Y.K. Kwon, L. Sario, and J. Schiff
Nagoya Math. J.
Vol. 42 (1971), 95–108

BOUNDED ENERGY-FINITE SOLUTIONS OF $\Delta u = Pu$ ON A RIEMANNIAN MANIFOLD

Y.K. KWON, L. SARIO, AND J. SCHIFF

Introduction

1. The classification of Riemann surfaces with respect to the equation $\Delta u = Pu$ ($P \geq 0$, $P \not\equiv 0$) was initiated by Ozawa [13] and further developed by L. Myrberg [8, 9], Royden [14], Nakai [10, 11], Sario-Nakai [15], Nakai-Sario [12], Glasner-Katz [3], and Kwon-Sario [7].

The objective of the present paper is to establish properties of bounded energy finite solutions of $\Delta u = Pu$ in terms of the P-harmonic boundary of a Riemannian manifold R. The occurrence of the P-singular point (Nakai-Sario [12]), at which all functions in the P-algebra vanish, necessitates delicate new arguments.

The P-algebra $M_P(R)$ is not, in general, uniformly dense in the space $B(R_P^*)$ of bounded continuous functions on the P-compactification R_P^*. However, we shall prove the following Urysohn-type theorem. Let K_0, K_1 be any disjoint compact subsets of R_P^* with the P-singular point $s \in K_0$. Then there exists a function $f \in M_P(R)$ such that $0 \leq f \leq 1$ on R_P^* and $f | K_i = i$ ($i = 0, 1$).

Although the standard maximum-minimum principle does not hold, the following modification can be established. Let u be P-superharmonic and bounded from below on a Riemannian manifold R such that $\liminf u \geq 0$ at the P-harmonic boundary Δ_P. Then $u \geq 0$ on R. As a consequence, $|u| \leq \limsup_{\Delta_P} |u|$ for every bounded P-harmonic function u on R.

This maximum principle together with the orthogonal decomposition enables us to prove the existence of a positive linear operator

$$\pi: B_s(\Delta_P) \to PB(R)$$

Received April 20, 1970.
The work was sponsored by the U.S. Army Research Office-Durham, Grant DA-ARO-D-31-124-70-G7, University of California, Los Angeles.

such that

$$\sup_R |\pi(f)| \leq \max_{\Delta_P} |f|$$

for all $f \in B_s(\Delta_P)$. Here $B_s(\Delta_P)$ is the space of bounded continuous functions on Δ_P which vanish at the P-singular point s, and $PB(R)$ is the space of bounded P-harmonic functions on R.

For functions $\pi(f)$ we deduce the following integral representation. There exist, for a fixed $x_0 \in R$, a regular Borel measure μ on Δ_P and a nonnegative measurable $K_P(x, t)$ on Δ_P such that

$$\pi(f)(x) = \int_{\Delta_P} f(p) K_P(x, p) d\mu(p)$$

on R for all $f \in B_s(\Delta_P)$ and $K_P(x_0, p) = 1$ on Δ_P. Here μ is unique up to a Dirac measure δ with $\delta(\Delta_P - s) = 0$. Consequently $u \in PBE(R)$ if and only if

$$u(x) = \int_{\Delta_P} f(p) K_P(x, p) d\mu(p)$$

on R for some $f \in M_P(R)$. In this case $u = f$ on Δ_P.

§1. *P*-algebra $M_P(R)$

2. On a connected, separable, oriented, smooth Riemannian manifold R of dimension N, consider the *P-algebra* $M_P(R)$ of bounded Tonelli functions f with finite energy integrals,

$$E_R(f) = \int_R \left[\sum_{i,j=1}^N g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} + Pf^2 \right] dV < \infty.$$

Here P ($\not\equiv 0$) is a fixed nonnegative continuous function on R, (g^{ij}) the inverse of the matrix (g_{ij}) of the fundamental metric tensor of R, $x = (x^1, \cdots, x^N)$ a local coordinate system, and $dV = {*1}$ the volume element of R (cf. Nakai-Sario [12] and Kwon-Sario [7]).

We endow $M_P(R)$ with the norm

$$\|f\| = \sup_R |f| + \sqrt{D_R(f) + \int_R Pf^2 dV}$$

where $D_R(f) = \int_R df \wedge {*df}$ is the Dirichlet integral of f over R.

We first show that the *P-algebra* $M_P(R)$ *with norm* $\|\cdot\|$ *is a Banach algebra, closed under the lattice operations* $f \cup g = \max(f, g)$ *and* $f \cap g = \min(f, g)$.

