Shûkichi Tanno
Nagoya Math. J.
Vol. 42 (1971), 67–77

A CLASS OF RIEMANNIAN MANIFOLDS SATISFYING R(X,Y)•R = 0

SHÛKICHI TANNO

1. Introduction

Let (M, g) be a Riemannian manifold and let R be its Riemannian curvature tensor. If (M, g) is a locally symmetric space, we have

$$(*) \qquad R(X, Y)\cdot R = 0 \qquad \text{for all tangent vectors } X, Y$$

where the endomorphism $R(X, Y)$ (i.e., the curvature transformation) operates on R as a derivation of the tensor algebra at each point of M. There is a question: Under what additional condition does this algebraic condition $(*)$ on R imply that (M, g) is locally symmetric (i.e., $\nabla R = 0$)? A conjecture by K. Nomizu [5] is as follows: $(*)$ implies $\nabla R = 0$ in the case where (M, g) is complete and irreducible, and $\dim M \geq 3$. He gave an affirmative answer in the case where (M, g) is a certain complete hypersurface in a Euclidean space ([5]).

With respect to this problem, K. Sekigawa and H. Takagi [8] proved that if (M, g) is a complete conformally flat Riemannian manifold with dim $M \geq 3$ and satisfies $(*)$, then (M, g) is locally symmetric.

On the other hand, R.L. Bishop and B.O'Neill [1] constructed a wide class of Riemannian manifolds of negative curvature by warped product using convex functions. For two Riemannian manifolds B and F, a warped product is denoted by $B \times_f F$, where f is a positive C^∞-function on B. The purpose of this paper is to prove

THEOREM A. *Let (F, g) be a Riemannian manifold of constant curvature $K \leq 0$. Let E^n be an n-dimensional Euclidean space and let f be a positive C^∞-function on E^n. On a warped product $E^n \times_f F$, assume that*

(i) *the condition $(*)$ is satisfied, and*

(ii) *the scalar curvature is constant.*

Received June 20, 1970.

Then $E^n \times {}_fF$ is locally symmetric. The converse is clear.

In theorem A, if $n \geq 2$, we see that $E^n \times {}_fF$ is not of constant curvature. If $n = 1$, we have

THEOREM B. *Let (F, g) be a Riemannian manifold of constant curvature $K \leq 0$. Let E^1 be a Euclidean 1-space and let f be a non-constant positive C^∞-function on E^1. Then $E^1 \times {}_fF$ satisfies the condition $(*)$ if and only if $E^1 \times {}_fF$ is of constant curvature.*

Concerning theorem B, it is remarked that, as is stated in [1], p. 28, a hyperbolic m-space is expressed as $H^m = E^1 \times_f E^{m-1}$ for $f = e^t$ or $= E^1 \times_f H^{m-1}$ for $f = \cosh t$.

The author is grateful to his colleague Dr. J. Kato with whom the author had serveral conversations on differential equations.

2. The Riemannian curvature tensor of $E^n \times {}_fF$

Let (F, g) be a Riemannian manifold and let E^n be a Euclidean n-space. We consider the product manifold $E^n \times F$. For vector fields A, B, C, etc. on E^n, we denote vector fields $(A, 0)$, $(B, 0)$, $(C, 0)$, etc. on $E^n \times F$ by also A, B, C, etc. Likewise, for vector fields X, Y, etc. on F, we denote vector fields $(0, X)$, $(0, Y)$, etc. on $E^n \times F$ by X, Y, etc.

