Seiya Sasao and Yutaka Ando
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KSO-GROUPS FOR 4-DIMENSIONAL CW-COMPLEXES

SEIYA SASAO AND YUTAKA ANDO

§ 0. In this paper we shall determine KSO-groups for 4-dimensional CW-complexes by their cohomology rings. We denote by $KSO(X)$ the group of orientable stable vector bundles over X. In 1959 A. Dold and H. Whitney [1] gave the classification of $SO(n)$-bundles over a 4-complex. It seems, however, to the authors that group structures of them are unknown. We shall give another definition of the difference bundles defined in [1], and we determine the group structure of $KSO(X)$.

§ 1. For a finite 4-dimensional CW-complex X, we denote by X_3 its 3-skeleton, by X/X_3 a complex obtained from X by contracting X_3 to a point in X, and by EX_3 the suspension of X_3. The following exact sequence is obtained from Puppe's sequence.

$$\text{(I)} \qquad \longrightarrow KSO(EX_3) \xrightarrow{j^*} KSO(X/X_3) \xrightarrow{p^*} KSO(X) \xrightarrow{i^*} KSO(X_3) \longrightarrow 0.$$

At first we define a map W_k: $KSO(X) \longrightarrow H^k(X; Z_2)$ which assigns to each bundle over X its k-th Whitney class. The following lemma is well known.

LEMMA 1-1. *The homomorphism* W_2: $KSO(X_3) \longrightarrow H^2(X_3; Z_2)$ *is an isomorphism.*

Secondly we define a map P_1: $KSO(X) \longrightarrow H^4(X; Z)$ which assigns to each element of $KSO(X)$ its first Pontrjagin class. Then we have

LEMMA 1-2.[1)] *For any finite CW-complex* X, *the map* P_1: $KSO(X) \longrightarrow H^4(X;Z)$ *is a group homomorphism.*

Proof. If ξ and η are orientable stable vector bundles over X, we can take $\tilde{\xi}$: $X \longrightarrow BSO(m)$ and $\tilde{\eta}$: $X \longrightarrow BSO(n)$ as their classifying maps for

1) This lemma and its proof are suggested to the authors by the referee, and the original lemma was proved under the condition that dim $X \leqq 4$.

sufficiently large m and n. Then we set f the composite map $\mu\circ(\tilde{\xi}\times\tilde{\eta})\circ\Delta$;

$$X \xrightarrow[\text{diagonal}]{\Delta} X\times X \xrightarrow{\tilde{\xi}\times\tilde{\eta}} BSO(m)\times BSO(n) \xrightarrow[\text{Whitney sum}]{\mu} BSO(m+n),$$

and we have

$$\xi\oplus\eta = f^*(\gamma_{m+n})$$

where γ_{m+n} is a universal $(m+n)$-plane bundle over $BSO(m+n)$. So we have

$$\begin{aligned}\xi\oplus\eta &= \Delta^*(\tilde{\xi}\times\tilde{\eta})^*(\gamma_m\times\gamma_n)\\ &= \Delta^*(\tilde{\xi}\times\tilde{\eta})^*(\pi_1^*\gamma_m\oplus\pi_2^*\gamma_n)\end{aligned}$$

where $\pi_1\colon BSO(m)\times BSO(n)\longrightarrow BSO(m)$ and $\pi_2\colon BSO(m)\times BSO(n)\longrightarrow BSO(n)$ are projections.

We set $\alpha=\pi_1^*\gamma_m$ and $\beta=\pi_2^*\gamma_n$, and we will prove

$$P_1(\alpha\oplus\beta)=P_1(\alpha)+P_1(\beta).$$

First, the following equations hold;

$$\begin{aligned}P_1(\alpha\oplus\beta) &= (-1)C_2((\alpha\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C})\oplus(\beta\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C}))\\ &= (-1)\{C_2(\alpha\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C})+C_2(\beta\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C})+C_1(\alpha\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C})C_1(\beta\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C})\},\end{aligned}$$

where $C_k(\zeta)$ denotes the k-th Chern class of ζ. On the other hand $H^2(BSO(m)\times BSO(n); Z)=0$. So we proved