The latter property is obvious by the definition of $M_P(R)$. To establish the former property choose a $\|\cdot\|$-Cauchy sequence $\{f_n\}$ in $M_P(R)$. Let f be a bounded continuous function on R with $\sup_R|f-f_n|\to 0$. In view of the BD-completeness of the Royden algebra $M(R)$ (cf. Sario-Nakai [15] and Chang-Sario [1]) we have

$$f\in M(R) \text{ and } D_R(f-f_n)\to 0.$$

Since the sequence $\{f_n\}$ is $\|\cdot\|$-Cauchy, the sequence of integrals $\int_R Pf_n^2 dV$ is, by the Schwarz inequality, a Cauchy sequence of real numbers, and consequently

$$\lim_{n\to\infty}\int_R Pf_n^2 dV = d < \infty .$$

Again by the Schwarz inequality

$$\lim_{n,m\to\infty}\int_R Pf_nf_m dV = d,$$

and therefore by Fatou's lemma

$$\int_R P(f-f_n)^2 dV = \int_R \lim_{m\to\infty} P(f_m-f_n)^2 dV \leq \varliminf_{m\to\infty}\int_R P(f_m-f_n)^2 dV$$

$$\leq \varlimsup_{m\to\infty}\int_R P(f_m-f_n)^2 dV \leq d - 2\varliminf_{m\to\infty}\int_R Pf_nf_m dV + \int_R Pf_n^2 dV.$$

On letting $n\to\infty$ we obtain $\lim_{n\to\infty}\int_R P(f-f_n)^2 dV = 0$. Thus $f\in M_p(R)$ and $\|f-f_n\|\to 0$. Since $\|fg\|\leq\|f\|\cdot\|g\|$ for $f, g\in M_P(R)$, the proof is complete.

3. Next we prove that every function in the P-algebra can be $\|\cdot\|$-approximated by smooth functions in it: *Given any $f\in M_P(R)$ and $\varepsilon>0$ there exists a function $f_\varepsilon\in C^\infty(R)\cap M_P(R)$ such that $\|f-f_\varepsilon\|<\varepsilon$.*

Set $|x|^2 = \sum_1^N (x^i)^2$ and consider first a function f with compact support in a ball V': $|x|<1/2$, with V: $|x|<1$ a parametric ball. Choose a sequence $\{f_n\}$ in $C^\infty(R)\cap M(R)$ such that $f_n = 0$ on $R-V$, $\sup_R|f-f_n|\to 0$ and $D_R(f-f_n)\to 0$ (cf. Sario-Nakai [15] and Chang-Sario [1]). It is easily seen that

$$f_n\in C^\infty(R)\cap M_P(R) \quad \text{and} \quad \|f-f_n\|\to 0.$$

For the general case consider a locally finite open covering of R by parametric balls $\{V_n: |x|<1\}$. Take a partition of unity $\{\varphi_n\}$ such that $\varphi_n\in C^\infty(R)$, $\varphi_n = 0$ on $R-V_n'$, and $\sum_1^\infty \varphi_n = 1$ on R.

Since $f\varphi_n \in M_P(R)$ and $f\varphi_n = 0$ on $R - V_n'$ we can find a function $f_n \in C^\infty(R) \cap M_P(R)$ such that $f_n = 0$ on $R - V_n$ and $\|f\varphi_n - f_n\| < \varepsilon/2^n$. Let $f_\varepsilon = \sum_1^\infty f_n$. Then $f_\varepsilon \in C^\infty(R)$, $\|f - f_\varepsilon\| \leq \sum_1^\infty \|f\varphi_n - f_n\| < \varepsilon$, and $f_\varepsilon \in C^\infty(R) \cap M_P(R)$.

§ 2. Subalgebra $M_{P\Delta}$

4. Set $f = BE\text{-}\lim_n f_n$ on R if $\{f_n\}$ is uniformly bounded on R, converges to f uniformly on compact subsets, and $E_R(f - f_n) \to 0$. Let $M_{PO}(R)$ be the family of functions in $M_P(R)$ which have compact supports in R, and $M_{P\Delta}(R)$ the family of BE-limits f of sequences $\{f_n\}$ in $M_{PO}(R)$.

By an argument similar to that in No. 2, it can be shown that $M_P(R)$ is complete in the BE-topology. We shall prove: *The family $M_{P\Delta}(R)$ is complete in the BE-topology and is an ideal of $M_P(R)$.*

For the proof consider a BE-Cauchy sequence $\{f_n\}$ in $M_{P\Delta}(R)$ and let f be its BE-limit in $M_P(R)$. For each n choose a sequence $\{f_{nm}\}$ in $M_{PO}(R)$ such that $f_n = BE\text{-}\lim_m f_{nm}$ on R.