We denote the inner product of A and B on E^n by $\langle A, B\rangle$. Let f be a positive C^∞-function on E^n. Then the (Riemannian) inner product $\langle\ ,\ \rangle$ for $A + X$ and $B + Y$ on the warped product $E^n \times {}_fF$ at (a, x) is given by (cf. [1])

$$\langle A + X,\ B + Y\rangle_{(a,x)} = \langle A, B\rangle_{(a)} + f^2(a) g_x(X, Y). \tag{2.1}$$

We extend the function f on E^n to that on $E^n \times {}_fF$ by $f(a, x) = f(a)$. The Riemannian connections defined by $\langle\ ,\ \rangle$ on E^n and $E^n \times {}_fF$ are denoted by ∇^0 and ∇, respectively. The Riemannian connection defined by g on F is denoted by D. Then we have the identities (cf. Lemma 7.3, [1])

$$\nabla_A B = \nabla^0_A B, \tag{2.2}$$

$$\nabla_A X = \nabla_X A = (Af/f)X, \tag{2.3}$$

$$\nabla_X Y = D_X Y - (\langle X, Y\rangle / f) \text{ grad } f. \tag{2.4}$$

By (2.2) we identify ∇^0 with ∇ in the sequel. In (2.4) $\text{grad} f$ on E^n is identified with grad f on $E^n \times {}_fF$ and we have

$$\langle \text{grad} f, A\rangle = df(A) = Af.$$

The Riemannian curvature tensors by ∇ and D are denoted by R and S respectively. We use both notations $R(X,Y)$ and R_{XY}, etc.:

$$R(X,Y)=R_{XY}=\nabla_{[X,Y]}-[\nabla_X,\nabla_Y], \text{ etc.}$$

Then, noticing that E^n is flat, we have (cf. Lemma 7.4, [1])

(2.5) $$R_{AB}C=0,$$

(2.6) $$R_{AX}B=-(1/f)\langle\nabla_A \operatorname{grad} f,\ B\rangle X,$$

(2.7) $$R_{AB}X=R_{XY}A=0,$$

(2.8) $$R_{AX}Y=R_{AY}X=(1/f)\langle X,Y\rangle\nabla_A \operatorname{grad} f,$$

(2.9) $$R_{XY}Z=S_{XY}Z-(\langle\operatorname{grad} f,\ \operatorname{grad} f\rangle/f^2)(\langle X,Z\rangle Y-\langle Y,Z\rangle X).$$

3. The condition (*)

From now on (§3 ~ §8) we assume that (F,g) is of constatn curvature $K\leq 0$. Then we have

$$\begin{aligned}S_{XY}Z&=K(g(X,Z)Y-g(Y,Z)X)\\&=(K/f^2)(\langle X,Z\rangle Y-\langle Y,Z\rangle X).\end{aligned}$$

In this case, (2.9) is written as

(3.1) $$R_{XY}Z=P(\langle X,Z\rangle Y-\langle Y,Z\rangle X),$$

where we have put

(3.2) $$P=(K-\langle\operatorname{grad} f,\ \operatorname{grad} f\rangle)/f^2\leq 0.$$

Now by definition we have

$$(R(X,Y)\cdot R)(Z,V)W=R_{XY}R_{ZV}W-R(R_{XY}Z,V)W-R(Z,R_{XY}V)W-R_{ZV}R_{XY}W$$

which vanishes by (3.1). Likewise, by (2.5) ~ (2.8), (3.1), we have

$$\begin{aligned}&(R(X,Y)\cdot R)(Z,A)W=0,\\&(R(X,Y)\cdot R)(Z,B)A=0,\\&(R(X,Y)\cdot R)(C,B)A=0,\end{aligned}$$

from which we have

$$(R(X,Y)\cdot R)(A,Z)W=-(R(X,Y)\cdot R)(Z,A)W=0,$$

(3.3) $$(R(X,Y)\cdot R)(Z,W)A=-(R(X,Y)\cdot R)(A,Z)W-(R(X,Y)\cdot R)(W,A)Z=0,$$

(3.4) $$(R(X,Y)\cdot R)(C,B)W=-(R(X,Y)\cdot R)(W,C)B-(R(X,Y)\cdot R)(B,W)C=0.$$

Next, by similar calculations we have

$$(3.5)\qquad (R(X,A)\cdot R)(Z,V)W = (fP\nabla_A \operatorname{grad} f + \nabla_Q \operatorname{grad} f)(\langle V,W\rangle\langle X,Z\rangle - \langle Z,W\rangle\langle X,V\rangle)/f^2,$$

where we have put $Q = \nabla_A \operatorname{grad} f$.

