. \tag{2.24}$$

On the other hand, it is proved that

$$d\omega_{ij}-\textstyle\sum_{k=1}^n\omega_{ik}\wedge\omega_{kj}=0,\quad i,j=1,\cdots,n. \tag{2.25}$$

We obtain from (2.23), (2.24) and (2.25),

$$K[E_{II}, D(N_{oII})]=-\Box_{II}. \tag{2.26}$$

Hence the right hand side of (2.17) equals zero. Indeed it follows that, for each k, $(1\leq k\leq n)$,

$$b_{n-k}^{n-1}((\kappa K[E_{II}, D(N_{oII})]:\ (l)\kappa\Box_{II}))=(-1)^l b_{n-k}^{n-1}((\kappa K[E_{II}, D(N_{oII})])$$
$$=(-1)^l C_{n-k}(E_{II}).$$

From $dC_{n-k}(E_{II})=0$, we find that $d''d'b_{n-k}^{n-1}((\kappa K[E_{II}]:(l)\kappa\Box_{II}))=0$ Thus we have from (2.17)

$$C_{n-k+1}(E)-C_1(E_I)\cdot C_{n-k}(E_{II})=C_{n-k+1}(E_{II}),\ \ k=1,\cdots,n. \tag{2.27}$$

It is trivial that $C(E)=1$, that is, $C_o(E)=1$ and $C_k(E)=0$, if $k\geq 1$. Therefore from (2.27)

$$C_l(E_{II})=-C_1(E_I)\cdot C_{l-1}(E_{II}) \qquad l=1,\cdots,n. \tag{2.28}$$

By noting $C_n(E_{II}) = 0$ and $C_o(E_{II}) = 1$, (2.22) follows directly from (2.28).

Q.E.D.

§3. The (n, k)-trivial bundle

3.1. Let E be a Hermitian vector bundle of fibre dimension n over a complex manifold X, which admits k linearly independent holomorphic sections, say $s_1, \cdots, s_k$, $(1 \leq k \leq n)$. At first, let us introduce the next notation: Let V be a complex vector space and let $v_1, \cdots, v_k$ be k vectors of V. Then we denote by $[v_1, \cdots, v_k]$ the linear subspace of V spanned by the vectors $v_1, \cdots, v_k$.

Since $s_1, \cdots, s_k$ are k linearly independent holomorphic sections of E, we can define, with the notation above, the following holomorphic vector bundles over X:

$$E_0^I = \bigcup_{x \in X} [s_1(x)] \tag{3.1}$$

$$E_i^I = \bigcup_{x \in X} [s_{i+1}(x)]/[s_1(x), \cdots, s_i(x)] \qquad i = 1, \cdots, k-1 \tag{3.2}$$

$$E_i^{II} = \bigcup_{x \in X} E_x/[s_1(x), \cdots, s_i(x)] \qquad i = 1, \cdots, k. \tag{3.3}$$

For convenience sake put $E_0^{II} = E$. Then one notes that each E_i^I is a subbundle of E_i^{II} of fibre dimension 1, and that E_i^{II} is of fibre dimension $(n-i)$ for $i = 0, \cdots, k$. Now let ξ_i: $E_{i-1}^{II} \longrightarrow E_i^{II}$ $(i = 1, \cdots, k)$ be homomorphisms defined by setting, for each $x \in X$

$$\xi_1(e) = e/[s_1(x)] \quad \text{and} \quad \xi_i(e/[s_1(x), \cdots, s_i(x)]) = e/[s_1(x), \cdots, s_{i+1}(x)], \qquad i = 2, \cdots, k,$$

for any $e \in E_x$. Then there exists a system of exact sequences:

$$0 \longrightarrow E_{i-1}^I \longrightarrow E_{i-1}^{II} \longrightarrow E_i^{II} \longrightarrow 0 \quad (i = 1, \cdots, k) \tag{3.4}$$

over X. Let N be a norm on E. First of all, in terms of the exact sequence: $0 \longrightarrow E_0^I \longrightarrow E_0^{II} \longrightarrow E_1^{II} \longrightarrow 0$, the norm N on $E = E_0^{II}$ induces norms N_0^I on E_0^I and N_1^{II} on E_1^{II} as defined in §2. Next N_1^{II} induces norms N_1^{II} on E_1^I and N_2^{II} on E_2^{II} from $0 \longrightarrow E_1^I \longrightarrow E_1^{II} \longrightarrow E_2^{II} \longrightarrow 0$. Thus the norm N on E induces norms N_{i-1}^I on E_{i-1}^I and N_i^{II} on E_i^{II} inductively. Here we write $C(E)$, $C(E_{i-1}^I)$ and $C(E_i^{II})$ $(i = 1, \cdots, k)$ for the Chern forms induced by the norm N. We shall now apply the duality formula (2.17) to each exact sequence of (3.4). Let $0 \longrightarrow E_{i-1}^I \longrightarrow E_{i-1}^{II} \longrightarrow E_i^{II} \longrightarrow 0$ be as in (3.4). Let $(E_{i-1}^I)^\perp$ denote the orthocomplement to E_{i-1}^I

and let $P^{II}_{i-1}: E^{II}_{i-1} \longrightarrow (E^{I}_{i-1})^{\perp}$ be the projection. Then we define an element $\square_i \in A^2(X: \mathrm{Hom}\,(E^{II}_i, E^{II}_i))$ by

$$(3.5) \qquad \square_i = P^{II}_{i-1} K[E^{II}_{i-1}, D(N^{II}_{i-1})] P^{II}_{i-1} - K[E^{II}_i, D(N^{II}_i)]$$

where $K(E^{II}_\alpha, D(N^{II}_\alpha)]$ is the curvature element of the cannonical connection $D(N^{II}_\alpha)$ induced by N^{II}_α $(\alpha = i-1, i)$. Then noting that $\dim E^{II}_{i-1} = (n-i+1)$, we have from (2.17)

$$(3.6) \qquad C_{n-k+1}(E^{II}_{i-1}) - C_1(E^{I}_{i-1}) \cdot C_{n-k}(E^{II}_i) - C_{n-k+1}(E^{II}_i)$$
$$= \kappa d''d' \textstyle\sum_{l=1}^{n-k} \frac{1}{l} \binom{n-k}{l} b^{n-1}_{n-k} ((\kappa K E^{II}_i, D(N^{II}_i)]: (l)\kappa \square_i)), \qquad i = 1, \cdots, k.$$

Let $\tilde{s}_i: X \to E^{I}_{i-1}$ $(i = 1, \cdots, k)$ be holomorphic sections defined as follows: For each $x \in X$,

$$(3.7) \qquad \tilde{s}_1(x) = s_1(x), \text{ and } \tilde{s}_i(x) = s_i(x)/[s_1(x), \cdots, s_{i-1}(x)] \text{ for } i = 2, \cdots, k.$$

Then these sections become global nonvanishing holomorphic sections, so that from (2.5)

$$(3.8) \qquad C_1(E^{I}_{i-1}) = \chi d''d' \log N^{I}_{i-1}(s_i) \qquad i = 1, \cdots, k.$$

As $\sum_{i=1}^{k} \{C_{n-k+1}(E^{II}_{i-1}) - C_{n-k+1}(E^{II}_i)\} = C_{n-k+1}(E)$ and $d'C_{n-k}(E^{II}_i) = 0$ $i = 1, \cdots, k$, it follows from (3.6) and (3.8) that

$$(3.9) \qquad C_{n-k+1}(E)$$
$$= \kappa d''d' \textstyle\sum_{i=1}^{k} \{\log N^{I}_{i-1}(\tilde{s}_i) C_{n-k}(E^{II}_i) + \sum_{l=1}^{n-k} \frac{1}{l} \binom{n-k}{l} b^{n-i}_{n-k}$$
$$((\kappa K E^{II}_i, DN^{II}_i)]: \kappa \square_i)).$$

Put

$$(3.10) \qquad \eta_{n-k+1}(E, N, \{s_i\}_{1 \le i \le k})$$
$$= -\frac{1}{4} d^c \textstyle\sum_{i=1}^{k} \Big\{ \log N^{I}_{i-1}(\tilde{s}_i) \cdot C_{n-k}(E^{II}_i) + \sum_{l=1}^{n-k} \frac{1}{l} \binom{n-k}{l} b^{n-i}_{n-k}$$
$$((\kappa K[E^{II}_i, D(N^{II}_i): (l)\kappa \square_i)) \Big\}.$$

where $d^c = i(d' - d')$.
Then from $dd^c = -2id''d'$, $C_{n-k+1}(E) = d\eta_{n-k+1}(E, N, \{s_i\}_{1 \le i \le k})$. One notes that $\eta_{n-k+1}(E, N, \{s_i\}_{1 \le i \le k})$ is an element of $A^{2(n-k)+1}(X)$.

DEFINITION 3.1. Let E be a holomorphic vector bundle of fibre dimension n with a norm N, over a complex manifold X. Suppose further

E admits k linearly independent holomorphic sections $s_1, \cdots, s_k$. Then E is called the *(n, k)-trivial bundle with the norm N and the k-frames* $= \{s_i\}_{1\leq i\leq k}$, over X, or simply the (n, k)-trivial bundle with (N, s) over X. Moreover the $2(n-k)+1$-form $\eta_{n-k+1}(E, N, s)$ on X defined by (3.10) is called the *boundary form* of the (n, k)-trivial bundle E.

With this definition, we resume discussions above as

PROPOSITION 3.1. *Let E be an (n, k)-trivial bundle with (N, s), over a complex manifold X, and let $\eta_{n-k+1}(E, N, s)$ be the boundary form of E. If $C_{n-k+1}(E)$ denotes the $(n-k+1)$ th Chern form induced by the norm N on E, then*

$$C_{n-k+1}(E) = d\eta_{n-k+1}(E, N, s). \tag{3.11}$$

3.2. The properties of boundary forms. We shall next study a local expression of the boundary form $\eta_{n-k+1}(E, N, s)$. Let E be an (n, k)-trivial bundle with $(N, s=\{s_i\})$ over X. Then a frame $u=\{u_i\}_{1\leq i\leq n}$ of E over an open set V of X, is called a *compatible frame* with the k-frame s if:

(i) u is an orthonormal frame of $E|V$.

(ii) For each $x\in X$, $[u_1(x), \cdots, u_i(x)] = [s_1(x), \cdots, s_i(x)]$ $i=1, \cdots, k$, i.e., $u_1, \cdots, u_k$ are global orthonormal sections constructed from the k-frame s, in terms of Schmidt's orthogonalization.