Let $\{R_n\}$ be a regular exhaustion of R. We may assume that

$$\sup_{R_n} |f_n - f_{nm}| < \frac{1}{n} \text{ and } E_R(f_n - f_{nm}) < \frac{1}{n^2}$$

for all $m \geq 1$ and $n \geq 1$. Upon truncating the f_{nm}, if necessary, by the uniform bound of $\{f_n\}$, we may assume that the sequence $\{f_{nm}\}$ is uniformly bounded. Since $f_{nn} \in M_{PO}(R)$ it suffices to prove that $f = CE\text{-}\lim_n f_{nn}$ on R. Now,

$$\begin{aligned} E_R(f - f_{nn})^{\frac{1}{2}} &\leq E_R(f - f_n)^{\frac{1}{2}} + E_R(f_n - f_{nn})^{\frac{1}{2}} \\ &< E_R(f - f_n)^{\frac{1}{2}} + \frac{1}{n} \to 0. \end{aligned}$$

For a compact set K of R choose k so large that $K \subset R_k$. Then for $n \geq k$,

$$\begin{aligned} \sup_K |f - f_{nn}| &\leq \sup_{R_k} |f - f_n| + \sup_{R_k} |f_n - f_{nn}| \\ &\leq \sup_{R_k} |f - f_n| + \frac{1}{n} \to 0, \end{aligned}$$

and we have $f = BE\text{-}\lim_n f_{nn}$ as desired.

The rest of the proof is obvious.

§ 3. *P*-compactification

5. By means of the P-algebra $M_P(R)$ we can construct a compactification R_P^* of R (cf. e.g. Constantinescu-Cornea [2] and Kwon-Sario [7]) with

the following properties:

(i) R_P^* is a compact Hausdorff space and contains R as an open dense subset.

(ii) Every $f \in M_P(R)$ has a continuous extension to R_P^*.

(iii) $M_P(R)$ separates points of R_P^*.

The space R_P^* is unique up to homeomorphisms which fix R elementwise. We shall refer to R_P^* as the *P-compactification*, and to $\Gamma_P = R_P^* - R$ as the *P-boundary* of R (Nakai-Sario [12]).

A point $s \in R_P^*$ is called a *P-singular point* if $f(s) = 0$ for all $f \in M_P(R)$ (loc. cit.). It exists and is unique if and only if $\int_R PdV = \infty$. It can be given a complete characterization (Kwon-Sario [7]): $s \in R_P^*$ *is P-singular if and only if for every neighborhood U of s in* R_P^*, $\int_{R \cap U} PdV = \infty$.

Points of R_P^* which are not P-singular will be called *P-regular*.

6. We turn to the question of the Urysohn property on R_P^*. First we prove:

LEMMA. *Let K be a compact subset of the P-compactification R_P^*, and N an open neighborhood of K in R_P^*. Then there exists a Dirichlet-finite Tonelli function f on R such that f is continuously extendable to R_P^*, $0 \leq f \leq 1$ on R_P^*, $f|K = 1$, and $f = 0$ on $R_P^* - N$.*

Proof. Let $\hat{M}_P(R)$ be the family of Dirichlet-finite bounded Tonelli functions on R with continuous extensions to R_P^*. Obviously $\hat{M}_P(R)$ is a subalgebra of $B(R_P^*)$, contains the constants, and is closed under $f \cup g$ and $f \cap g$.

Since $M_P(R) \subset \hat{M}_P(R)$, the Stone-Weierstrass theorem is applicable and we conclude that $\hat{M}_P(R)$ is uniformly dense in $B(R_P^*)$.

Choose an open set U in R_P^* with $K \subset U \subset \bar{U} \subset N$, and a function $g \in B(R_P^*)$ such that $-1 \leq g \leq 2$ on R_P^*, $g|K = 2$, and $g|R_P^* - U = -1$. By the above argument there exists a function $h \in \hat{M}_P(R)$ such that $|g - h| < 1$ on R_P^*. Then $f = (h \cup 0) \cap 1$ has the required properties.