$$P_1(\alpha\oplus\beta)=P_1(\alpha)+P_1(\beta).$$

Now we have

$$\begin{aligned}P_1(\xi\oplus\eta) &= \Delta^*(\tilde{\xi}\times\tilde{\eta})^*(P_1(\pi_1^*\gamma_m)+P_1(\pi_2^*\gamma_n))\\ &= \Delta^*\{(\tilde{\xi}\times 0)^*P_1(\gamma_m)+(0\times\tilde{\eta})^*P_1(\gamma_n)\}\\ &= \Delta^*\{P_1(\bar{\pi}_1^*\tilde{\xi}^*\gamma_m)+P_1(\bar{\pi}_2^*\tilde{\eta}^*\gamma_n)\}\\ &= \Delta^*\{\bar{\pi}_1^*P_1(\xi)+\bar{\pi}_2^*P_1(\eta)\}\\ &= P_1(\xi)+P_1(\eta),\end{aligned}$$

where 0 is a constant map and $\bar{\pi}_1\colon X\times X\longrightarrow X$ and $\bar{\pi}_2\colon X\times X\longrightarrow X$ are projections.

Thus we proved

$$P_1(\xi \oplus \eta) = P_1(\xi) + P_1(\eta)$$

for each orientable stable vector bundles ξ and η over X.

LEMMA 1–3. *The homomorphism* P_1: $KSO(X/X_3) \longrightarrow H^4(X/X_3; Z)$ *is a monomorphism, the image of* P_1 *coincides with* $2H^4(X/X_3; Z)$, *and the following diagram is commutative up to sign.*

$$\begin{array}{ccccc} KSO(EX_3) & \xrightarrow{P_1} & H^4(EX_3; Z) & \underset{\approx}{\xrightarrow{E}} & H^3(X_3; Z) \\ \downarrow j^* & & \downarrow \tilde{j} & & \downarrow \delta \\ KSO(X/X_3) & \xrightarrow{P_1} & H^4(X/X_3; Z) & \underset{\approx}{\longrightarrow} & H^4(X, X_3; Z). \end{array}$$

Proof. These are well known results.

§ 2. We take two elements η_1 and η_2 in $KSO(X)$ which satisfy $W_2(\eta_1) = W_2(\eta_2)$. As $i^*(\eta_1) = i^*(\eta_2)$ in the sequence (I), we can take ξ in $KSO(X/X_3)$ such that $p^*(\xi) = \eta_1 - \eta_2$. The homotopy type of X/X_3 is a finite wedge sum of 4-spheres. So we find α_ξ uniquely in $H^4(X/X_3; Z)$ which satisfies $P_1(\xi) = 2\alpha_\xi$. We can regard α_ξ as an element of $H^4(X, X_3; Z)$. We define $d(\eta_1, \eta_2)$ to be the image of α_ξ by the inclusion homomorphism j: $H^4(X, X_3; Z) \longrightarrow H^4(X; Z)$. The following lemma assures the uniqueness of $d(\eta_1, \eta_1)$.

LEMMA 2–1. *For every* ξ *in* $KSO(EX_3)$, $P_1(\xi)$ *is contained in* $2H^4(EX_3; Z)$. *Conversely, for any* α *in* $H^4(EX_3; Z)$ *and any* β *in* $H^2(EX_3; Z_2)$, *there exists an element* ξ *in* $KSO(EX_3)$ *so that* $W_2(\xi) = \beta$ *and* $P_1(\xi) = 2\alpha$.

Proof. At first we consider the Bockstein exact sequence;

$$\longrightarrow H^4(EX_3; Z) \xrightarrow{(2)} H^4(EX_3; Z) \xrightarrow{i_1} H^4(EX_3; Z_2) \longrightarrow .$$

Then we have $i_1(P_1(\xi)) = (W_2(\xi))^2 = 0$. So $P_1(\xi)$ is contained in $2H^4(EX_3; Z)$.

Conversely, if we take any element ξ_1 in $KSO(EX_3)$ as $W_2(\xi_1) = \beta$, then we can find α_1 so that $P_1(\xi_1) = 2\alpha_1$. By the method of A. Dold and H. Whitney [1] we can take an element ξ in $KSO(EX_3)$ so that $\tilde{d}(\xi, \xi_1) = \alpha - \alpha_1$.[2] And the equalities

$$P_1(\xi) - P_1(\xi_1) = 2\tilde{d}(\xi, \xi_1) = 2(\alpha - \alpha_1)$$

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[2] Here $\tilde{d}(\xi, \xi_1)$ is the difference bundle defined by A. Dold and H. Whitney [1].

imply

$$P_1(\xi) = 2\alpha - 2\alpha_1 + P_1(\xi_1) = 2\alpha - 2\alpha_1 + 2\alpha_1 = 2\alpha.$$