Let $0\to E_{i-1}^{I}\to E_{i-1}^{II}\xrightarrow{\xi_i} E_i^{II}\to 0$ be as defined in (3.4) and put $\xi_0=$ identity mapping of E. Let $u=\{u_i\}_{1\leq i\leq n}$ be a compatible frame of $E|V$ with the k-frame s. Then for each i, $(1\leq i\leq k)$, $\{\xi_{i-1}\cdots\xi_0 u_t\}_{i\leq t\leq n}$ becomes an orthonormal frame of E_{i-1}^{II} such that $\xi_{i-1}\cdots\xi_0 u_i$ and $\{\xi_i\cdots\xi_1 u_t\}_{i+1\leq t\leq n}$ form orthonormal frames of $E_{i-1}^{I}|V$ and $E_i^{II}|V$ respectively. Moreover if $\hat{\xi}_i\colon E_i^{II}\to(E_{i-1}^{I})^{\perp}$ denotes the inverse mapping of $\xi_i|(E_{i-1}^{I})^{\perp}$, $i=1, \cdots, k$, then from (ii) in Lemma 2.3 it follows that $D(N_i^{II})=\xi_i\cdots\xi_1 D\hat{\xi}_1\cdots\hat{\xi}_i$, $i=1, \cdots, k$. Combining these facts with Lemma 2.6, we can prove inductively

LEMMA 3.2. *Let u be a compatible frame of $E|V$ with the k-frame θ and let $\theta(u, D(N))=\|\theta_{ij}\|$ be the connection matrix of the connection $D(N)$ relative to the frame u. Let us put, for each i, $(i=1, \cdots, k)$,*

$$\Theta_{ii} = \|d\theta_{st} - \textstyle\sum_{l=i+1}^{n}\theta_{sl}\wedge\theta_{lt}\|_{i+1\leq s,t\leq n} \tag{3.12}$$

$$\Theta_i = \|-\theta_{si}\wedge\theta_{it}\|_{i+1\leq s,t\leq n} \tag{3.13}$$

(3.14) $$s_i = \sum_{j=1}^{i} g_{ij}u_j, \quad g_{ij} \in A^0(X).$$

Then relative to the frame $\{\xi_{i-1}\cdots\xi_1 u_t\}_{i+1\leq t\leq n}$,

(3.15) $$K[E_i^{II}, D(N_i^{II})] = \Theta_{ii}$$

(3.16) $$\square_i = \Theta_i$$

(3.17) $$N_{i-1}^I(\tilde{s}_i) = |g_{ii}|^2, \ \textit{for} \ i = 1, \cdots, k.$$

Therefore we obtain from (3.10)

(3.18) $$\eta_{n-k+1}(E, N, s)|V$$
$$= \frac{-1}{4\pi} d^c \sum_{i=1}^{k} \Big\{ \log |g_{ii}|^2 b_{n-k}^{n-i}((\kappa\Theta_{ii})) + \sum_{l=1}^{n-k} \frac{1}{l} \binom{n-k}{l} b_{n-k}^{n-i}((\kappa\Theta_{ii} : (l)\kappa\Theta_0)) \Big\}.$$

From this lemma we have

COROLLARY 3.3. *The boundary form* $\eta_{n-k+1}(E, N, s)$ *is a real form on* X.

Proof. At first, let $u = \{u_i\}_{1\leq i\leq n}$ be a compatible frame of $E|V$ with s and put $\theta(u, D(N)) = \|\theta_{ij}\|$. Then since $D(N)$ preserves the inner product $\langle\ ,\ \rangle_N$ and $\langle u_i, u_j\rangle_N = \delta_{ij}$, $i, j = 1, \cdots, n$, we observe that $\bar{\theta}_{ij} = -\theta_{ji}$ $i, j=1, \cdots, n$. Therefore if Θ_{ii} and Θ_i are as defined by (3.12) and (3.13) respectively, then $\bar{\Theta}_{ii} = -{}^t\Theta_{ii}$ and $\bar{\Theta}_i = -{}^t\Theta_i$ for each i. On the other hand, from the definition (1.9) of b_k^n,

$$b_k^n(A_1, \cdots, A_k) = b_k^n({}^tA_1, \cdots, {}^tA_k) \qquad A_i \in M_n.$$

Hence, $b_{n-k}^{n-i}((\bar{\kappa}\bar{\Theta}_{ii})) = b_{n-k}^{n-i}((\kappa {}^t\Theta_{ii})) = b_{n-k}^{n-i}((\kappa\Theta_{ii}))$, and $b_{n-k}^{n-i}((\bar{\kappa}\bar{\Theta}_{ii}; (l)\bar{\kappa}\bar{\Theta}_i)) = b_{n-k}^{n-i}((\kappa\Theta_{ii} : (l)\kappa\Theta_i))$. Further, as $\bar{d}^c = d^c$ this corollary is proved. Q.E.D.

3.3 Naturality of boundary forms. We shall next state the naturality of the boundary form. For this purpose, in general, let E be a Hermitian vector bundle over a complex manifold X, and let Y be a complex manifold. Now given a holomorphic mapping $f: Y \longrightarrow X$, we have the induced bundle, denoted by $f^\sharp E$, of E under f defined as follows: Let $\Pi: E \longrightarrow X$ be the projection. Then

$$f^\sharp E = \{(y, e) \in Y \times E : f(y) = \Pi(e)\}.$$

If $t \in \Gamma(E)$, then $t.f$ is considered as an element of $\Gamma(f^\sharp E)$. Let N be a norm on E. Then a norm $f^\sharp N$ on $f^\sharp E$ is denfined by, $f^\sharp N(y, e) = N(e)$, $(y, e) \in f^\sharp E$. This norm $f^\sharp N$ is called the *induced norm* of N under f. It is trivial from definition that

$$(3.20) \qquad f^*\langle t, t'\rangle_N = \langle t.f, t'.f\rangle_{f^\sharp N}, \quad t, t' \in \Gamma(E).$$

Moreover we can define a connection $f^\sharp D$ on $f^\sharp E$ as follows: Let $t \in \Gamma(f^\sharp E)$. For each $x \in X$, we take a neighborhood V of x such that there exists a frame $s = \{s_i\}$ of $E|V$. Then there exist elements such $f_i \in A^0(f^{-1}(V))$ that $t = \sum_i f_i \cdot (s \cdot f)$ on $f^{-1}(V)$. If $\theta(s, D(N)) = \|\theta_{ij}\|$ the connection matrix relative to the frame s, then put

$$(3.21) \qquad f^\sharp D \cdot t = \sum_i df_i \cdot (s_i f) + \sum_{i,j} f_i \cdot f^*\theta_{ij}(s_j f) \text{ on } V.$$

That this definition is well-defined need not the assumption that f is holomorphic. However the next Lemma 3.4 follows from the facts that f is holomorphic and that $D(N) = D$ is the cannonical connection induced by the norm N on E.

LEMMA 3.4. *The connection $f^\sharp D$ is equal to the canonical connection $D(f^\sharp N)$, i.e., $f^\sharp D(N) = D(f^\sharp N)$.*

This is proved as (ii) in Lemma 2.3. Let $u = \{u_i\}$ be a frame of $E|V$. Then we denote by $f^\sharp u = \{u_i \cdot f\}$ the induced frame of $f^\sharp E | f^{-1}(V)$. Then we observe from Lemma 3.4 that

$$(3.22) \qquad f^*\theta(u, D(N)) = \theta\ (f^\sharp u, D(f^\sharp N)).$$

If $C(E)$ and $C(f^\sharp E)$ denote the Chern form induced by norms N and $f^\sharp N$, respectively, then

$$(3.23) \qquad f^*C(E) = C(f^\sharp E).$$

Now let E be an (n, k)-trivial bundle with (N, s) over a complex manifold X. Let Y be a complex manifold and let $f: Y \longrightarrow X$ be a holomorphic mapping. Then the induced bundle $f^\sharp E$ becomes the (n, k)-trivial bundle with $(f^\sharp N, f^\sharp s)$ over Y. Hence if $\eta_{n-k+1}(E, N, s)$ and $\eta_{n-k+1}(f^\sharp E, f^\sharp N, f^\sharp s)$ denote the boundary forms of E and $f^\sharp E$ respectively, then we obtain

PROPOSITION 3.5. (*Naturality of boundary form*)

$$(3.24) \qquad f^*\eta_{n-k+1}(E, N, s) = \eta_{n-k+1}(f^\sharp E, f^\sharp N, f^\sharp s)$$

Proof. As $d^c f^* = f^* d^c$, this proposition follows directly from (3.18), (3.20) and (3.22). Q.E.D.

3.4. The k-general Stiefel bundle. We shall study properties of the boundary form of an (n,k)-trivial bundle constructed from a Hermitian vector bundle. At first let V be a complex vector space of dimension n. Then we denote by $F_k(V)$ the k-general Stiefel manifold consisting of all the k-frames $(v_1, \cdots, v_k)$ of V. Now let E be a Hermitian vector bundle of fibre dimension n over a complex manifold X. Then let E_k be a holomorphic bundle defined by

$$(3.25) \qquad E_k = \bigcup_{x\in X} F_k(E_x).$$

This bundle E_k is called the *k-general Stiefel bundle* of E. Clearly E_k has the k-general Stiefel manifold $F_k(\boldsymbol{C}^n)$ as fibre. Let $\pi_k\colon E_k \longrightarrow X$ be the projection. Then we obtain the induced bundle $\pi_k^{\#}E$ of E under π_k. This induced bundle $\pi_k^{\#}E$ is a holomorphic vector bundle of fibre dinension n over E_k, which admits k linearly independent holomorphic sections of $\pi_k^{\#}E$, say $s_1, \cdots, s_k$, defined by setting

$$(3.26) \qquad s_i(v_1, \cdots, v_k) = \{(v_1, \cdots, v_k), v_i\}, \ (v_1, \cdots, v_k)\in E_k \ i = 1, \cdots, k.$$