$$(3.6)\qquad (R(X,A)\cdot R)(Z,B)W = (\langle fP\nabla_A \operatorname{grad} f, B\rangle + \langle \nabla_A \operatorname{grad} f,\ \nabla_B \operatorname{grad} f\rangle)(\langle X,W\rangle Z - \langle Z,W\rangle X)/f^2,$$

$$(3.7)\qquad (R(X,A)\cdot R)(Z,B)C = \langle X,Z\rangle(\langle \nabla_B \operatorname{grad} f, C\rangle \nabla_A \operatorname{grad} f - \langle \nabla_A \operatorname{grad} f, C\rangle \nabla_B \operatorname{grad} f)/f^2,$$

$$(3.8)\qquad (R(X,A)\cdot R)(C,B)G = (\langle \nabla_A \operatorname{grad} f, B\rangle\langle \nabla_C \operatorname{grad} f, G\rangle - \langle \nabla_A \operatorname{grad} f, C\rangle\langle \nabla_B \operatorname{grad} f, G\rangle)X/f^2.$$

Finally we have $R(A,B)\cdot R = 0$, since $R_{AB} = 0$.

LEMMA 3.1. *On $E^n \times {}_fF$, the condition* (*) *is equivalent to*

$$(3.9)\qquad fP\nabla_A \operatorname{grad} f + \nabla_Q \operatorname{grad} f = 0, \qquad Q = \nabla_A \operatorname{grad} f, \qquad \textit{and}$$

$$(3.10)\qquad \langle \nabla_B \operatorname{grad} f, C\rangle \nabla_A \operatorname{grad} f = \langle \nabla_A \operatorname{grad} f,\ C > \nabla_B \operatorname{grad} f.$$

Proof. $R(X,Y)\cdot R = 0$ and $R(A,B)\cdot R = 0$ hold always. If (*) holds, then (3.5) and (3.7) imply (3.9) and (3.10). Conversely, (3.5) and (3.9) imply $(R(X,A)\cdot R)(Z,V)W = 0$. Since

$$\langle \nabla_Q \operatorname{grad} f, B\rangle = \langle \nabla_B \operatorname{grad} f, Q\rangle = \langle \nabla_B \operatorname{grad} f,\ \nabla_A \operatorname{grad} f\rangle,$$

(3.6) and (3.9) imply $(R(X,A)\cdot R)(Z,B)W = 0$. (3.7) and (3.10) imply $(R(X,A)\cdot R)(Z,B)C = 0$. Similarly, (3.8) and (3.10), together with the fact that $\langle \nabla_A \operatorname{grad} f, B\rangle = \langle \nabla_B \operatorname{grad} f, A\rangle$, imply $(R(X,A)\cdot R)(C,B)G = 0$. Finally we have $(R(X,A)\cdot R)(Z,V)B = 0$ and $(R(X,A)\cdot R)(C,B)W = 0$ in the same way as (3.3) and (3.4).

4. The condition for $\nabla R = 0$

Using the identity

$$(\nabla_X R)(Y,Z)W = \nabla_X(R_{YZ}W) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R_{YZ}(\nabla_X W),$$

together with (2.3), (2.4) and (2.8), we get

$$(4.1)\qquad (\nabla_X R)(Y,Z)W =$$

$$(\langle X,Y\rangle\langle Z,W\rangle-\langle X,Z\rangle\langle Y,W\rangle)(fP\,\mathrm{grad}\,f+\nabla_{\mathrm{grad}\,f}\,\mathrm{grad}\,f)/f^2,$$