7. The occurrence of the P-singular point s entails that the Urysohn property is only valid in the following modified form:

THEOREM. *For disjoint compact subsets K_0 and K_1 of R_P^* such that K_0 contains the P-singular point s, there exists a function $f \in M_P(R)$ such that $0 \leq f \leq 1$ on R_P^* and $f|K_i = i$ $(i = 0, 1)$.*

Proof. Since every $x \in K_1$ is P-regular there exists an open set N_x in R_P^* such that $x \in N_x$, $K_0 \cap N_x = \phi$, and $\int_{N_x \cap R} PdV < \infty$. By virtue of the compactness of K_1 we can choose a finite set $\{x_1, \cdots, x_m\} \subset K_1$ such that $K_1 \subset N = \cup_1^m N_{x_i}$, $N \cap K_0 = \phi$, and $\int_{N \cap R} PdV < \infty$.

By the above lemma there exists a function $f \in \hat{M}_P(R)$ such that $0 \leq f \leq 1$ on R_P^*, $f|K_1 = 1$, and $f|R_P^* - N = 0$. Then $E_R(f) \leq D_R(f) + \int_{N \cap R} PdV < \infty$ and f has the desired property.

§ 4. *P*-superharmonic functions

8. A function v on R is called *P-superharmonic* if

(i) v is lower semicontinuous on R, $-\infty < v \leq \infty$, $v \not\equiv \infty$ on R,

(ii) for any parametric ball V,

$$v(x) \geq -\int_{\partial V} v(y)^* dg_V(y, x)$$

on V, where $g_V(y, x)$ is the P-harmonic Green's function on V with pole x. A function v is *P-subharmonic* if $-v$ is P-superharmonic.

Let Ω be a regular subregion of R and v a C^2-function on $\bar{\Omega}$. We shall make use of the following basic property of P-harmonic and P-superharmonic functions (Nakai [11]): *If $\Delta v \leq Pv$ on Ω, then v dominates any P-harmonic function u on Ω, continuous on $\bar{\Omega}$ with $u|\partial\Omega \leq v|\partial\Omega$, that is, v is P-superharmonic on Ω.*

For the proof set $w = v - u$ on Ω. Then $\Delta w \leq Pw$ on Ω and $w|\partial\Omega \geq 0$. Let Ω_0 be a component of the open set $\{x \in \Omega | w(x) < 0\}$. Since w is superharmonic on Ω_0, we have

$$0 > w(x) \geq \inf_{\Omega_0} w = \min_{\partial \Omega_0} w = 0,$$

which implies that $\Omega_0 = \phi$, hence $w \geq 0$ on Ω as desired.

We also have at once: *If a sequence $\{v_i\}$ of continuous P-superharmonic functions on R converges to a function v uniformly on compact subsets, then v is also P-superharmonic.*

§ 5. *P*-harmonic projection

9. Next we shall establish the orthogonal decomposition theorem which plays an important role in our discussion (cf. Nakai-Sario [12]): *Every* $f \in M_P(R)$ *possesses the following properties:*

(i) *f has the unique decomposition* $f = u + g$, $u \in PBE(R)$, $g \in M_{P\Delta}(R)$.

(ii) $E(f) = E(u) + E(g)$.

(iii) *If* $f \geq 0$, *then* $u \geq 0$.

(iv) *If f is P-superharmonic (resp. P-subharmonic), then* $u \leq f$ (*resp.* $u \geq f$).

For the sake of completeness we shall sketch the proof. Take a regular exhaustion $\{R_n\}$ of R and let u_n^+ (resp. u_n^-) be the continuous function on R which is P-harmonic on R_n with $u_n^+ | R - R_n = f^+$ (resp. $u_n^- | R - R_n = f^-$). Since $0 \leq u_n^+ \leq \sup_R |f|$ and $0 \leq u_n^- \leq \sup_R |f|$ on R, we may assume that both $\{u_n^+\}$ and $\{u_n^-\}$ converge to u^+ and u^-, say, uniformly on compact subsets of R (cf. Royden [14]). Since these sequences are E-Cauchy, we have

$$u^+ = BE\text{-}\lim_n u_n^+, \quad u^- = BE\text{-}\lim_n u_n^-$$

on R and u^+, $u^- \in PBE(R)$.

Set $u = u^+ - u^- \in PBE(R)$ and $g = f - u \in M_{P\Delta}(R)$. Then $f = u + g$ is the desired decomposition. Its uniqueness and property (ii) are immediate consequences of the energy principle (cf. Royden [14]).

If $f \geq 0$ then $u_n^- \equiv 0$ and hence $u^- \equiv 0$ on R. Consequently $u = u^+ - u^- = u^+ \geq 0$ as asserted. If f is P-superhamonic on R then $u_n \leq f$ since $u_n = f$ on $R - R_n$. Therefore $u \leq f$.

The function u is called the *P-harmonic projection* of f.