LEMMA 2-2. *The cohomology class $d(\eta_1, \eta_2)$ is well defined.*

Proof. We ascertain that the cohomology class $j(\alpha_\xi)$ is independent of the choice of a bundle ξ. We take ξ' so that $p^*(\xi) = \eta_1 - \eta_2 = p^*(\xi')$. These equalities imply that in the sequence (I) there exists ξ'' in $KSO(EX_3)$ so that $\xi - \xi' = j^*(\xi'')$. As $P_1(\xi) - P_1(\xi') = P_1 \circ j^*(\xi'') = \tilde{j} \circ P_1(\xi'')$, where $\tilde{j}$ is as in Lemma 1-3, Lemma 2-1 shows that there exists $\alpha_{\xi''}$ in $H^4(EX_3; Z)$ such that $2\alpha_\xi - 2\alpha_{\xi'} = \tilde{j} \circ P_1(\xi'') = \tilde{j}(2\alpha_{\xi''}) = 2\tilde{j}(\alpha_{\xi''})$. The group $H^4(X/X_3; Z)$, however, is torsion free, and hence the equality $\alpha_\xi - \alpha_{\xi'} = \tilde{j}(\alpha_{\xi''})$ holds. As $j(\alpha_\xi) - j(\alpha_{\xi'}) = j \circ \tilde{j}(\alpha_{\xi''}) = j\delta E(\alpha_{\xi''}) = 0$, we have the equation $j(\alpha_\xi) = j(\alpha_{\xi'})$.

The properties of $d(\eta_1, \eta_2)$ are following;

LEMMA 2-3. (1) *If $W_2(\eta_1) = W_2(\eta_2)$ for η_1 and η_2 in $KSO(X)$, then $d(\eta_1, \eta_2) = 0$ if and only if $\eta_1 = \eta_2$.*

(2) *For η_1 in $KSO(X)$ and α in $H^4(X; Z)$, there exists an element η_2 in $KSO(X)$ so that $W_2(\eta_1) = W_2(\eta_2)$ and $d(\eta_1, \eta_2) = \alpha$.*

(3) *$P_1(\eta_1) - P_1(\eta_2) = 2d(\eta_1, \eta_2)$, if $W_2(\eta_1) = W_2(\eta_2)$.*

(4) *$d(\eta_1, \eta_2) + d(\eta_2, \eta_3) = d(\eta_1, \eta_3)$.*

(5) *$d(n\eta_1, n\eta_2) = nd(\eta_1, \eta_2)$.*

(6) *$W_4(\eta_1) - W_4(\eta_2) = d(\eta_1, \eta_2)_2$.*[3]

Proof. (1) If $d(\eta_1, \eta_2) = 0$, there exists ξ in $KSO(X/X_3)$ such that $P_1(\xi) = 2\alpha_\xi$, where α_ξ is in $\delta H^3(X_3; Z)$. So we have $P_1(\xi) = \delta(2\beta_\xi)$ where β_ξ is in $H^3(X_3; Z)$. By Lemmas 1-3 and 2-1 we can take ξ' in $KSO(EX_3)$ so that $P_1(\xi') = 2(E^{-1}\beta_\xi)$. As the homomorphism $P_1 \colon KSO(X/X_3) \longrightarrow H^4(X/X_3; Z)$ is a monomorphism, the equation $j^*(\xi') = \xi$ is obtained from $P_1 \circ j^*(\xi') = 2\alpha_\xi = P_1(\xi)$. Thus we proved that $\eta_1 = \eta_2$. The proofs of other properties are similar, so they are omitted.

§ 3. To begin with we represent cohomology groups of X so that they satisfy the following properties i) – ii):

$$H^2(X; Z_2) = \sum_{i=0} \sum_{j=1}^{s_i} Z_2[x_{ij}] + \sum_{k=1}^{s} Z_2[x_k],$$

[3] $d(\eta_1, \eta_2)_2$ is the reduction mod 2 of $d(\eta_1, \eta_2)$.

$$H^4(X; Z_2) = \sum_{i=1}^{r_0} Z_2[\tilde{y}_i] + \sum_{i=1} \sum_{j=1}^{r_i} Z_2[\tilde{z}_{ij}],$$

$$H^4(X; Z) = \sum_{i=1}^{r_0} Z[y_i] + \sum_{i=1} \sum_{j=1}^{r_i} Z_{2^i}[z_{ij}]$$

$$+ \sum_{p:\ \text{odd prime}} \sum_{i=1} \sum_{j=1}^{t_i} Z_{p^i}[v_{pij}].$$

Here [ ] denotes a generator of the group, and following properties are satisfied.

i) $[x_k]^2 = 0; [x_{0j}]^2 = [\tilde{y}_j],\ j = 1, \cdots, s_0;\ [x_{ij}]^2 = [\tilde{z}_{ij}],\ i = 1, \cdots,\ j = 1, \cdots, s_i.$

ii) $[\tilde{y}_i] = i_1[y_i],\ [\tilde{z}_{ij}] = i_1[z_{ij}]$, where $i_1\colon H^4(X; Z) \longrightarrow H^4(X; Z_2)$.