Moreover let N be a norm on E. Then $\pi_k^{\#}E$ becomes the (n,k)-trivial bundle with the induced norm $\pi_k^{\#}N$ and the k-frame $s = \{s_i\}_{1\leq i\leq k}$, over E_k. Therefore if $\eta_{n-k+1}(\pi_k^{\#}E\ \pi_k^{\#}N, s)$ denotes the boundary form of $\pi_k^{\#}E$, and if $C_{n-k+1}(\pi_k^{\#}E)$ is the $(n-k+1)$ th Chern form induced by the norm $\pi_k^{\#}N$ on $\pi_k^{\#}E$, then from Proposition 3.1, $C_{n-k+1}(\pi_k^{\#}E) = d\eta_{n-k+1}(\pi_k^{\#}E,\ \pi_k^{\#}N, s)$. Further let $C_{n-k+1}(E)$ be the $(n-k+1)$ the Chern form induced by the norm N on E. Then it follows from (3.23) that $\pi_k^{*}C_{n-k+1}(E) = C_{n-k+1}(\pi_k^{\#}E)$. We have

$$(3.27) \qquad \pi_k^{*}C_{n-k+1}(E) = d\eta_{n-k+1}(\pi_k^{\#}E,\ \ \pi_k^{\#}N, s) \qquad \text{on } E_k.$$

Let x be any fixed point of X, and let us take a neighborhood V of x such that $\varphi\colon V\times F_k(\boldsymbol{C}^n) \longrightarrow \pi_k^{-1}(V)$ is a trivialization of $E_k|V$. Then we define a holomorphic mapping $\varphi_x\colon F_k(\boldsymbol{C}^n). \longrightarrow E_k$ by

$$(3.28) \qquad \varphi_x(v_1, \cdots, v_k) = \varphi\{x, (v_1, \cdots, v_k)\}\ (v_1, \cdots, v_k)\in F_k(\boldsymbol{C}^n).$$

This mapping φ_x is called the inclusion map at x. Then it is obvious from (3.27) that a $2(n-k)+1$-form

$\varphi_x^* \eta_{n-k+1}(\pi_k^\# E,\ \pi_k^\# N, s)$ on $F_k(\boldsymbol{C}^n)$ is a closed form, i.e.,

$$d\varphi_x^* \eta_{n-k+1}(\pi_k^\# E,\ \pi_k^\# N, s),\ = 0,$$

and that $\varphi_x^\#\ \pi_k^\# E = (\pi_k \cdot \varphi_x)^\# E$ is the product bundle $F_k(\boldsymbol{C}^n) \times E_x$ over $F_k(\boldsymbol{C}^n)$. Let us consider the product bundle $F_k(\boldsymbol{C}^n) \times \boldsymbol{C}^n$ over $F_k(\boldsymbol{C}^n)$. We consider $F_k(\boldsymbol{C}^n) \times \boldsymbol{C}^n$ as the (n, k)-trivial bundle with (No. s^o) defined as follows: We take a norm No to be one induced by the inner product (,) of $\boldsymbol{C}^n$ as defined in § 2, and we define a k-frame $s^o = \{s_i^o\}_{1 \leq i \leq k}$ by $s_i^o(v_1, \cdots, v_k) = \{(v_1, \cdots, v_k), v_i\}$ for $(v_1, \cdots, v_k) \in F_k(\boldsymbol{C}^n)$, $i = 1, \cdots, k$.

Then the boundary form of $F_k(\boldsymbol{C}^n) \times \boldsymbol{C}^n$ is also a cocycle form.

DEFINITION 3.2. Let $-\Phi_k$ be the boundary form of the (n, k)-trivial bundle $F_k(\boldsymbol{C}^n) \times \boldsymbol{C}^n$ with (No, s^o). Then Φ_k is called *the obstruction form* of $F_k(\boldsymbol{C}^n)$.

PROPOSITION 3.6. *Notations being as above, let* $\{\varphi_x^* \eta_{n-k+1}(\pi_k^\# E,\ \pi_k^\# N, s)\}$ *and* $\{\Phi\}_k$, *respectively, denote the cohomology class of* $\varphi_x^* \eta_{n-k+1}(\pi_k^\# E,\ \pi_k^\# N, s)$ *and* Φ_k. *Then*

(3.29) $$-\{\Phi_k\} = \{\varphi_x^* \eta_{n-k+1}(\pi_k^\# E,\ \pi_k^\# N, s)\}$$

(3.30) $\{\Phi_k\}$ *is a generator of* $2(n-k)+1$*-dimensional cohomology group of* $F_k(\boldsymbol{C}^n)$, $H^{2(n-k)+1}(F_k(\boldsymbol{C}^n); \boldsymbol{Z}) = \boldsymbol{Z}$.

Proof. At first we shall prove (3.29). Since φ_x is a holomorphic map, it follows from (3.24) that

$$\varphi_x^* \eta_{n-k+1}(\pi_k^\# E,\ \pi_k^\# N, s) = \eta_{n-k+1}((\pi_k \varphi_x)^\# E,\ (\pi_k \varphi_x)^\# N,\ \varphi_x^\# s).$$

There exists an element $g \in GL(n: \boldsymbol{C})$ such that the (n, k)-trivial bundle $(\pi_k \varphi_x)^\# E$ with $\{(\pi_k \varphi_x)^\# N, \varphi_x^\# s\}$ is identified with the (n, k)-trivial bundle $F_k(\boldsymbol{C}^n) \times \boldsymbol{C}^n$ with (No, s^o) under the transformation T_g of $F_k(\boldsymbol{C}^n)$ defined by, $T_g(v_1, \cdots, v_k) = (g \cdot v_1, \cdots, g \cdot v_k)$ for any $(v_1, \cdots, v_k) \in F_k(\boldsymbol{C}^n)$, that is,

$$\begin{aligned} &T_g^* \eta_{n-k+1}((\pi_k \varphi_x)^\# E,\quad (\pi_k \varphi_x)^\# N,\quad \varphi_x^\# s) \\ &= \eta_{n-k+1}(T_g^\# (\pi_k \varphi_x)^\# E,\quad T_g^\# (\pi_k \varphi_x)^\# N,\quad T_g^\# \varphi_x^\# s) \\ &= \eta_{n-k+1}(F_k(\boldsymbol{C}^n) \times \boldsymbol{C}^n,\quad \text{No},\ s^o) = -\Phi_k. \end{aligned}$$

However T_g is homotopic to the identity mapping of $F_k(\boldsymbol{C}^n)$. Thus, (3.29) is proved. On the other hand, (3.30) follows from the next lemma.

LEMMA 3.7. *Let* $F\colon \boldsymbol{C}^{n-k+1}-\{0\}$——$F_k(\boldsymbol{C}^n)$ *be a mapping defined by*

$$F(v)=(e_1,\cdots,e_{k-1},v) \text{ for any } v\in\boldsymbol{C}^{n-k+1}-\{0\}$$

where C^{n-k+1} *is regarded as the subspace* $\overbrace{0\times\cdots\times 0}^{k-1}\times\boldsymbol{C}^{n-k+1}$ of $\boldsymbol{C}^n$, *and* $e_1,\cdots,e_n$ *is the natural basis of* $\boldsymbol{C}^n$.
Then if $S_{n-k+1}(\boldsymbol{C})$ *is the unit sphare about the origin in* $\boldsymbol{C}^{n-k+1}$, *it follows that the restriction of* $F^*\Phi_k$ *to* $S_{n-k+1}(\boldsymbol{C})$ *becomes the normalized volume element of* $S_{n-k+1}(\boldsymbol{C})$, i.e.,

$$\int_{S_{n-k+1}(c)} F^*\Phi_k=1\cdot \tag{3.31}$$

Proof. For simplicity put $E=F_k(\boldsymbol{C}^n)\times\boldsymbol{C}^n$. Since $-\Phi_k$ is the boundary form of E with (No, s^o) and $F\colon\boldsymbol{C}^{n-k+1}-\{0\}\longrightarrow F_k(\boldsymbol{C}^n)$ is holomorphic, $F^*(-\Phi_k)$ is the bounadry form of the (n,k)-trivial bundle $F^\sharp E$ with ($F^\sharp$ No, $F^\sharp s^o$), over $\boldsymbol{C}^{n-k+1}-\{0\}$. In terms of the definitions of F and the k-frame s^o, we have

$$s_i^oF(v)=e_i \qquad i=1,\cdots,k-1, \text{ and } s_k^oF(v)=v \text{ for } v\in\boldsymbol{C}^{n-k+1}-\{0\}.$$

Hence $F^*(-\Phi_k)$ is equal to the boundary form of the $(n-k+1,1)$-trivial bundle $E(\boldsymbol{C})=(\boldsymbol{C}^{n-k+1}-\{0\})\times\boldsymbol{C}^{n-k+1}$ with the norm No and the 1-frame s_1 defined by $s_1(v)=v\times v$, $v\in\boldsymbol{C}^{n-k+1}-\{0\}$. Here let us consider the following exact sequence:

$$0\longrightarrow E(\boldsymbol{C})_0^I\longrightarrow E(\boldsymbol{C})\longrightarrow E(\boldsymbol{C})_1^{II}\longrightarrow 0$$

where $E(\boldsymbol{C})_0^I=\bigcup_{v\in\boldsymbol{C}^{n-k+1}-\{0\}}[s_1(v)]$ and $E(\boldsymbol{C})_1^{II}=\bigcup_{v\in\boldsymbol{C}^{n-k+1}-\{0\}}\boldsymbol{C}^{n-k+1}/[s_1(v)]$. Then $C_1(E(\boldsymbol{C})_0^I)=\frac{i}{2\pi}d''d'\log\mathrm{No}(s_1)$, so that, from Corollary 2.7, $C_{n-k}(E(\boldsymbol{C})_1^{II})=\left(-\frac{i}{2\pi}d''d'\log\mathrm{No}(s_1)\right)^{n-k}$. Let $z^1,\cdots,z^{n-k+1}$ be complex coordinates of $\boldsymbol{C}^{n-k+1}$. Then as $\mathrm{No}(s_1(v))=(v,v)=\sum_{j=1}^{n-k+1}z^j(v)\bar z^j(v)$, we obtain

$$\begin{aligned}F^*(-\Phi_k)&=-\frac{1}{4\pi}d^c\log\mathrm{No}(s_1)\cdot C_{n-k}(E(\boldsymbol{C})_1^{II})\\&=-\frac{1}{4\pi}d^c\log\textstyle\sum_{j=1}^{n-k+1}|z^j|^2\cdot\left(-\frac{i}{2\pi}d''d'\log\sum_{j=1}^{n-k+1}|z^j|^2\right)^{n-k}.\end{aligned}$$

Therefore $F^*\Phi_k$ is the normalized volume element of $S_{n-k+1}(\boldsymbol{C})$, [2]. Q.E.D.