§ 6. *P*-harmonic boundary

10. The set $\Delta_P = \{x \in R_P^* | f(x) = 0 \text{ for all } f \in M_{P\Delta}(R)\}$ is a compact subset of Γ_P, called the *P-harmonic boundary* of R (Nakai-Sario [12]). If $\Delta_P = \phi$, it is easily seen that $1 \in M_{P\Delta}(R)$ and hence $M_{P\Delta}(R) = M_P(R)$.

The following two properties of Δ_P are fundamental (cf. Kwon-Sario [6, 7]):

(i) $M_{P\Delta}(R) = \{f \in M_P(R) | f \equiv 0$ *on* $\Delta_P\}$.

(ii) *If* $u \in PBE(R)$ *and* $u | \Delta_P \equiv 0$, *then* $u \equiv 0$ *on* R.

11. We are now ready to establish the existence of an Evans P-superharmonic function on R. It brings forth the P-harmonically negligible nature of the set $\Gamma_P - \Delta_P$.

THEOREM. *Let F be a nonempty compact subset of $\Gamma_P - \Delta_P$. Then there exists a nonnegative continuous P-superharmonic function v on R such that $v|\Delta_P = 0$, $v|F = \infty$, and $E_R(v) < \infty$.*

Proof. There exists a compact subset K of R_P^* such that $K = \overline{K \cap R}$, $K \cap \Delta_P = \phi$, $\partial(K \cap R)$ is smooth, and F is contained in the interior K^0 of K. Choose a function $f \in M_P(R)$ such that $0 \leq f \leq 1$ on R, $f|K \equiv 1$, and $f|\Delta_P \equiv 0$. For a fixed regular exhaustion $\{R_n\}$ of R set $K_n = K - R_n$.

Construct continuous functions u_{nm} on R such that $u_{nm} = f$ on $R - (R_m - K_n)$ and $u_{nm} \in P(R_m - K_n)$. Since $\{u_{nm}\}$ is E-Cauchy for each fixed n, and $0 \leq u_{nm} \leq 1$, we may assume that $\{u_{nm}\}$ is BE-Cauchy for each n. Let $u_n = BE\text{-}\lim_m u_{nm}$. Then $u_n \in PBE(R - K_n)$, $0 \leq u_n \leq 1$, and $u_n|K_n = 1$.

Let $g_n = BE\text{-}\lim_m (f - u_{nm})$ on R. Since $g_n \in M_{P\Delta}(R)$, $g_n|\Delta_P = 0$. Thus $u_n = f = 0$ on Δ_P and $u_n \in PBE(R - K_n) \cap M_{P\Delta}(R)$. It is not difficult to see that the sequence $\{u_n\}$ has a BE-convergent subsequence, again denoted by $\{u_n\}$. Let $u = BE\text{-}\lim_n u_n$ on R. Since $u \in PBE(R) \cap M_{P\Delta}(R)$, $u \equiv 0$ on R.

For a fixed point $x_0 \in R$, we can choose a subsequence, say again $\{u_n\}$, such that

$$u_n(x_0) < 2^{-n}, \quad E_R(u_n) < 2^{-n}.$$

Let $v_m = \sum_1^m u_i$ and $v = \sum_1^\infty u_i$. Then $E_R(v - v_m) \to 0$. By Harnack's inequality $\{v_m\}$ converges to v uniformly on compact subsets of R, and v is a continuous P-superharmonic function on R.

The remainder of the proof is obvious.

12. We claim:

THEOREM. *Suppose u is P-superharmonic (resp. P-subharmonic), bounded from below (resp. above) on R, and satisfies*

$$\liminf_{x \to p,\ x \in R} u(x) \geq 0 \qquad (\text{resp. } \limsup_{x \to p,\ x \in R} u(x) \leq 0)$$

for every $p \in \Delta_P$. Then $u \geq 0$ (resp. $u \leq 0$) on R.

Proof. It suffices to consider the case in which u is P-superharmonic on R. For each $n \geq 1$ the set

$$F_n = \left\{ p \in \Gamma_P \mid \liminf_{x \to p,\ x \in R} u(x) \leq -\frac{1}{n} \right\}$$

is compact and $F_n \cap \Delta_P = \phi$. Let v_n be Evans' P-superharmonic function corresponding to F_n. Then

$$\liminf_{x \to p,\ x \in R} (u + \varepsilon v_n)(x) > -\frac{1}{n}$$

for all $\varepsilon > 0$ and $p \in \Gamma_P$. Since $u + \varepsilon v_n$ is P-superharmonic and bounded from below on R we have

$$u + \varepsilon v_n > -\frac{1}{n}$$

on R. On letting $\varepsilon \to 0$ and then $n \to \infty$ we obtain the desired conclusion.