LEMMA 3-1. *There exists η_k in $KSO(X)$ so that $W_2(\eta_k) = [x_k]$ and $2\eta_k = 0$ $(1 \leqq k \leqq s)$.*

Proof. As $[x_k]^2 = 0$, we have $P_1(\xi) \equiv 0 \pmod 2$ for any ξ in $KSO(X)$ which satisfies $W_2(\xi) = [x_k]$. And Lemma 2-3 shows that there exists η_k in $KSO(X)$ so that $W_2(\eta_k) = [x_k]$ and $P_1(\eta_k) = 0$. On the other hand the qualities

$$2d(2\eta_k, 0) = P_1(2\eta_k) = 2P_1(\eta_k) = 0$$

imply that $d(2\eta_k, 0)$ is of order 2. And the equalities

$$d(2\eta_k, 0)_2 = W_4(2\eta_k) = (W_2(\eta_k))^2 = [x_k]^2 = 0$$

hold, so we have $d(2\eta_k, 0) = 0$. This shows that $2\eta_k = 0$.

We know that the reduction mod 2 of the first Pontrjagin class is a squaring of the second Whitney class. Consequently, we can ignore elements in $H^4(X; Z)$ which are divisible by 2, because we proved (3) of Lemma 2-3. We have the following

LEMMA 3-2. *There exists an element η_{ij} in $KSO(X)$ so that*

$$W_2(\eta_{ij}) = [x_{ij}] \qquad (i \geqq 0),$$

$$P_1(\eta_{ij}) = \begin{cases} [y_j] & (i = 0), \\ [z_{ij}] & (i \geqq 1). \end{cases}$$

Moreover we can determine the order of η_{ij} as follows;

If $i = 0$, $P_1(l\eta_{ij}) = l[y_j]$ for any integer l.

If $i \geqq 1$, $d(2^{i+1}\eta_{ij}, 0) = 2d(2^i\eta_{ij}, 0) = P_1(2^i\eta_{ij}) = 2^i P_1(\eta_{ij}) = 2^i[z_{ij}] = 0$.
If $i \geqq 2$, $d(2^i\eta_{ij}, 0) = 2d(2^{i-1}\eta_{ij}, 0) = P_1(2^{i-1}\eta_{ij}) = 2^{i-1}[z_{ij}] \neq 0$.
If $i = 1$, $d(2\eta_{ij}, 0)_2 = W_4(2\eta_{ij}) = W_2(\eta_{ij})^2 = [x_{ij}]^2 = [\tilde{z}_{ij}] \neq 0$.

By Lemma 1-3 we can determine the map j^* in the sequence (I). So we have

LEMMA 3-3. i) *$KSO(X/X_3) = \sum_{i=1}^{r_0} Z[\tilde{y}_i] + \sum_{i=1} \sum_{j=1}^{r_i} Z[\tilde{\tilde{z}}_{ij}] + \sum_{p:\text{ odd prime}} \sum_i \sum_{j=1}^{t_i} Z[\tilde{v}_{pij}] + \sum_i Z[u_i]$, $j \circ P_1(\tilde{y}_i) = 2[y_i]$, $j \circ P_1(\tilde{\tilde{z}}_{ij}) = 2[z_{ij}]$, and $j \circ P_1(\tilde{v}_{pij}) = 2[v_{pij}]$ for a natural homomorphism $j: H^4(X/X_3; Z) \longrightarrow H^4(X; Z)$, and u_i is a bundle which corresponds to a 4-cell homologicaly trivial.*

ii) *$p^*(KSO(X/X_3)) = \sum_{i=1}^{r_0} Z[\tilde{y}_i'] + \sum_{i=1} \sum_{j=1}^{r_i} z_{2^i}[\tilde{\tilde{z}}_{ij}'] + \sum_{p:\text{ odd prime}} \sum_i \sum_{j=1}^{t_i} Z_{p^i}[\tilde{v}_{pij}']$, where $\tilde{y}'$ denotes $p^*(\tilde{y})$.*