One notes that in the case of $k=1$, the mapping F defined in Lemma 3.7 becomes the identity mapping of $\boldsymbol{C}^n-\{0\}$, so that, the restriction of the obstruction from Φ_1 of $F_2(\boldsymbol{C}^n)=\boldsymbol{C}^n-\{0\}$ to the unit sphere $S_{n-1}(\boldsymbol{C}^n), \Phi_1|S_{n-1(C)}$, is the normalized volume element of $S_{n-1}(\boldsymbol{C})$.

§4. The generalized relative Gauss-Bonnet formula.

4.1. In this section we shall establish an integral formula for the i th Chern form $C_i(E)$. In the case of $i=\dim E=\dim X$, Bott and Chern established the integral formula of $C_n(E)$ as the relative Gauss-Bonnet theorem. Here we want to extend this theorem.

Let E be a holomorphic vector bundle of fibre dimension n with a norm N, over an m-dimensional complex manifold X, and let E_k be the k-general Stiefel bundle of E with the projection $\pi_k: E_k \longrightarrow X$. Let $\pi_k^\# E$ be the (n, k)-trivial bundle with the induced norm $\pi_k^\# N$ and the k-frame defined by (3.26). We denote by $\eta_{n-k+1}(\pi_k^\# E)$ the boundary form of $\Pi_k^\# E$ and by $C_{n-k+1}(E)$ the $(n-k+1)$ th Chern form induced by the norm N on E. Now let A be a real $2(m-n+k-1)$-dimensional oriented submanifold of X with boundary ∂A, and let $s:(X-A) \longrightarrow E_k$ be a smooth section. Moreover let V be a real $2(n-k+1)$-dimensional (non-compact) oriented manifold and let $D \subset V$ be a compact domain with the smooth boundary ∂D. Then we obtain

THEOREM 4.1. *Let us suppose that there exists a smooth mapping $f: V \longrightarrow X$ such that $f^{-1}(A) \cap D=\{p_1, \cdots, p_l\}$ is a set of isolated points, $f^{-1}(A) \cap \partial D=\phi$, and $f(D) \cap \partial A=\phi$. If $n(p_j, f, A)$ denotes the intersection number at $(p_j; f(p_j))$ of the singular chains $f: D \longrightarrow X$ and $\iota_A: A \longrightarrow X$ ($\iota_A=$ the inclusion map), for each j, then*

$$\int_D f^* C_{n-k+1}(E)=\cdot\int_{\partial D} f^* \quad s^* \eta_{n-k+1}(\pi_k^\# E)+\textstyle\sum_{j=1}^l obs_k(p_j, sf, D) \tag{4.1}$$

$$obs_k(p_j, sf, D)=obs_k^{\perp}(f(p_j), s, A) n(p_j, f, A), \quad j=1, \cdots, \tag{4.2}$$

$$\int_D f^* C_{n-k+1}(E)=\int_{\partial D} f^* \quad s^* \eta_{n-k+1}(\pi_k^\# E)+\textstyle\sum_{j=1}^l obs_k^{\perp}(f(p_j), s, A) n(p_j, f, A), \tag{4.3}$$

where $obs_k(p_j, sf, D)$ and $obs_k^{\perp}(f(p_j), s, A)$ are integers defined in Definition 4.1 *and* 4.2, *respectively.*

4.2. Definition of obstruction numbers. Before the proof of this theorem we define $obs_k(p_j, sf, D)$ and $obs_k^{\perp}(f(p_j), s, A)$. Let Φ_k be the obstruction form of the k-general Stiefel manifold $F_k(\boldsymbol{C}^n)$. Let Y be a real $2(n-k+1)$-

demensional oriented manifold Y with boundary ∂Y. Let p be any point in $(Y-\partial Y)$. Now, given a smooth mapping $t: Y-\{p\} \longrightarrow E_k$ such that $\pi_k t$ can be regarded as the smooth mapping from Y into X, we define an integer, denoted by $obs_k(p, t, Y)$ as follows: Let $\pi_k t(p) = q \in X$ and choose a neighborhood $V(q)$ of q which admits a trivialization $\varphi: V(q) \times F_k(\boldsymbol{C}^n) \longrightarrow \pi_k^{-1}(V(q))$ of $E_k|V(q)$. Then let $\psi: \pi_k^{-1}(V(q)) \longrightarrow F_k(\boldsymbol{C}^n)$ be a holomorphic mapping defined by

$$(4.4) \qquad \psi \cdot \varphi\{q', (v_1, \cdots, v_k)\} = (v_1, \cdots, v_k), \ q' \in V(q), \ (v^1, \cdots, v^k) \in F_k(\boldsymbol{C}^n).$$

Next take a chart $(U_\delta(p),\ h = (y^1, \cdots, y^{2(n-k+1)}))$ of Y at p such that $h(p)=0$, $h(U_\delta(p))$ is the ball of raduis $U\delta$, $(\delta > 0)$ and $\pi_k t\ (U_\delta(p)) \subset V(q)$. For an ε-ball $U_\varepsilon(p)$, $0 < \varepsilon < \delta$, let us take the normalized volume element ω_k of $\partial U_\varepsilon(p)$. Further let $\gamma: U_\delta(p) - \{p\} \longrightarrow \partial U_\varepsilon(p)$ be a smooth mapping defined by

$$(4.5) \qquad \gamma_\varepsilon(p') = h^{-1}\left(\varepsilon \frac{y^1(p')}{\|h(p')\|}, \cdots, \varepsilon \frac{y^{2(p-k+1)}(p')}{\|h(p')\|}\right) \ p' \in U_\delta(p),$$

where $\|h(p')\| = (\sum_{j=1}^{2(n-k+1)} (y^j(p')^2)$

Then $\gamma_\varepsilon^* \omega_k$ becomes a cocycle form on $(U_\delta(p) - \{p\})$ whose cohomology class $\{\gamma^* \omega_k\}$ is a generator of $H^{2(n-k)+1}(U_\delta(p) - \{p\} : \boldsymbol{Z}) = \boldsymbol{Z}$. On the other hand as $\{\Phi_k\}$ is also a generator of $H^{2(n-k)+1}(F_k(\boldsymbol{C}^n) : \boldsymbol{Z}) = \boldsymbol{Z}$, it follows from the fact that $\psi \cdot t$ is a smooth mapping of $(U_\delta(p) - \{p\})$ into $F_k(\boldsymbol{C}^n)$ that there exists an integer n such that

$$(4.6) \qquad \{(\psi \cdot t)^* \Phi_k\} = n\{\gamma_\varepsilon^* \omega_k\}, \text{ i.e.,}$$

$$(4.6)' \qquad n = \int_{\partial U_\varepsilon(p)} (\psi t)^* \Phi_k .$$

Here put, $obs_k(p, t, Y) = n = \int_{\partial U_\varepsilon(p)} (\psi \cdot t)^* \Phi$

DEFINITION 4.1. The integer $obs_k(p, t, Y)$ defined by (4.6) or (4.6)$'$ is called *the kth obstruction number of t at p relative to Y*. We show that (4.6)$'$ is independent of $U_\varepsilon(p)$ and ψ. It is clear from $d\Phi_k = 0$ and Stockes formula that

$$(4.7) \qquad \int_{\partial U_\varepsilon(p)} (\psi t)^* \Phi_k = \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} (\psi t)^* \Phi_k$$

We have

LEMMA 4.2. *Let notations be as above. Then*

$$\int_{\partial U_\varepsilon(p)} (\phi \cdot t)^* \Phi_k = \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} t^* \eta_{n-k+1}(\pi_k E), \quad 0 < \varepsilon < \delta. \tag{4.8}$$

Proof. Let $\varphi_q: F_k(\boldsymbol{C}^n) \longrightarrow E_k$ be the inclusion map at $q = \pi_k t(p)$ defined from the trivialization $\varphi: V(q) \times F_k(\boldsymbol{C}^n) \longrightarrow \pi_k^{-1}(V(q))$. From $d\eta_{n-k+1}(\pi_k^\# E) = \pi_k^\# C_{n-k+1}(E)$, we have

$$d\varphi^* \eta_{n-k+1}(\pi_k^\# E) = C_{n-k+1}(E) \quad \text{on} \quad V(q) \times F_k(\boldsymbol{C}^n).$$

Moreover, as $dC_{n-k+1}(E) = 0$, we obtain a $2(n-k)+1$-form ω on $U(q)$ such that $C_{n-k+1}(E) | V(q) = d\omega$. Then $\varphi^* \eta_{n-k+1}(\pi_k E) - \omega$ is a cocycle form on $V(q) \times F_k(\boldsymbol{C}^n)$. However $H^{2(n-k)+1}(V(q) \times F_k(\boldsymbol{C}^n)) = H^{2(n-k)+1}(F_k(\boldsymbol{C}^n)) = \boldsymbol{R}$. Therefore there exists a real number a such that

$$\{\varphi^* \eta_{n-k+1}(\pi_k E) - \omega\} = a\{\Phi_k\} \quad \text{on} \quad V(q) \times F_k(\boldsymbol{C}^n).$$

Let $j_q: F_k(\boldsymbol{C}^n) \longrightarrow V(q) \times F_k(\boldsymbol{C}^n)$ be a mapping defined by

$$j_q(v_1, \cdots, v_k) = \{q, (v_1, \cdots, v_k)\} \quad (v_1, \cdots, v_k) \in F_k(\boldsymbol{C}^n).$$

Then from (3.29), $a\{\Phi_k\} = a\{j_q^* \Phi_k\} = \{(\varphi j_q)^* \eta_{n-k+1}(\pi_k^\# E) - j_q^* \omega\} = \{\varphi_q^* \eta_{n-k+1}(\pi_k^\# E)\} = -\{\Phi_k\}$. Hence $a = -1$. Therefore we have

$$\{\varphi^* \eta_{n-k+1}(\pi_k E) - \omega\} = -\{\Phi_k\} \text{ on } V(q) \times F_k(\boldsymbol{C}^n). \tag{4.9}$$

Since $\pi_k t$ is a smooth mapping of $U_\delta(p)$ into $V(q)$, Lemma 4.2 follows directly from (4.7) and (4.9) as follows:

$$\begin{aligned}
\int_{\partial U_\varepsilon(p)} (\phi t)^* \Phi_k &= \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} (\phi t)^* \Phi_k = \lim_{\varepsilon \to 0} \int (\varphi^{-1} t)^* \Phi_k \\
&= \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} (\varphi^{-1} t)^* (\omega - \varphi^* \eta_{n-k+1}(\pi_k^\# E)) \\
&= \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} (\pi_k t)^* \omega - \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} t^* \eta_{n-k+1}(\pi_k^\# E) \\
&= -\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon(p)} t^* \eta_{n-k+1}(\pi_k^\# E).
\end{aligned}$$

Q.E.D.