13. We are now able to prove:

THEOREM. *If $u \in PB(R)$, then*

$$|u| \leq \sup_{p \in \Delta_P} \limsup_{x \to p,\ x \in R} |u(x)|$$

on R.

Proof. Set $M = \sup_{p \in \Delta_P} \limsup_{x \to p,\ x \in R} |u(x)| < \infty$. Then $M - u$ is P-superharmonic on R and has the property

$$\inf_{p \in \Delta_P} \liminf_{x \to p,\ x \in R} (M - u(x)) \geq 0.$$

Therefore $M - u \geq 0$ on R. By considering $-u$ we similarly obtain $M + u \geq 0$.

14. We turn to the problem of determining the dimension of the vector space $PBE(R)$ in terms of the P-harmonic boundary Δ_P. Note that Δ_P here is different from that in Kwon-Sario [7], where it was defined as a quotient space of the Royden harmonic boundary Δ. In the present case the P-singular point always lies on Δ_P.

PROPOSITION. *The dimension of the space $PBE(R)$ of bounded energy-finite P-harmonic functions on R is equal to the cardinality of the set $\Delta_P - s$ in the sense that either both are infinite, or finite and equal.*

The proof is the same as in Kwon-Sario [7].

§ 7. Type problem

15. For a regular exhaustion $\{R_n\}$ of R we consider continuous functions e_n on R such that $e_n \in P(R_n)$ and $e_n = 1$ on $R - R_n$. Since $0 < e_{n+p} \leq e_n \leq 1$ on R, the sequence $\{e_n\}$ converges to a P-harmonic function e, uniformly on compact subsets of R. The function e is called the *elliptic measure* of the ideal boundary of R (Royden [14]). It is known (loc. cit.) that the vanishing of e on R is independent of the choice of the exhaustion. We shall denote by O_e the class of pairs (R, P) for which $e \equiv 0$.

The class O_e has the following relation to the P-harmonic boundary:

THEOREM. *If $\Delta_P = \phi$, then $(R, P) \in O_e$. Conversely if $(R, P) \in O_e$, then either $\Delta_P = \phi$ or $\Delta_P = \{s\}$.*

Proof. If $\Delta_P = \phi$, $1 \in M_{P\Delta}(R)$ and hence $1 = BE\text{-}\lim_n f_n$ on R of a sequence $\{f_n\}$ in $M_{PO}(R)$. The elliptic measure e has a finite energy integral in this case and $e = BE\text{-}\lim_n e f_n$ on R in view of $\int_R P dV < \infty$. Thus

$$E_R(e) = \lim_{n \to \infty} E_R(e f_n, e) = 0$$

by virtue of the energy principle. We conclude that $e \equiv 0$ and $(R, P) \in O_e$. Conversely if $(R, P) \in O_e$, then $\dim PBE(R) = 0$ since $|u| \leq e \sup_R |u|$ for each $u \in PBE(R)$. A fortiori either $\Delta_P = \phi$ or $\Delta_P = \{s\}$.

16. Consider the sequence $\{w_n\}$ of continuous functions w_n on R such that $w_n \in P(R_n - \bar{R}_0)$, $w_n | \bar{R}_0 = 1$, and $w_n | R - R_n = 0$. Then $w = B\text{-}\lim_n w_n$ exists on R and $w \in PB(R - \bar{R}_0)$.

COROLLARY 1. *If $\inf_R w > 0$, then $(R, P) \in O_e$.*

Proof. In view of

$$E_R(w_{n+p} - w_n, w_{n+p}) = E_{R_{n+p} - R_0}(w_{n+p} - w_n, w_{n+p}) = 0,$$

we conclude that $w = BE\text{-}\lim_n w_n$ and $w \in M_{P\Delta}(R)$. Therefore $\inf_R w > 0$ implies that $\Delta_P = \phi$ and $(R, P) \in O_e$.

COROLLARY 2 (Ozawa [13]). *A Riemannian manifold R is parabolic if and only if $\inf_R w > 0$ for some density P on R.*

§ 8. Dirichlet problem

17. Let $B(R_P^*)$ be the space of bounded continuous functions on R_P^* and $B_s(R_P^*)$ the space of functions in $B(R_P^*)$ which vanish at the P-singular point s. In view of the construction of R_P^*, the P-algebra $M_P(R)$ is a subalgebra of $B(R_P^*)$. It is natural to ask what is the uniform closure of $M_P(R)$ in the space $B(R_P^*)$.