where we have used $\nabla_X P=XP=0$. Similarly we get

$$(\nabla_X R)(A,Y)W= \tag{4.2}$$
$$((\nabla_A\,\mathrm{grad}\,f)f+fPAf)(\langle Y,W\rangle X-\langle X,W\rangle Y)/f^2,$$

$$(\nabla_A R)(B,Y)W= \tag{4.3}$$
$$\langle Y,W\rangle(f\nabla_A\nabla_B\,\mathrm{grad}\,f-f\,\nabla_T\,\mathrm{grad}\,f-Af\,\nabla_B\,\mathrm{grad}\,f)/f^2,\quad T=\nabla_A B,$$

$$(\nabla_X R)(Y,A)B= \tag{4.4}$$
$$\langle X,Y\rangle(Bf\,\nabla_A\,\mathrm{grad}\,f-\langle\nabla_A\,\mathrm{grad}\,f,B\rangle\,\mathrm{grad}\,f)/f^2,$$

$$(\nabla_A R)(B,X)C= \tag{4.5}$$
$$(Af\,\langle\nabla_B\,\mathrm{grad}\,f,C\rangle+f\,\langle\nabla_T\,\mathrm{grad}\,f,C\rangle-f\,\langle\nabla_A\nabla_B\,\mathrm{grad}\,f,C\rangle)X/f^2.$$

LEMMA 4.1. *On* $E^n\times{}_fF$, $\nabla R=0$ *if and only if*

$$fP\,\mathrm{grad}\,f+\nabla_{\mathrm{grad}\,f}\,\mathrm{grad}\,f=0, \tag{4.6}$$

$$f\,\nabla_A\nabla_B\,\mathrm{grad}\,f-f\,\nabla_T\,\mathrm{grad}\,f-Af\,\nabla_B\,\mathrm{grad}\,f=0,\quad T=\nabla_A B,\ \textit{and} \tag{4.7}$$

$$Bf\,\nabla_A\,\mathrm{grad}\,f-\langle\nabla_A\,\mathrm{grad}\,f,B\rangle\,\mathrm{grad}\,f=0. \tag{4.8}$$

Proof. Necessity comes from (4.1), (4.3) and (4.4). Conversely, assume that (4.6) ~ (4.8) hold. Then, we have $(\nabla_X R)(Y,Z)W=0$ and $(\nabla_A R)(B,Y)W=0$ by (4.1) and (4.3). We take the inner products of A and both sides of (4.6) to get

$$\begin{aligned}0&=f\,PAf+\langle\nabla_{\mathrm{grad}\,f}\,\mathrm{grad}\,f,A\rangle\\&=f\,PAf+\langle\nabla_A\,\mathrm{grad}\,f,\ \mathrm{grad}\,f\rangle\\&=f\,PAf+(\nabla_A\,\mathrm{grad}\,f)f.\end{aligned}$$

Therefore, we have $(\nabla_X R)(A,Y)W=0$ by (4.2). Next we take the inner products of C and both sides of (4.7). Then we have $(\nabla_A R)(B,X)C=0$ by (4.5). By (4.4) and (4.8) we have $(\nabla_X R)(Y,A)B=0$. These, together with the first and second Bianchi identities, imply $(\nabla_X R)(Y,W)A=(\nabla_A R)(X,Y)W=(\nabla_A R)(Y,W)B=(\nabla_Y R)(A,B)W=(\nabla_X R)(A,B)C=(\nabla_A R)(B,C)X=0$.
Finally, $(\nabla_A R)(B,C)G=0$ follows from (2.5).