Thus Definition 4.1. is well-defined. This definition is extended as follows: Let $p \in Y - \partial Y$. If p is an isolated singular point of a smooth mapping t, that is, there exists a neighborhood $U(p)$ of p such that t is a smooth mapping of $(U(p) - \{p\})$ into E_k, and $\pi_k t$ is differentiable on $U(p)$, then we can define $obs_k(p, t, U(p))$. Then put

$$obs_k(p, t, Y) = obs_k(p, t, U(p)).$$

In particular, the 1 th obstruction, $obs_1(p, t, Y)$, becomes the degree of t at p because Φ_1 is regarded as the normalized volume element of the unit sphere in $\boldsymbol{C}^n$. If t is a smooth mapping of Y into E such that

α) $t \neq 0$ on ∂Y

β) t has isolated zeroes only, say $p_1, \cdots, p_l$,

then for each point p_j, $obs_1(p_j, t, Y)$ is the order of vanishing of t, so that we write by zero (p_j, t, Y) the 1 th obstruction of t at p_j relative to Y.

4.3. Let A be the submanifold of X as defined in Theorem 5.1. Let q be a point in $(A - \partial A)$. Then *a complemental submanifold to A at q*, denoted by $A_q^{\perp}$, is a real $2(n-k+1)$-dimensional oriented submanifold of X (with boundary $A_q^{\perp}$) satisfying the following conditions:

(4.10) $\qquad A_q^{\perp} \cap A = \{q\} \qquad q \in A_q^{\perp} - \partial A_q^{\perp}$

(4.11) $\qquad$ There exists a chart $(U, h = (z^1, \cdots, z^{2(n-m+k-1)}$
$y^1, \cdots, y^{2(n-k+1)}))$ at q in X such that, $h(q) = (0, \cdots, 0)$

$$A_q^{\perp} \cap U = \{q' \in U : z^1(q') = \cdots = z^{2(m-n+k-1)}(q') = 0\}$$

$$A \cap U = \{q' \in U : y^1(q') = \cdots = y^{2(n-k+1)}(q') = 0\}$$

(4.12) $\qquad A_q^{\perp}$ is compact.

Then we choose the orientation of $A_q^{\perp}$ as follows: Put $u = (z^1, \cdots, z^{2(m-n+k-1)})$ and $u = (y^1, \cdots, y^{2(n-k+1)})$. If h and u are positive coordinates systems on U and $A \cap U$ respectively, then v is also the positive coordinates system on $A_q^{\perp} \cap U$.

Since A is the submanifold of X, there exists, of course, such a submanifold of X. Now let $s : (X - A) \longrightarrow E_k$ be the smooth cross section and let $q \in (A - \partial A)$. Then taking a complemental submanifold $A_q^{\perp}$ to A at q, we can define the kth obstruction number $obs_k(q, s, A_q^{\perp})$. It will be shown in the proof of Theorem 4.1 that $obs_k(q, s, A_q^{\perp})$ is independent of $A_q^{\perp}$.

DEFINITION 4.2. For any point $q \in (A - \partial A)$, $obs_k^{\perp}(q, s, A)$ which is called *the kth obstruction number of s at q corresponding to A*, is defined as follows:

Let $A_q^{\perp}$ be a complemental sybmanifold to A at q. Then put

$$obs_k^{\perp}(q, s, A) = obs_k(q, s, A_q^{\perp}). \tag{4.13}$$

4.4. Proof of Theorem 4.1. Withoutloss of generality we can assume that $f^{-1}(A)\cap D = \{p\}$, $p\notin\partial D$ and $f(p)\notin\partial A$. and that $f(d)$ is contained in a coordinate δ_1-ball U_{δ_1} of $f(p)$ which admits a trivialization $\varphi: U_{\delta_1}\times F_k(\boldsymbol{C}^n)\longrightarrow \pi_k^{-1}(U_{\delta_1})$ of $E_k|U_{\delta_1}$. Let $V_{\varepsilon_1}(p)$ be an ε_1-ball of p contained completely in D and let put $D_{\varepsilon_1} = D - V_{\varepsilon_1}(p)$. Since $s.f: D_{\varepsilon_1}\longrightarrow E_k$ is the smooth mapping and $\pi_k^* C_{n-k+1}(E) = d\eta_{n-k+1}(\pi_k^{\#}E)$ on E_k, we obtain from Stokes formula

$$\int_{D\varepsilon_1} f^* C_{n-k+1}(E) = \int_{\partial D} f^*\{s^*\eta_{n-k+1}(\pi_k^{\#}E)\} - \int_{\partial\varepsilon V_1(p)} (s\cdot f)^*\eta_{n-k+1}(\pi_k^{\#}E).$$

Here let $\psi: \pi_k^{-1}(U_{\delta_1})\longrightarrow F_k(\boldsymbol{C}^n)$ be as defined by (4.4). Then from (4.7),

$$-\lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon(p)} (s\cdot f)^*\eta_{n-k+1}(\pi_k E) = \int_{\partial V_\varepsilon(p)} (\psi(sf))^*\Phi_k \qquad 0<\varepsilon<\varepsilon_1.$$

Therefore

$$\int_D f^* C_{n-k+1}(E) = \int_{\partial D} f^*\{s^*\eta_{n-k+1}(\pi_k E)\} + \int_{\partial V_\varepsilon(p)} (\psi(sf))^*\Phi_k.$$

This relation implies (4.1) because of $\int_{\partial V_\varepsilon(p)} (\psi(s\cdot f)^*\Phi_k = obs_k(p, s.f, D)$. In order to prove (4.2) and (4.3), we calculate the integration $\int_{\partial V_\varepsilon(p)} (\psi(sf))^*\Phi_k$, Let ε be fixed ($0<\varepsilon<\varepsilon_1$). Let us put $q = f(p)\in X$ and take a complemental submanifold $A_q^{\perp}$ to A at q. Then from the conditions (4.10) and (4.11) it follows that $A_q^{\perp}\cap A = \{q\}$ and that there exists a chart $\{U, h = (z^1, \cdots, z^{2(m-n+k-1)}, y^1, \cdots, y^{2(n-k+1)})\}$ in X at q such that $h(q) = 0$

$$A\cap U = \{q'\in U: y^1(q') = \cdots = y^{2(n-k+1)}(q') = 0\}$$
$$A_q^{\perp}\cap U = \{q'\in U: z^1(q') = z^{2(m-n+k-1)}(q') = 0\}$$

Assume $U = U_{\delta_1}$ and put $U_{\delta_1}(q) = U_{\delta_1}$. Further we assume that $f(V_\varepsilon(p))\subset U_\delta(q) \subsetneqq U_{\delta_1}(q)$, $0<\delta<\delta_1$. Let put $u = (z^1, \cdots, z^{2(m-n+k-1)})$ and $v = (y^1, \cdots, y^{2(n-k+1)})$. Then let us consider a homotopy mapping H_t given by

$$H_t = h^{-1}\{(1-t)u\times v)f\}: V_{\varepsilon_1}(p)\longrightarrow U_{\delta_1}(q), \text{ for all } t\in[0,1].$$

For $t = 1$, H_1 is the smooth mapping of $V_{\varepsilon_1}(p)$ into $A_q^{\perp}\cap U_{\delta_1}(q)$, and for each $t\in[0,1]$, $V(p)\cap H_t^{-1}(A) = \phi$ and $H_t(V_\varepsilon(p))\cap A = \{q\}$. Hence, as $f = H_0$ is homotopic to H_1, we obtain

$$(4.14) \qquad \int_{\partial V_\varepsilon(p)} (\psi s f)^* \Phi_k = \int_{\partial V_\varepsilon(p)} H_1^* (\psi s)^* \Phi_k$$

If $\iota_{A_q^\perp}: A_q^\perp \longrightarrow X$ denotes the inclusion mapping, then from $H_1(V_\varepsilon(p)) \subset A_q^\perp \cap U_\delta(q)$, (note $f(V_\varepsilon(p)) \subset U_\delta(q)$),

$$(4.15) \qquad \int_{\partial V_\varepsilon(p)} H^*(\psi s)^* \Phi_k = \int_{\partial V_\varepsilon(p)} H_1^* (\psi s \iota_{A_q^\perp})^* \Phi_k,$$

Here if ω_k denotes the normalized volume element of $\partial(A_q^\perp \cap U_\delta(q))$, and if $\gamma_\delta: (A_q^\perp \cap U_{\delta_1}(q) - \{q\}) \longrightarrow \partial(A_q^\perp \cap U_\delta(q))$ denotes a smooth mapping as defined by (4.5), then from $\{(\psi s \iota_{A_q^\perp})^* \Phi_k\} = obs_k(q, s, A_q^\perp)\{\gamma_\delta^* \omega_k\}$,

$$(4.16) \qquad \int_{\partial V_\varepsilon(p)} H_1^*(\psi . s . A_q^\perp)^* \Phi_k = obs_k(q, s, A_q^\perp) \int_{\partial V_\varepsilon(p)} (\gamma_\delta H_1)^* \omega_k$$

It follows from (4.14), (4.16) and (4.16) that

$$(4.17) \qquad \int_{\partial V_\varepsilon(p)} (\psi . s f)^* \Phi_k = obs_k(q, s, A_q^\perp) \int_{\partial V_\varepsilon(p)} (\gamma_\delta H_1)^* \omega_k$$

where H_1 is homotopic to f.

To prove that $\int_{\partial V_\varepsilon(p)} (\gamma_\delta H_1)^* \omega_k$ is equal to the intersection number at $(p, H_1(p) = q)$ of the singular chains $H_1 = h^{-1}(0 \times vf): V_\varepsilon(p) \longrightarrow X$ and $\iota_A: A \longrightarrow X$, we change the mapping $v.f$ for a mapping g_1. $V_{\varepsilon_1}(p) \longrightarrow v(U_{\delta_1}(q)) \subset \boldsymbol{R}^{2(n-k+1)}$ which agrees with $v.f$ on a neighborhood of the boundary $\partial V_\varepsilon(p)$, which is homotopic to $v.f$, and which has a maximal rank at each $p' \in g_1^{-1}(0)$. In terms of Thom's Transversality Lemma [6], there exists such a mapping g_1. Hence put $G_1 = h^{-1}(0 \times g_1)$. Then G_1 is, of course, homotopic to H_1. Thus from (4.17),

$$(4.18) \qquad \int_{\partial V_\varepsilon(p)} (\psi s f)^* \Phi_k = obs_k(q, s, s, A_q^\perp) \int_{\partial V_\varepsilon(p)} (\gamma_\delta G_1)^* \omega_k$$

(4.19) $\quad G_1 = h^{-1}(0 \times g_1): V_{\varepsilon_1}(p) \longrightarrow U_{\delta_1}(q)$, has a maximal rank at each $p' \in G_1^{-1}(q)$.