We maintain:

THEOREM. *With respect to the sup-norm topology, the P-algebra $M_P(R)$ is dense in $B_s(R_P^*)$ or $B(R_P^*)$ according as there does or does not exist a P-singular point s.*

Proof. The uniform closure $\overline{M_P(R)}$ of $M_P(R)$ is a closed subalgebra of $B(R_P^*)$ and separates points in the compact Hausdorff space R_P^*. Hence $\overline{M_P(R)}$ is either $B(R_P^*)$ or $B_x(R_P^*)$ for some $x \in R_P^*$ (see e.g. Hewitt-Stromberg [4, p. 98]), as asserted.

18. Let $B_s(\Delta_P)$ and $B(\Delta_P)$ be the families of functions on Δ_P defined as above. If there exists no P-singular point s we understand that $B_s(\Delta_P) = B(\Delta_P)$ and $B_s(R_P^*) = B(R_P^*)$.

THEOREM. *There exists a positive linear mapping π: $B_s(\Delta_P) \to PB(R)$ such that* $\sup_R |\pi(f)| \leq \max_{\Delta_P} |f|$ *for all* $f \in B_s(\Delta_P)$.

Proof. By Tietze's extension theorem every $f \in B_s(\Delta_P)$ has a continuous extension $\hat{f}$ to R_P^* with

$$\max_{R_P^*} |\hat{f}| = \max_{\Delta_P} |f|.$$

Choose $f_n \in M_P(R)$ such that $\max_{R_P^*} |\hat{f} - f_n| < 1/n$, and let u_n be the P-harmonic projection of f_n on R (cf. No. 9). Then

$$\sup_R |u_n - u_m| = \max_{\Delta_P} |u_n - u_m| < \frac{1}{n} + \frac{1}{m}.$$

Thus there exists a function $u \in PB(R)$ such that $\sup_R |u - u_n| \to 0$ as $n \to \infty$.

Set $\pi(f) = u$. Since $\pi(f) = f$ on Δ_P and $\pi(f) \in PB(R)$ the mapping π: $B_s(\Delta_P) \to PB(R)$ is well-defined. Theorem 13 yields

$$\sup_R |\pi(f)| \leq \sup_{p \in \Delta_P} \lim_{x \to p,\ x \in R} \sup |\pi(f)(x)| = \max_{\Delta_P} |f|$$

as required. The positiveness and linearity of π follow immediately from Theorem 12 and No. 9.

§ 9. Integral representation

19. For a fixed point $x \in R$ consider the functional L_x on $B_s(\Delta_P)$ defined by $L_x(f) = \pi(f)(x)$. Clearly L_x belongs to the class $B_s(\Delta_P)^*$ of bounded linear functionals on $B_s(\Delta_P)$. By the Hahn-Banach theorem we may extend L_x to an element of $B(\Delta_P)^*$. Thus the restriction mapping φ: $B(\Delta_P)^* \to B_s(\Delta_P)^*$ is a surjective homomorphism with kernel

$$\varphi^{-1}(0) = \{L \in B(\Delta_P)^* \mid L(f) = 0 \text{ for all } f \in B_s(\Delta_P)\}.$$

Hence we have a canonical isomorphism

$$B(\Delta_P)^*/\varphi^{-1}(0) \cong B_s(\Delta_P)^*.$$

We are ready to state:

THEOREM. *To each $x \in R$ there corresponds a regular Borel measure μ_x on Δ_P such that*

$$\pi(f)(x) = \int_{\Delta_P} f(p) d\mu_x(p)$$

for all $f \in B_s(\Delta_P)$. The measure μ_x is unique up to a Dirac measure δ_x with $\delta_x(\Delta_P - s) = 0$.

The measure μ_x is called the *P-harmonic measure* with *center x*.

Proof. We have seen that

$$L_x = L_1 + L_2$$

for some $L_1, L_2 \in B(\Delta_P)^*$ with $L_2(f) = 0$ for all $f \in B_s(\Delta_P)$. By the Riesz representation theorem there exist regular (signed) Borel measures μ_x, δ_x on Δ_P such that

$$L_1(f) = \int_{\Delta_P} f \, d\mu_x, \quad L_2(f) = \int_{\Delta_P} f \, d\delta_x$$

for all $f \in B(\Delta_P)$. Thus we have

$$L_x(f) = \int_{\Delta_P} f \, d\mu_x + \int_{\Delta_P} f \, d\delta_x = \int_{\Delta_P} f \, d\mu_x$$

for all $f \in B_s(\Delta_P)$. Since L_x is a positive functional, μ_x is a measure on Δ_P, unique up to a Dirac measure δ_x with $\delta_x(\Delta_P - s) = 0$.