5. The scalar curvature

In this section, we obtain the expression of the scalar curvature. Let $(A_\alpha, X_i;\ \alpha=1,\cdots,n;\ i=1,\cdots,r=\dim F)$ be vector fields on some open set on $E^n \times F$ such that they make an orthonormal basis at each point of the open set. We denote by R_1 the Ricci curvature tensor. Then we have

$$R_1(Y, Z) = \sum_i \langle R(Y, X_i)Z, X_i\rangle + \sum_\alpha \langle R(Y, A_\alpha)Z, A_\alpha\rangle,$$

which is calculated by (2.8) and (3.1), and we get

$$\begin{aligned} R_1(Y, Z) &= P\sum_i \langle\langle Y, Z\rangle X_i - \langle X_i, Z\rangle Y, X_i\rangle \\ &\quad + \sum_\alpha \langle -(1/f)\langle Z, Y\rangle \nabla_{A_\alpha} \operatorname{grad} f, A_\alpha\rangle \\ &= [(r-1)P - (1/f)\sum_\alpha \langle \nabla_{A_\alpha} \operatorname{grad} f, A_\alpha\rangle]\langle Y, Z\rangle, \end{aligned}$$

where we have used

$$\begin{aligned} \sum_i \langle \langle X_i, Z\rangle Y, X_i\rangle &= \sum_i \langle Y, X_i\rangle \langle X_i, Z\rangle \\ &= \sum_i \langle \langle Y, X_i\rangle X_i, Z\rangle = \langle Y, Z\rangle. \end{aligned}$$

Similarly we have

$$\begin{aligned} R_1(B, C) &= \sum_i \langle R(B, X_i)C, X_i\rangle + \sum_\alpha \langle R(B, A_\alpha)C, A_\alpha\rangle \\ &= -(r/f)\langle \nabla_B \operatorname{grad} f, C\rangle. \end{aligned}$$

Therefore we get

$$\begin{aligned} \text{The scalar curvature } &= \sum_i R_1(X_i, X_i) + \sum_\alpha R_1(A_\alpha, A_\alpha) \\ &= r[(r-1)P - (2/f)\sum_\alpha \langle \nabla_{A_\alpha} \operatorname{grad} f, A_\alpha\rangle]. \qquad (5.1) \end{aligned}$$

6. Two lemmas

LEMMA 6.1. *On $E^n \times {}_f F$, (4.6) is equivalent to* $P =$ constant.

Proof. By (3.2) and (4.6) we have

$$(1/f)(K - \langle \operatorname{grad} f, \operatorname{grad} f\rangle) \operatorname{grad} f + \nabla_{\operatorname{grad} f} \operatorname{grad} f = 0.$$

Since this equation is considered as an equation on E^n, we introduce the natural coordinate symtem $(x^\alpha; \alpha = 1, \cdots, n)$ on E^n. Then the last equation is nothing but

$$\left(K - \sum_\alpha \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\alpha}\right) \frac{\partial f}{\partial x^\beta} + f \sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} \frac{\partial f}{\partial x^\alpha} = 0,$$

which implies that each partial derivative of

$$P = \left[K - \sum_\alpha \left(\frac{\partial f}{\partial x^\alpha}\right)^2\right] \Big/ f^2 \qquad (6.1)$$

vanishes. Thus, P is constant. The converse is clear.

LEMMA 6.2. *On $E^n \times {}_f F$, if the condition* (*) *is satisfied and the scalar curvature is constant, then P is constant.*

Proof. If f is constant, Lemma 6.2 is trivial. Therefore we assume that f is not constant. We put $A = \partial/\partial x^\alpha$, $B = \partial/\partial x^\beta$ and $C = \partial/\partial x^\gamma$, which are parallel on E^n. Then (3.9) and (3.10) are written as

$$(6.2) \qquad fP \frac{\partial^2 f}{\partial x^\alpha \partial x^\delta} + \sum_\theta \frac{\partial^2 f}{\partial x^\theta \partial x^\delta} \frac{\partial^2 f}{\partial x^\theta \partial x^\alpha} = 0,$$

$$(6.3) \qquad \frac{\partial^2 f}{\partial x^\beta \partial x^\gamma} \frac{\partial^2 f}{\partial x^\alpha \partial x^\delta} = \frac{\partial^2 f}{\partial x^\alpha \partial x^\gamma} \frac{\partial^2 f}{\partial x^\beta \partial x^\delta}.$$