(4.20) $\quad G_1$ is homotopic to f, and each $p' \in G_1^{-1}(q)$ belongs to $V_\varepsilon(p) - \partial V_\varepsilon(p)$

Then we have

LEMMA 4.3.

$$\int_{\partial V_\varepsilon(p)} (\gamma_\delta G_1)^* \omega_k = n(q, f, A). \tag{4.21}$$

Proof. From definition of G_1 it is clear that $G_1(V_\varepsilon(p)) \cap A = \{q\}$, $G_1(\partial V_\varepsilon(p)) \cap A = \phi$ and $G_1(V_\varepsilon(p)) \cap \partial A = \phi$. Therefore from (4.20), $n(p, f, A) = n(V_\varepsilon(p), G_1, A)$. Hence, at first, we compute $n(V_\varepsilon(p), G_1, A)$. Let put $\alpha = 2(m-n+k-1)$ and $\beta = 2(n-k+1)$. Let $h = (z^1, \cdots, z^\alpha, y^1, \cdots, y^\beta)$, $u = (z^1, \cdots, z^\alpha)$ and $v = (y^1, \cdots, y^\beta)$, respectively, be coordinate systems on $U_{\delta_1}(q)$, $A \cap U_{\delta_1}(q)$ and $A_q^\perp \cap U_{\delta_1}(q)$, as before. Assume now that h and u are positive coordinate systems. Then, from the choice of the orientation of $A_q^\perp$, v is also the positive coordinate system. Let $(x^1, \cdots, x^\beta)$ be a coordinate system of $V_{\varepsilon_1}(p)$ which is positive. Let us put $G_1^{-1}(q) = \{p_1', \cdots, p_s'\}$, that is, $g_g^{-1}(0) = \{p_1', \cdots, p_s'\}$. Then we define a mapping $\iota_A \times G_1 \colon (A \cap U_{\delta_1}(q)) \times V_{\varepsilon_1}(p) \longrightarrow X$ by

$$x^i(\iota_A \times G_1)(q', p') = z^i \iota_A(q') \qquad i = 1, \cdots, \alpha$$
$$y^i(\iota_A \times G_1)(q', p') = y^i G_1(p') \qquad i = 1, \cdots, \beta$$

Here for each $p_j' \in G_1^{-1}(q)$, let $J_{(p_j', q)}(\iota_A \times G_1)$ be the Jacobian of the mapping $\iota_A \times G_1$ at (p_j', q), that is,

$$J_{(p_j', q)}(\iota_A \times G) = \left| \frac{\partial z^1(\iota_A \times G_1), \cdots, z^\alpha(\iota_A \times G_1), y^1(\iota_A \times G_1), \cdots, y^\beta(\iota_A \times G_1))}{\partial(z^1, \cdots \cdot z^\alpha \quad x^1, \cdots \cdot x^\beta)} \right|_{(p_j', q)}$$

Then it follows from $z^i(\iota_A \times G_1) = z^i$ that

$$J_{(p_p', p)}(\iota_A \times G_1) = \left| \frac{\partial(y^1(\iota_A \times G_1), \cdots, y^\beta(\iota_A \times G_1))}{\partial(x^1, \cdots \quad \cdots x^\beta)} \right|_{(p_j', q)} \tag{4.22}$$
$$= \left| \frac{\partial(y^1 \cdot g_1, \cdots, y^\beta \cdot g_1)}{\partial(x^1, \cdots, x^\beta)} \right|_{p_j'} \qquad \text{for each } p_j' \in g_1^{-1} \in (0)$$

so that, from (4.19), $J_{(p_j', q)}(\iota_A \times G_1) \neq 0$ for each p_j'. Since the right hand side of (4.22) is the Jacobian $J_{p_j'}(g_1)$ of the mapping $g_1 \colon V_\varepsilon(p) \longrightarrow \boldsymbol{R}^{2(n-k+1)}$ at p_j', it follows from definition of the intersection number ([5]) that

$$n(V_\varepsilon(p), G_1, A) = \textstyle\sum_{j=1}^s \operatorname{sign} J_{p_j'}(g_1) \tag{4.23}$$

Thus we have: $n(p, f, A) = \sum_{j=1}^s \operatorname{sign} J_{p_j'}(g_1)$ where the p_j' are points of $g_1^{-1}(0)$.

Next we shall calculate $\int_{\partial V_\varepsilon(p)} (\gamma_\delta G_1)^* \omega_k$. Since ω_k is the normalized volume element of $\partial(A_{q}^{\perp}\cap U_{\delta}(q))$, and for each $p'\in(V_{\varepsilon_1}(p)-g_1^{-1}(0))$

$$\gamma_{\delta}G_1(p') = h^{-1}\Big(\overbrace{0,\cdots\cdots,0}^{\alpha},\ \delta\frac{y^1g_1(p')}{\|g_1(p')\|},\cdots,\delta\frac{y^{\beta}g_1(p')}{\|g_1(p')\|}\Big)$$

where $\|g_1(p')\| = \sqrt{\sum_{j=1}^{\beta}(y^j(p'))^2}$,

We can reformulate $\int_{\partial V_{\varepsilon}(p)}(\gamma_{\delta}G_1)^{*}\omega_k$ as follows: Let $y^1,\cdots,y^n$ be coordinaes of $\boldsymbol{R}^n$ and let S_{n-1} be the unit sphere about the origin in $\boldsymbol{R}^n$. We denote by ω the normalized volume element of S_{n-1}. Let $\gamma\colon \boldsymbol{R}^n-\{0\}\longrightarrow S_{n-1}$ be the boundary mapping defined by

$$\gamma(y^1,\cdots,y^n) = (y^1/(\sqrt{\sum(y^i)^2}),\cdots,y^n/(\sqrt{\sum(y^i)^2}).$$

Further let D_1 be a compact domain of $\boldsymbol{R}^n$. Now, given a smooth mapping $g\colon \boldsymbol{R}^n\longrightarrow\boldsymbol{R}^n$ such that $g_1^{-1}(0)\cap D_1=\{p_1',\cdots,p_s'\}$, $g_1^{-1}(0)\cap\partial D_1=\phi$ and for each p_j', $J_{p_j'}(g_1)\neq 0$.

Under this situation, we show that

$$\int_{\partial D_1}(\gamma g_1)^{*}\omega = \sum_{j=1}^{s}\operatorname{sign} J_{p_j'}(g_1). \tag{4.24}$$

Indeed, let $V_{\varepsilon}(p_j')$ be ε-balls about p_j' in D_1 which are pairwise disjoint. Put $D_{1,\varepsilon}=D-\cup V_{\varepsilon}(p_j')$. Then, as $\gamma\cdot g_1=g/\|g_1\|$ is differentiable on $D_{1,\varepsilon}$, we have from Stokes formula, $\int_{\partial D}(\gamma g_1)^{*}\omega=\sum_{j=1}^{s}\int_{\partial V_{\varepsilon}(p_j')}(\gamma g_1)^{*}\omega$. In terms of $J_{p_j'}(g_1)=0$, $(j,\cdots,s)$, we can assume that for each j, $\|g_1\|=\varepsilon$ on $\partial V_{\varepsilon}(p_j')$, and $J(g_1)\neq 0$ on $V_{\varepsilon}(p_j')$. Now let $\operatorname{vol}(S_{n-1})$ denote the volume of S_{n-1} and let put $\tau=\sum_{j=1}^{n}(-1)^{j-1}y^jdy^1\wedge\cdots\wedge dy^{j-1}\wedge dy^{j+1}\cdots\wedge dy^n$. Then $\omega=\frac{1}{\operatorname{vol}(S_{n-1})}\tau|_{S_{n-1}}$. By noting that $y^i\left(\frac{1}{\varepsilon}g_1\right)=\frac{1}{\varepsilon}y^i(g_1)$, $(i=1,\cdots,n)$, we have: for each j,

$$\begin{aligned}\int_{\partial V_{\varepsilon}(p_j')}(\gamma g_1)^{*}\omega &= \int_{\partial V_{\varepsilon}(p_j')}\left(\frac{g_1}{\varepsilon}\right)^{*}\omega = \frac{1}{\operatorname{vol}(S_{n-1})}\int_{\partial V_{\varepsilon}(p_j')}\left(\frac{g_1}{\varepsilon}\right)^{*}\tau\\ &= \frac{1}{\varepsilon^n\operatorname{vol}(S_{n-1})}\int_{\partial V_{\varepsilon}(p_j')}g_1^{*}\tau\\ &= \frac{n}{\varepsilon^n\operatorname{vol}(S_{n-1})}\int_{V_{\varepsilon}(p_j')}g_1^{*}(dy^1\wedge\cdots\wedge dy^n)\\ &= \frac{n}{\varepsilon^n\operatorname{vol}(S_{n-1})}\operatorname{sign} J_{p_j'}(g_1)\int_{(y^1g_1)^2+\cdots+(y^ng_1)^2\leq\varepsilon^2}d(y'g_1)\cdots d(y^ng_1)\\ &= \operatorname{sign} J_{p_j'}(g_1).\end{aligned}$$

Tnus (4.24) is proved, so that, we have proved Lemma 4.3. Q.E.D.

Now we return to the proof of Theorem 4.1. At first it follows from (4.18), (4.21) and $q=f(p)$ that

$$\int_{\partial V_\varepsilon(p)} (\psi s f)^* \Phi_k = obs_k(f(p), s, A^{\perp}_{f(p)})\, n(p, f, A),$$

that is,

$$obs_k(p, sf, D) = obs_k(f(p), s, A^{\perp}_{f(p)})\, n(p, f, A). \tag{4.25}$$

In particular, let us take any complemental submanifold $A'^{\perp}_q$ to A at $q \in (A-\partial A)$ as a compact domain D and the inclusion mapping $\iota_{A'^{\perp}_q} \longrightarrow X$. Then clearly $n(q, \iota_{A'^{\perp}_q}, A)=1$, so that, from (4.25) we have

$$obs_k(q, s, A'^{\perp}_q) = obs_k(q, s, A^{\perp}_q).$$

Thus $obs^{\perp}_k(q, s, A)$ is independent of $A^{\perp}_q$. Therefore

$$obs_k(p, sf, D) = obs^{\perp}_k(f(p), s, A) \cdot n(p, f, A).$$

Hence (4.2) is proved. On the other hand, (4.3) follows immediately from (4.1) and (4.2). Q.E.D.