20. Let $\mu = \mu_{x_0}$ be the P-harmonic measure centered at a fixed point $x_0 \in R$.

THEOREM. *There exists a function $K_p(x, p)$ on $R \times \Delta_P$ with the following properties:*

(i) *$K_P(x, p)$ is a Borel measurable function on Δ_P for each $x \in R$, nonnegative* μ-a.e. *on Δ_P,* and $K_P(x_0, p) = 1$ on Δ_P,

(ii) *for any $f \in B_s(\Delta_P)$* and $x \in R$,

$$\int_{\Delta_P} f(p) d\mu_x(p) = \int_{\Delta_P} f(p) K_P(x, p) d\mu(p),$$

(iii) *$K_P(x, p)$ is essentially bounded on Δ_P, uniformly on every compact subset of R,*

(iv) *$K_P(x, p)$ is uniquely determined* μ-a.e. *on Δ_P.*

The proof of the theorem is essentially the same as in the case $P \equiv 0$ (cf. Kwon-Sario [6]).

COROLLARY 1. *A function u belongs to the vector space $PBE(R)$ if and only if*

$$u(x) = \int_{\Delta_P} f(p) K_P(x, p) d\mu(p)$$

on R for some $f \in M_P(R)$. In this case $u \equiv f$ on Δ_P.

COROLLARY 2. *Let $u, v \in PBE(R)$. Then the least P-harmonic majorant $u \vee v$ and the greatest P-harmonic minorant $u \wedge v$ exist and have the expressions*

$$(u \vee v)(x) = \int_{\Delta_P} (u \cup v)(p) K_P(x, p) d\mu(p),$$

$$(n \wedge v)(x) = \int_{\Delta_P} (u \cap v)(p) K_p(x, p) d\mu(p)$$

on R.

BIBLIOGRAPHY

[1] J. Chang-L. Sario, *Royden's algebra on Riemannian spaces*, Math. Scand. 27 (1970).
[2] C. Constantinescu-A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer, 1963, 244 pp.
[3] M. Glasner-R. Katz, *On the behavior of solutions of* $\Delta u = Pu$ *at the Royden boundary*, J. Analyse Math. 22 (1969), 343–354.

[4] E. Hewitt-K. Stromberg, *Real and abstract analysis*, Springer, 1965, 476 pp.
[5] Y.K. Kwon-L. Sario, *A maximum principle for bounded harmonic functions on Riemannian spaces*, Canad. J. Math. 22 (1970), 847-854.
[6] ——————, *Harmonic functions on a subregion of a Riemannian manifold*, J. Ind. Math. Soc. (to appear).
[7] ——————, *The P-singular point of the P-compactification for* $\Delta u = Pu$, Bull. Amer. Math. Soc. (to appear).
[8] L. Myrberg, *Über die Integration der Differentialgleichung* $\Delta u = c(P)u$ *auf offenen Riemannschen Flächen*, Math. Scand. 2 (1954), 142-152.
[9] ————, *Über die Existenz der Greenschen Funktion der Gleichung* $\Delta u = c(P)u$ *auf Riemannschen Flächen*, Ann. Acad. Sci. Fenn. Ser. A.I. 170 (1954), 8 pp.
[10] M. Nakai, *The space of non-negative solutions of the equation* $\Delta u = pu$ *on a Riemann surface*, Kōdai Math. Sem. Rep. 12 (1960), 151-178.
[11] ————, *The space of Dirichlet-finite solutions of the equation* $\Delta u = Pu$ *on a Riemann surface*, Nagoya Math. J. 18 (1961), 111-131.
[12] M. Nakai-L. Sario, *A new operator for elliptic equations and the P-compatification for* $\Delta u = Pu$, Math. Ann. 189 (1970), 242-256.
[13] M. Ozawa, *A set of capacity zero and the equation* $\Delta u = Pu$, Kōdai Math. Sem. Rep. 12 (1960), 76-81.
[14] H.L. Royden, *The equation* $\Delta u = Pu$ *and the classification of open Riemann surfaces*, Ann. Acad. Sci. Fenn. Ser. A.I. 271 (1959), 27 pp.
[15] L. Sario-M. Nakai, *Classification theory of Riemann surfaces*, Springer, 1970, 446 pp.

University of California, Los Angeles