Summing with respect to α and γ in (6.3), and substituting the result into (6.2), we have

$$(6.4) \qquad \left(fP + \sum_\theta \frac{\partial^2 f}{\partial x^\theta \partial x^\theta}\right) \frac{\partial^2 f}{\partial x^\alpha \partial x^\delta} = 0.$$

Define a subset Θ of E^n by

$$\Theta = \left\{x \in E^n \,;\, \left(\frac{\partial^2 f}{\partial x^\alpha \partial x^\delta}\right)(x) = 0 \text{ for all } \alpha, \delta\right\}.$$

Let Θ_0 be a component of Θ. If Θ_0 contains an open set, f is of the form $f = a_\alpha x^\alpha + b$ on the interior of Θ_0 for some constant a_α, b (if the same letter appears as a subscript and as a superscript, we abbreviate $\sum$). Since f is positive and C^∞-differentiable, $\Psi = E^n - \Theta = E^n \cap \Theta^c$ can not be empty. Since Θ is closed, Ψ is a non-empty open set. On Ψ we have

$$(6.5) \qquad fP + \sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^\alpha} = 0.$$

On the other hand, the scalar curvature is given by (5.1), which is also written as

$$(6.6) \qquad \text{the scalar curvature} = r\left[(r-1)P - (2/f)\sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^\alpha}\right].$$

By (6.5) and (6.6), we get

$$(6.7) \qquad r(r+1)P = \text{the scalar curvature} = \text{constant},$$

which shows that P is constant on Ψ.

On Θ_0, if $a_\alpha = 0$ for all $\alpha = 1, \cdots, n$, then P is constant on Θ_0 too. So we assume that at least one of a_α is not zero. Then, by (6.1) and $K \leq 0$, we get

$$P=(K-\textstyle\sum_\alpha a_\alpha^2)/(a_\beta x^\beta+b)^2<0. \tag{6.8}$$

We easily see that the function P on Θ_0 given by (6.8) can not be C^∞-differentiably extended to P on $\Theta_0 \cup \Psi$ so that P is constant on Ψ. Therefore Θ can not contain any open set where f is not constant. Hence, we have (6.7) on E^n.

7. Proof of Theorem A

Since $E^n \times {}_fF$ satisfies the condition (*) and the scalar curvature is constant, P is constant by Lemma 6.2. By Lemma 6.1 we see that (4.6) is equivalent to (6.1) with $P=$ constant. Now we solve (6.1) and show that the solution f satisfies (4.7) and (4.8). Then $E^n \times {}_fF$ is locally symmetric by Lemma 4.1. (6.1) is

$$K-\textstyle\sum_\alpha \left(\dfrac{\partial f}{\partial x^\alpha}\right)^2 - Pf^2=0. \tag{7.1}$$

We solve the last partial differential equation by Lagrange-Charpit method. First we put

$$p_\alpha=\frac{\partial f}{\partial x^\alpha}, \quad \alpha=1,\cdots,n.$$

Then the characteristic differential equations of (7.1) are

$$\begin{aligned}\frac{dx^1}{-2p_1}=\frac{dx^2}{-2p_2}&=\cdots=\frac{dx^n}{-2p_n}\\ &=\frac{df}{-2(p_1)^2-\cdots-2(p_n)^2}\\ &=\frac{-dp_1}{-2fPp_1}=\cdots=\frac{-dp_n}{-2fPp_n}.\end{aligned} \tag{7.2}$$