4.5. COROLLARY 4.4, (c.f. [1]). *Let E be a Hermitian vector bundle of fibre dimension n over an m-dimensional complex manifold X, $(n \leq m)$ and let $s: X \longrightarrow E$ be a smooth section of E which is $\neq 0$ on ∂X, and which is transversal to the zero of s. Let zero (s) be the set of zeroes of s. Then zero (s) becomes a real $2(m-n)$-dimensional oriented closed submanifold of X and the proper homology class of* zero (s) *is the Poincaré dual of $C_n(E)$.*

Proof. Notice that the 1-general Stiefel bundle E_1 of E is the subbundle of E, i.e., $E_1=\{e \in E: e \neq 0\}$. Let q be any point of zero(s). From $q \in X - \partial X$, we can take a neighborhood V in X about q, which admits a trivialization $\varphi: V \times \boldsymbol{C}^n \longrightarrow E|V$. Here let $\psi: E|V \longrightarrow \boldsymbol{C}^n$ be a holomorphic mapipng defined by,

$$\psi \cdot \varphi(q', v) = v, \qquad q' \in V, \ v \in \boldsymbol{C}^n. \tag{4.26}$$

Then put $\psi s=(s_1, \cdots, s_n)$ and $s_i = s^i + \sqrt{-1}\, s^{n-i}$, $i=1, \cdots, n$. That s is transversal to the zero section of X in E, implies that $ds^1_{q'} \wedge \cdots \wedge ds^{2n}_{q'} = 0$

for each $q' \in V \cap \mathrm{zero}(s)$. We obtain a family of charts $\{V_\alpha, h_\alpha = (s_\alpha^1, \cdots, s_\alpha^{2n}, t_\alpha^1, \cdots, t_\alpha^{2(m-n)})\}$ of X such that $\{V_\alpha\}$ cover zero(s), and for each α,

(i) V_α admits a trivialization $\varphi_\alpha \colon V \times \boldsymbol{C}^n \longrightarrow E|V_\alpha$, and so,

$$\psi_\alpha \colon E|V_\alpha \longrightarrow \boldsymbol{C}^n \text{ defined by (4.26).}$$

(ii) $s_\alpha^1, \cdots, s_\alpha^{2n}$ are real-valued functions defined by ψ_α and s,

i.e., $\psi_\alpha s = (s_\alpha^1 + \sqrt{-1}\, s_\alpha^n, \cdots, s_\alpha^n + -1 s_\alpha^{2n})$.

(iii) $V_\alpha \cap \mathrm{zero}(s) = \{q \in V_\alpha \colon s_\alpha^1(q) = \cdots = s_\alpha^{2n}(q) = 0\}$

(iv) h_α is the positive coordinate system on V_α.

Therefore zero(s) is a real $2(m-n)$-dimensional closed submanifold of X, which admits charts $\{V_\alpha \cap \mathrm{zero}(s), (t_\alpha^1, \cdots, t_\alpha^{2(m-n)})\}$. We want to prove that zero(s) is orientable. Let us suppose $V_\alpha \cap V_\beta \cap \mathrm{zero}(s) \neq \phi$. Then there exists a translation function $g_{\alpha\beta} = \|(g_{\alpha\beta})_j^i\|$ on $V_\alpha \cap V_\beta$ such that

$$s_\alpha^i = \textstyle\sum_{j=1}^{2n} (g_{\alpha\beta})_j^i s_\beta^j \qquad i = 1, \cdots\cdot, 2n, \text{ and } \det(g_{\alpha\beta}) > 0.$$

Let us put $a(q) = \det$
$$\begin{pmatrix} \dfrac{\partial t_\alpha^1}{\partial t_\beta^1}, \cdots\cdots, \dfrac{\partial t_\alpha^1}{\partial t_\beta^{2(m-n)}} & \dfrac{\partial t_\alpha^1}{\partial s_\beta^1}, \cdots\cdots, \dfrac{\partial t_\alpha^1}{\partial s_\beta^{2n}} \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ \dfrac{\partial t_\alpha^{2(m-n)}}{\partial t_\beta^1}, \cdots, \dfrac{\partial t_\alpha^1}{\partial t_\beta^{2(m-n)}} & \dfrac{\partial t_\alpha^{2(m-n)}}{\partial s_\beta^1}, \cdots, \dfrac{\partial t_\alpha^{2(m-n)}}{\partial s_\beta^{2n}} \\ \dfrac{\partial s_\alpha^1}{\partial t_\beta^1}, \cdots\cdots, \dfrac{\partial s_\alpha^1}{\partial t_\beta^{2(m-n)}} & \dfrac{\partial s_\alpha^1}{\partial s_\beta^1}, \cdots\cdots, \dfrac{\partial s_\alpha^1}{\partial s_\beta^{2n}} \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ \dfrac{\partial s_\alpha^{2n}}{\partial t_\beta^1}, \cdots\cdots, \dfrac{\partial s_\alpha^{2n}}{\partial t_\beta^{2(m-n)}} & \dfrac{\partial s_\alpha^{2n}}{\partial s_\beta^1}, \cdots\cdots, \dfrac{\partial s_\alpha^{2n}}{\partial s_\beta^{2n}} \end{pmatrix}_q$$

for each $q \in V_\alpha \cap V_\beta$. Hence, as $\partial s_\alpha^i / \partial t_\beta^j(q) = 0$ for any $q \in V_\alpha \cap V_\beta \cap$ zero(s), $i = 1, \cdots, 2n$, $j = 1, \cdots, 2(m-n)$, it follows from (iv) that $a(q) = \det\left(\dfrac{\partial t_\alpha^i}{\partial t_\beta^j}\right)$ $\det(g_{\alpha\beta}) > 0$ $q \in V_\alpha \cap V_\beta \cap \mathrm{zero}(s)$, so that, from $\det(g_{\alpha,\beta}) > 0$, we find that

$$\det\left(\frac{\partial t_\alpha^i}{\partial t_\beta^j}\right) > 0 \text{ on } V_\alpha \cap V_\beta \cap \mathrm{zero}(s).$$

Therefore zero(s) is orientable. As $s \neq 0$ on ∂X, zero(s) has not the boundary. We shall next prove the second statement. For simplicity put $A = \mathrm{zero}(s)$. Since s is the smooth cross-section of $E|(X-A)$. and $\partial A = \phi$, we can define $obs_1^\perp(q, s, A)$ for any $q \in A$. Let $q \in V_\alpha \cap A$. Then we calculate $obs_{1}^{\perp}(q, s, A)$. From the condition (iii) the set $A_{q}^{\perp} = \{q' \in V_\alpha : t_\alpha^1(q') = \cdots = t_\alpha^{2(m-n)}(q') = 0\}$ becomes a complemental submanifold to A at q. Then, of course, $(s_\alpha^1, \cdots, s_\alpha^{2n})$ is the coordinate system of $A_q^\perp \cap V_\alpha$. Hence the restriction of $\psi_\alpha \cdot s$ to $A_q^\perp$ is consider as the inclusion mapping as follows: Let us put $v_\alpha(s_\alpha^1, \cdots, s_\alpha^{2n})$ and let $z^1, \cdots, z^n$ be complex coordinates of $\boldsymbol{C}^n$. If $x^1, \cdots, x^{2n}$ are coordinates of $\boldsymbol{R}^{2n}$ with $x^i + \sqrt{-1}\, x^{n+1} = z_i$, then from definition of s_α^i, $(i = 1, \cdots, 2n)$,

$$x^i \psi_\alpha s v_\alpha^{-1}(s_\alpha^1, \cdots, s_\alpha^{2n}) = s_\alpha^i \qquad i = 1, \cdots, 2n.$$

Therefore we have from $obs_1(q, s, A_q^\perp) = \mathrm{zero}\,(q, s, A_q^\perp)$, $obs_1(q, , s, A_q^\perp) = 1$. Thus for any $q \in A$, we obtain

$$obs_{1}^{\perp}(q, s, A) = 1 \qquad A = \mathrm{zero}(s). \tag{4.27}$$

Now let γ be a smooth singular $2n$-cycle in the interior of X such that every singular chain σ in γ which intersects zero(s), meets σ in an isolated intersior point. Hence we can apply Theorem 4.1 to each singular chain σ in γ. Then from (4.3) and (4.27),

$$\int_\sigma C_n(E) = \int_{\partial\sigma} s^* \eta_n(\pi_1^\# E) + n(\sigma, \ \mathrm{zero(s)})$$

where $n(\sigma$, zero(s) is the intersection number of σ and zero(s). Hence summing over σ in γ, we find

$$\int_\gamma C_n(E) = n(\gamma, \ \mathrm{zero(s)}). \qquad \text{Q.E.D.}$$

COROLLARY 4.5. [1]. (*The relative Causs-Bonnet theorem*). *Let E be a Hermitian n-bundle over an n-complex manifold X with the boundary ∂X. Now, given a smooth section s of E such that*

i) $s \neq 0$ *on* ∂X, ii) s *has isolated zeroes only, then we have*

$$\sum_{j=1}^{l} \mathrm{zero}\ (p_j;\, s) = \int_X C_n(E) - \int_{\partial X} s^* \eta_n(\pi_1^\# E)$$

where the p_j are zeroes of s.

Indeed, if we apply (4.1) to the case when $k = 1$, $\dim X = \dim E = n$, $D = X$, and $f =$ the identity mapping of X, then this corollary follows from the fact that $obs_1(p_j, s, X) = \mathrm{zero}\,(p_j : s)$ $j = 1, \cdots, l$ Q.E.D.

§ 5. An application to complex projective space

In this section we will inverstigate Levine's "The First Main Theorem" for holomorphic mappings of non-compact, complex manifolds into complex projective space [2].