The element $d(\tilde{\tilde{z}}_{ij}', 2\eta_{ij})$ is defined since $W_2(\tilde{\tilde{z}}_{ij}') = 0 = W_2(2\eta_{ij})$ for $1 \leqq j \leqq s_i$. We have

$$d(\tilde{\tilde{z}}_{ij}', 2\eta_{ij})_2 = W_4(\tilde{\tilde{z}}_{ij}') - W_4(2\eta_{ij}) = W_4(\tilde{\tilde{z}}_{ij}') - W_2(\eta_{ij})^2 = [\tilde{z}_{ij}] - [\tilde{z}_{ij}] = 0.$$

Hence we can choose an element β_{ij} in $H^4(X; Z)$ such that $d(\tilde{\tilde{z}}_{ij}', 2\eta_{ij}) = 2\beta_{ij}$. Then $4\beta_{ij} = 2d(\tilde{\tilde{z}}_{ij}', 2\eta_{ij}) = P_1(\tilde{\tilde{z}}_{ij}) - P_1(2\eta_{ij}) = 2[z_{ij}] - 2[z_{ij}] = 0$. Lemma 2-3 shows that we can take η_{ij}' so that $d(\eta_{ij}', \eta_{ij}) = \beta_{ij}$. Then we have $d(\tilde{\tilde{z}}_{ij}', 2\eta_{ij}') = 0$, and $P_1(\eta_{ij}') = [z_{ij}] + 2\beta_{ij}$, $4\beta_{ij} = 0$. This shows that $\tilde{\tilde{z}}_{ij}' = 2\eta_{ij}'$. Thus we may use η_{ij}' in place of η_{ij}.

The above results are summarized as follows:

Elements of $KSO(X)$	Number	Order
η_k	$1 \leqq k \leqq s$	2
η_{ij}'	$1 \leqq i,\ 1 \leqq j \leqq s_i$	2^{i+1}
η_{oj}'	$1 \leqq j \leqq s_0$	∞
$\tilde{\tilde{y}}_j'$	$s_0 < j \leqq r_0$	∞
$\tilde{\tilde{z}}_{ij}'$	$1 \leqq i,\ s_i < j \leqq r_i$	2^i
$\tilde{\tilde{v}}_{pij}'$	$1 \leqq i,\ p \neq 2$	p^i

Now if we use the sequence (I), we can easily prove that $KSO(X)$ is an abelian group generated by the above elements. Thus we have

THEOREM.

$$KSO(X) = \sum_1^s Z_2 + \sum_{i=1} \sum_{j=1}^{s_i} Z_{2^{i+1}} + \sum_1^{r_0} Z$$

$$+ \sum_{i=1} \sum_{j=s_i+1}^{r_i} Z_{2^i} + \sum_{p:\text{ odd prime}} \sum_i \sum_1^{t_i} Z_{p^i},$$

where s, s_i, r_i *and* t_i *are as in the first part of this section.*

COROLLARY 1. *If* Y *is a* 3-*dimensional CW-complex, then we have that*

$$KSO(EY) \cong H^1(Y; Z_2) + H^3(Y; Z).$$

COROLLARY 2. *If* M *is an orientable, closed, topological* 4-*manifold, we have;*

$$KSO(M) \cong H^2(M; Z_2) + Z, \quad \textit{if } S_q^2 H^2(M; Z_2) = 0,$$

and

$$KSO(M) = \sum_1^r Z_2 + Z \quad (r = \dim H^2(M; Z_2) - 1), \quad \textit{if } S_q^2 H^2(M; Z_2) \neq 0.$$

COROLLARY 3. *If* M *is a non-orientable, closed, topological* 4-*manifold, we have that*

$$KSO(M) \cong H^2(M; Z_2) + Z_2, \quad \textit{if } S_q^2 H^2(M; Z_2) = 0$$

and

$$KSO(M) = \sum_1^r Z_2 + Z_4 \quad (r = \dim\ H^2(M; Z_2) - 1), \quad \textit{if } S_q^2 H^2(M; Z_2) \neq 0.$$

We give a few examples.

X	$P_2(C)$	$P_4(R)$	$S^2 \times S^2$	$S^3 \overset{2^i}{\cup} e^4\ (i \geqq 1)$
$KSO(X)$	Z	Z_4	$Z + Z_2 + Z_2$	Z_{2^i}

REFERENCE

[1] A. Dold and H. Whitney, "Classifications of oriented sphere bundles over a 4-complex," Ann. of Math., 69 (1959), 667–677.

Department of Mathematics
Tokyo Institute of Technology