If f is constant, Theorem A is trivial. Hence, we assume that f is not constant. Then at least one of $p_1,\cdots,p_n$ does not vanish. So we assume $p_1\neq 0$ (locally, if necessary) and furthermore we can assume that p_1 is positive. In this case, the last $(n-1)$ equatoins of (7.2) give the first integrals

$$p_\alpha=s_\alpha p_1, \quad \alpha=2,\cdots,n,$$

where s_α are constants. Then (7.1) is

$$K-(p_1)^2(1+s_2^2+\cdots+s_n^2)-Pf^2=0.$$

If we put $s_1 = 1$, we have

$$p_1 = \left[\frac{K - P f^2}{\sum_\alpha s_\alpha^2}\right]^{1/2},$$

$$df = p_\alpha dx^\alpha = p_1 s_\alpha dx^\alpha.$$

Then we get

$$\frac{df}{[K - P f^2]^{1/2}} = \frac{d(s_\beta x^\beta)}{[\sum s_\alpha^2]^{1/2}}. \tag{7.3}$$

By putting $[K - P f^2]^{1/2} = \sqrt{-P} f + y$, we have

$$f = \frac{K - y^2}{2\sqrt{-P}\, y}, \tag{7.4}$$

$$\frac{-(K + y^2) dy / (2\sqrt{-P}\, y^2)}{(K + y^2)/2y} = \frac{d(s_\beta x^\beta)}{[\sum_\alpha s_\alpha^2]^{1/2}}. \tag{7.5}$$

Therefore we have

$$y = b \exp [-(-P/\textstyle\sum s_\alpha^2)^{1/2} (s_\beta x^\beta)]. \tag{7.6}$$

If we put $[-P/\sum s_\alpha^2]^{1/2} s_\beta = c_\beta$, then, by (7.4) and (7.6), we have

$$f = \frac{1}{2\sqrt{-P}} \left[\frac{K}{b} \exp (c_\beta x^\beta) - b \exp (- c_\beta x^\beta)\right], \tag{7.7}$$

which is a solution of (7.1). Consequently, we see that f satisfies (4.7) and (4.8), which are written as

$$f \frac{\partial^3 f}{\partial x^\alpha \partial x^\beta \partial x^\gamma} - \frac{\partial f}{\partial x^\alpha} \frac{\partial^2 f}{\partial x^\beta \partial x^\gamma} = 0,$$

$$\frac{\partial f}{\partial x^\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\gamma} - \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} \frac{\partial f}{\partial x^\gamma} = 0.$$

8. Proof of Theorem B

Assume that $E^1 \times {}_f F$ satisfies condition (*). Then (3.9) or (6.2) is written as

$$\left(f P + \frac{d^2 f}{dx^2}\right) \frac{d^2 f}{dx^2} = 0, \tag{8.1}$$

where x is the natural coordinate system of E^1. Similarly as in §6, we define Θ and Ψ. Then on Ψ, by (6.1) and (8.1), we have

(8.2) $$\left[K-\left(\frac{df}{dx}\right)^2\right]+f\frac{d^2f}{dx^2}=0,$$

which implies that the derivative of $P=[K-(df/dx)^2]/f^2$ is zero so that P is constant on each component of Ψ. On the other hand, on an open interval contained in Θ, f is of the form $f=cx+d$ for some constants c and d. Since P is C^∞-differentiable, Θ can not contain any open interval where f is not constant. Thus, P is constant on E^1. Since f is non-constant, P is a negative constant. Now we have

$$K-\left(\frac{df}{dx}\right)^2-Pf^2=0,$$

whose solution f is

$$f=\frac{1}{2\sqrt{-P}}\left[\frac{K}{b}\exp\sqrt{-P}\,x-b\exp(\sqrt{-P}\,x)\right]$$

where $b<0$ is a constant. Then we have

$$\nabla_A \operatorname{grad} f=-fPA,$$

and hence, (2.6), (2.8) are expressed as

(8.3) $$\begin{aligned} R_{AX}B &= (-1/f)<-PfA,\ B>X \\ &= P(\langle A,B\rangle X-\langle X,B\rangle A), \end{aligned}$$

(8.4) $$\begin{aligned} R_{AX}Y &= (1/f)\langle X,Y\rangle(-PfA) \\ &= P(\langle A,Y\rangle X-\langle X,Y\rangle A). \end{aligned}$$

Thus, (8.3), (2.7), (8.4) and (3.1) show that $E^1\times{}_fF$ is of constant curvature $P<0$.