Let $\boldsymbol{p}^n(\boldsymbol{C})$ be n-dimensional complex projective space of all the 1-dimensional subspaces of $\boldsymbol{C}^{n+1}$, and let V be a non-compact real $2(n-k+1)$-dimensional oriented manifold. Let $D\subset V$ be a compact domain with the smooth boundary ∂D. We assume that there exists a smooth mapping f of V into $\boldsymbol{p}^n(\boldsymbol{C})$.

THEOREM 5.1, ([2]). *Let A be a complex $(k-1)$-dimensional linear subspace of $\boldsymbol{p}^n(\boldsymbol{C})$ such that $f^{-1}(A)\cap D$ is a set of isolated points in $(D-\partial D)$. Let ι denoted the inclusion mapping of A into $\boldsymbol{p}^n(\boldsymbol{C})$. If $n(D,f,A)$ denotes the intersection number of the singular chains $f: D\longrightarrow \boldsymbol{p}^n(\boldsymbol{C})$ and $\iota: A\longrightarrow \boldsymbol{p}^n(\boldsymbol{C})$, and if $V(D)$ denotes the volume of $f(D)$, then*

$$(5.1) \qquad V(D)-n(D,f,A)=\int_{\partial D} f^*\Lambda$$

where Λ is a real $2(n-k)+1$-form on $(\boldsymbol{p}^n(\boldsymbol{C})-A)$, which is given by (5.11).

The volume element of $\boldsymbol{p}^n(\boldsymbol{C})$ is the one induced by the standard unitary invariant Kähler metric, normalized so that the volume of $\boldsymbol{p}^n(\boldsymbol{C})$ equals 1.

(Levine assumes in [2] that V is a complex manifold and that f is holomorphic.)

Proof. In order to prove this by using Theorem 4.1, let us consider the canonical holomorphic vector bundles L, T, and E over $\boldsymbol{p}^n(\boldsymbol{C})$, defined a as follows, ([1]):

(5.2) T is the product bundle $\boldsymbol{p}^n(\boldsymbol{C})\times \boldsymbol{C}^{n+1}$

(5.3) L is the subbundle of T consisting of all the pairs (l,v), where $v\in l$.

(5.4) E is the quotient bundle T/L (Note $\dim E=n$). Then, over $\boldsymbol{p}^n(\boldsymbol{C})$ we obtain the following exact sequence:

$$(5.5) \qquad 0\longrightarrow L\longrightarrow T\longrightarrow E\longrightarrow 0.$$

Let N_0 be the norm on T induced by the inner product (,) of $\boldsymbol{C}^{n+1}$ as before. In terms of (5.5), the norm N_0 on T induces norms N_1 on L and N_2 on E as stated in § 2. We shall apply Theorem 4.1 to this holomorphic

n-bundle E with the norm N_0, over $\boldsymbol{p}^n(\boldsymbol{C})$. Let $C(E)$, E_k and $\eta_{n-k+1}(\pi_k^{\#}E)$ be as defined in previous sections. Now let $z^0, \cdots z^n$ be homogeneous coordinates of $\boldsymbol{p}^n(\boldsymbol{C})$ coorresponding to the natural basis $e_0, \cdots, e_n$ of $\boldsymbol{C}^{n+1}$. Here put

$$(5.6) \qquad \Omega = \frac{i}{2\pi} d' d'' \log \textstyle\sum_{j=0}^n z^j \bar{z}^j.$$

It is well-known ([5]) that Ω is the real 2-form on $\boldsymbol{p}^n(\boldsymbol{C})$ induced by the standard, unitary invariant, Kähler metric. Then we have

LEMMA 5.2. *Let $C_l(E)$ be the l th Chern form of E. Then we obtain*

$$(5.7) \qquad C_l(E) = \Omega^l, \qquad (l = 1, \cdots, n)$$

Proof. Let V_j be open sets defined by $V_j = \{l \in \boldsymbol{p}^n(\boldsymbol{C})\colon z^j(l) \neq 0\}$, $i = 0, \cdots, n$. For each j let $(\xi^0, \cdots, \xi^{j-1}, \xi^{j+1}, \cdots, \xi^n)$ be the coordinate system on V_j defined by $\xi^i = z^i/z^j$, $i = 0, \cdots, j-1, j+1, \cdots, n$. Then we obtain a holomorphic nonvanishing section $s_j; V_j \longrightarrow L$ given by

$$s_j(l) = \{l, (\xi^0(l), \cdots, \xi^{j-1}(l), 1, \xi^{j+1}(l), \cdots, \xi^n(l))\}.$$

Of course, from definition of the norm N_1 on L,

$$N_1(s_j(l)) = 1 + (\xi(l), \xi(l))_j \qquad \text{for each } l \in V_j$$

where $(\xi(l), \xi(l)_j = \xi^0(l)\bar{\xi}^0(l) + \cdots + \xi^{j-1}(l)\bar{\xi}^{j-1}(l) + \xi^{j+1}(l)\bar{\xi}^{j+1}(l) + \cdots + \xi^n(l)\bar{\xi}^n(l)$. Therefore it follows from (2.5) that $C_1(L)|V_j = -\frac{i}{2\pi} d' d'' \log(1 + (\xi, \xi)_j)$, so that, from (5.6) we have $C_1(L) = -\Omega$. However in terms of Corollary 2.7, $C_l(E) = (-C_1(L))^l$. Hence (5.7) is proved. Q.E.D.

Further we can prove

LEMMA 5.3.

$$(5.8) \qquad \int_{\boldsymbol{p}^n(\boldsymbol{C})} C_n(E) = 1$$

Proof. Let $v \in \boldsymbol{C}^{n+1}$ and let $\hat{s}_v\colon \boldsymbol{p}^n(\boldsymbol{C}) - [v] \longrightarrow E_1 \subset E$ be a holomorphic section defined by $\hat{s}_v(l) = (l, v/l)$, $l \in \boldsymbol{p}^n(\boldsymbol{C}) - [v]$. Then from Corollary 4.5 we have

$$\int_{\boldsymbol{p}^n(\boldsymbol{C})} C_n(E) = \text{zero } ([v], \hat{s}_v).$$

It is sufficient to prove zero $([v], \hat{s}_v) = 1$. For convernience sake we assume

$v = e_0$. Then we obtain a frame $t = \{t_i\}_{1 \leq i \leq n}$ of $E|V_0$ given by $t_i(l) = (l, -e_i/l)$ $l \in V_0$. Let $\varphi: V_0 \times C^n \longrightarrow E|V_0$ be the trivialization defined by

$$\varphi(l, v) = \sum_{i=1}^n z^i(v) t_i(l) \qquad (l, v) \in V_0 \times C^n$$

where $z^1, \cdots, z^n$ are complex coordinates of C^n. Further let $\psi: E|V_0 \longrightarrow C^n$ be a holomorphic mapping defined by φ, i.e., $\psi\varphi(l, v) = v$, for $(l, v) \in V_0 \times C^n$. To show zero $([e_0], \hat{s}_{e_0}) = 1$, we estimate the mapping $\psi \cdot \hat{s}_{e_0}: V_0 \longrightarrow C^n$. If $\xi^1, \cdots, \xi^n$ denote the coordinates on V_0, as before, then it is easy to prove that

$$\psi(l) = (\xi^1(l), \cdots, \xi^n(l)) \qquad \text{for each } l \in V_0$$

Therefore $$\text{zero}\,([e_0], \hat{s}_{e_0}) = 1.$$ Q.E.D.

From Lemma 5.2 and 5.3, $C_n(E) = \Omega^n$ becomes the normalized volume element of $\boldsymbol{p}^n(\boldsymbol{C})$. Moreover from the fact that $C(E)$ (or Ω) is invariant under unitary transformations it follows that: Let $A^\perp$ be any complex $(n-k+1)$-dimensional linear subspace of $\boldsymbol{p}^n(\boldsymbol{C})$. Then

$$\int_A C_{n-k+1}(E) = \int_A \Omega^{n-k+1} = 1. \tag{5.9}$$

Now let $f, D, V(D)$ and A be as described in Theorem 5.1. Then, of course, we have

$$V(D) = \int_D f^* \Omega^{n-k+1} = \int_D f^* C_{n-k+1}(E). \tag{5.10}$$

Let l be any fixed point in A and let us take an orthonormal basis $v_0, \cdots, v_n$ of C^{n+1} such that

(α) $v_0, \cdots, v_{k-1}$ belong to A

(β) $v_{k-1} \in l$.

Then we denote by $A_l^\perp$ the complex $(n-k+1)$-dimensional projective space consisting of all the 1-dimensional subspace of $[v_{k-1}, \cdots, v_n]$. Note $A \cap A_l^\perp = \{l\}$. It is obvious that $A_l^\perp$ is a complemental submanifold to A at l without boundary. Moreover we define a holomorphic s section $s: (\boldsymbol{p}^n(\boldsymbol{C}) - A) \longrightarrow E_k$ by $s(l) = \{l, (v_0/l, \cdots, v_{k-1}/l)\}$ for all $l \in (\boldsymbol{p}^n(\boldsymbol{C}) - A)$. It is clear that s is the well-defined section. Here put

$$(5.11) \qquad \Lambda = s^*\eta_{n-k+1}(\pi_k^\# E) \qquad \text{on } \boldsymbol{p}^n(\boldsymbol{C}) - A.$$

The boundary form $\eta_{n-k+1}(\pi_k^\# E)$ is a real $2(n-k)+1$-form, and so is. Hence, from (4.3) we have: $\int_{A_l^\perp} C_{n-k+1}(E) = \int_{A_l^\perp} \Lambda + obs_k^\perp(l, s, A) n(l, A_l^\perp, A)$ where $\iota_{A_l^\perp}$: $A_l^\perp \longrightarrow \boldsymbol{p}^n(\boldsymbol{C})$ is the inclusion mapping. However $\partial A_l^\perp = \phi$, $n(l, A_l^\perp, A) = 1$, and from (5.9), $\int_{A_l^\perp} C_{n-k+1}(E) = 1$. so that, we have: for any $l \in A$ $obs_k^\perp(l, s, A) = 1$. Again using (4.3) we have

$$(5.12) \qquad \int_D f^* C_{n-k+1}(E) = \int_{\partial D} f^* \Lambda + \sum_{j=1}^l n(p_j, f, A)$$

where $\qquad f^{-1}(A) \cap D = \{p_1, \cdots, p_l\}.$

But, from definition of $n(D, f, A)$, $\sum_{j=1}^l n(p_j, f, A) = n(D, f, A)$. (5.1) follows from (5.10) and (5.12). Q.E.D.

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