9. Remarks

(i) If (F,g) is a complete Riemannian manifold, then $E^n\times{}_fF$ is also a complete Riemannian manifold (cf. Lemma 7.2, [1]).

(ii) Assume that (F,g) is of constant curvature $K<0$. If $(\partial^2 f/\partial x^\alpha\partial x^\beta)$ is non-singular at some point of E^n and n is sufficiently small with respect to $r=\dim F$ (for example, $n=2$), then $E^n\times{}_fF$ is irreducible. In fact, by a result due to D. Montgomery and H. Samelson [3] we see that there is no proper subgroup of the orthogonal group $0(n+r)$ of order greater than $(n+r-1)(n+r-2)/2$, provided $n+r\neq 4$. On the other hand, the holonomy algebra is generated by (cf. [4])

$$R_{AX},\ R_{XY},\ \cdots,\ \text{etc.}$$

which are given by (2.5) ~ (2.8), (3.1). And under the circumstance stated above the restricted homogeneous holonomy group at the point is $SO(n+r)$.

(iii) It is an open question if one can get complete solutions of non-linear partial differential equations (6.1), (6.2) and (6.3) (i.e., the condition (*) on $E^n \times {}_fF$, $n \geqq 2$). If one can get the complete solutions, then one sees whether the assumption on the scalar curvature is necessary or not in Theorem A.

(iv) The condition (*) is expressed in local coordinates as

$$\nabla_r \nabla_s R^h{}_{ijk} - \nabla_s \nabla_r R^h{}_{ijk} = 0.$$

In [6], K. Nomizu and H. Ozeki showed that if $\nabla\nabla R = 0$ (more generally, $\nabla^k R = 0$ for some k) on a (complete) Riemannian manifold, then $\nabla R = 0$.

(v) Studies concerning $R(X,Y)\cdot R$ were made also by A. Lichnerowich [2], p. 11, P. J. Ryan [7], K. Sekigawa and S. Tanno [9], J. Simons [10], S. Tanno and T. Takahashi [11], etc.

REFERENCES

[1] R.L. Bishop and B.O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc., 145 (1969), 1–49.

[2] A. Lichnerowich, Géométrie des groupes de transformations, Dunod Paris, 1958.

[3] D. Montgomery and H. Samelson, Transformation groups of spheres, Ann. of Math., (2) 44 (1943), 454–470.

[4] A. Nijenhuis, On the holonomy groups of linear connections IA, IB, Proc. Kon. Ned. Akad. Amsterdam, 15 (1953), 233–249.

[5] K. Nomizu, On hypersurfaces satisfying a certain condition on the curvature tensor, Tôhoku Math. Journ., 20 (1968), 46–59.

[6] K. Nomizu and H. Ozeki, A theorem on curvature tensor fields, Proc. Nat. Acad. Sci. U.S.A., 48 (1962), 206–207.

[7] R.J. Ryan, Homogeneity and some curvature conditions for hypersurfaces, Tôhoku Math. Journ., 22 (1969), 363–388.

[8] K. Sekigawa and H. Takagi, On the conformally flat spaces satisfying a certain condition on the Ricci tensor, Pacific Journ. of Math., 34(1970), 157–162.

[9] K. Sekigawa and S. Tanno, Sufficient conditions for a Riemannian manifold to be locally symmetric, Pacific Journ. of Math., 34(1970), 157–162.

[10] J. Simons, On the transitivity of holonomy systems, Ann. of Math., 76 (1962), 213–234.

[11] S. Tanno and T. Takahashi, Some hypersurfaces of a sphere, Tôhoku Math. Journ., 22(1970), 212–219.

Mathematical Institute
Tôhoku University