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ON ASYMPTOTIC VALUES OF SLOWLY GROWING ALGEBROID FUNCTIONS

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1. Let $f(z)$ be a k-valued algebroid function in $|z| < \infty$ and

(1) $$F(z,f) \equiv A_0(z)f^k + A_1(z)f^{k-1} + \cdots A_k(z) = 0$$

be its defining equation such that the coefficients $A_i(z)$ $(i = 0, 1, \cdots, k)$ are entire functions without any common zero and the left hand side is irreducible. We denote by $\mathfrak{X}$ the k-sheeted covering surface over $|z| < \infty$ generated by $f(z)$ and by $\mathfrak{X}(r)$ and $\Gamma(r)$ the part of $\mathfrak{X}$ over $|z| \leq r$ and the curves on $\mathfrak{X}$ over $|z| = r$, respectively. We use the standard notations of the Nevanlinna-Selberg theory [4]:

$$m(r,a) = \frac{1}{2k\pi}\int_{\Gamma(r)} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta, \quad m(r,f) = \frac{1}{2k\pi}\int_{\Gamma(r)} \log^+ |f(re^{i\theta})|\, d\theta$$

$$N(r,a) = \frac{1}{k}\int_0^r \frac{n(t,a) - n(0,a)}{t} + \frac{n(0,a)}{k}\log r, \quad N(r,\infty) = N(r,f)$$

$$T(r,f) = m(r,f) + N(r,f), \qquad \delta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r,a)}{T(r,f)},$$

where $n(r,a)$ is the number of zeros of $f(z) - a$ on $\mathfrak{X}(r)$ and $n(r,\infty) = n(r,f)$.

From now on, we consider the functions with the slow growth:

(2) $$T(r,f) = O[(\log r)^2].$$

For such functions both of the number of deficient values and that of asymptotic values are at most k (Valiron [7], [9] and Tumura [5]). Especially, when $k = 1$ i.e. the function is single-valued and meromorphic, it can prossess no deficient value without that value being an asymptotic value (Valiron [9] and Anderson-Clunie [1]).

For an algebroid function $f(z)$, a value α is an asymptotic value, if there exists a path $L_{\mathfrak{X}}$ on $\mathfrak{X}$ stretching to the point at infinity such that $f(z)$

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tends to α along $L_{\mathfrak{X}}$, in other words, if there exists a path L on the z-plane stretching to the point at infinity such that at least one branch of $f(z)$ can be continued analytically along L and the value taken by the branch tends to α along L.

Our main aim in this note is to give an extension of the above result of Anderson-Clunie to the case of an algebroid function:

THEOREM 1. *Let $f(z)$ be a k-valued algebroid function in $|z|<\infty$ satisfying* (2). *If $f(z)$ has k deficient values α_i $(i=1,2,\cdots,k)$, then each of α_i $(=1,2,\cdots,k)$ is an asymptotic value of $f(z)$.*

This theorem will be obtained as an immediate corollary of Theorem 2 stated in §5. In the last section, we shall give a condition for a deficient value to be an asymptotic value without the restriction that $f(z)$ has k deficient values.

2. First we shall give some lemmas. To prove them, we use the following results.

I. (*Valiron* [6]) *If $f(z)$ is a k-valued algebroid function in $|z|<\infty$, then*

$$\left|T(r,f)+\frac{1}{k}\log|C_\lambda|-\mu(r,A)\right|<\log 2, \tag{3}$$

where $\mu(r,A)=\dfrac{1}{2k\pi}\displaystyle\int_0^{2\pi}\log A(re^{i\theta})d\theta$ *with* $A(z)=\max\limits_{0\le i\le k}|A_i(z)|$ *and $C_\lambda z^\lambda$ is the first non-zero term of the Taylor development of $A_0(z)$ at the origin.*

II. (*Valiron* [9]) *If $f(z)$ is a k-valued algebroid function in $|z|<\infty$ satisfying* (2), *and if a_i $(i=1,2,\cdots,k+1)$ are $k+1$ distinct complex numbers (may be infinity), then we have*

$$\lim_{r\to\infty}\frac{N(r,a_1,a_2,\cdots,a_{k+1})}{kT(r,f)}=1$$

where $N(r,a_1,a_2,\cdots,a_{k+1})=\max\limits_{1\le i\le k+1}N\left(r,\dfrac{1}{F(z,a_i)}\right)$ *for each* $r>0$.

III. (*Valiron* [8]) *If $g(z)$ is an entire function of order zero with* $g(0)=1$[1], *then*

$$\log M(r,g)=N\left(r,\frac{1}{g}\right)+\Theta(r)W\left(r,\frac{1}{g}\right)\quad(0<\Theta(r)<1),$$

[1] This condition is not essential to obtain (4).

where $M(r,g)=\max_{|z|=r}|g(z)|$ *and* $W\left(r,\frac{1}{g}\right)=r\int_0^\infty n\left(t,\frac{1}{g}\right)\frac{dt}{t^2}$.

In particular, if $\log M(r,g)=O[(\log r)^2]$, then

$$\log M(r,g)<K(\log r)^2 \qquad (K\text{: constant})$$

$$n\left(r,\frac{1}{g}\right)\log r=\int_r^{r^2}n\left(r,\frac{1}{g}\right)\frac{dt}{t}\leq\int_r^{r^2}n\left(t,\frac{1}{g}\right)\frac{dt}{t}<K(\log r^2)^2$$

$$=K'(\log r)^2$$

$$W\left(r,\frac{1}{g}\right)<K'r\int_0^\infty\frac{\log t}{t^2}\,dt=K'r\,\frac{\log r+1}{r}=O(\log r),$$

so that we have

$$\log M(r,g)\sim N\left(r,\frac{1}{g}\right) \qquad (r\to\infty). \tag{4}$$

IV. (*Hayman* [3]) *If an entire function* $g(z)$ *satisfies*

$$\log M(r,g)=O[(\log r)^2],$$

then

$$\log M(r,g)\sim\log|g(z)|, \tag{5}$$

uniformly in θ *as* $z=re^{i\theta}\to\infty$ *outside an* $\mathscr{E}$*-set.*

Here we call an $\mathscr{E}$-set any countable set of circles not containing the origin and subtending angles at the origin whose sum s is finite. We note the following two facts about $\mathscr{E}$-sets.

a) The union of two $\mathscr{E}$-sets in again an $\mathscr{E}$-set.

b) Given any $\mathscr{E}$-set then for almost all fixed θ and any $r>r_0(\theta)$, where $r_0(\theta)$ depends only on θ, $z=re^{i\theta}$ lies outside the $\mathscr{E}$-set.

We consider a system $\mathfrak{S}(z)=(S_0(z),S_1(z),\cdots,S_k(z))$ of $k+1$ entire functions $S_i(z)$ $(i=0,1,\cdots,k)$ having no common zero and satisfying

$$\log M(r,S_i)=O[(\log r)^2] \qquad (i=0,1,\cdots,k). \tag{6}$$

We define $\mu(r,S)$ by

$$\mu(r,S)=\frac{1}{2k\pi}\int_0^{2\pi}\log S(re^{i\theta})d\theta,$$

where $S(z)=\max_{0\leq i\leq k}|S_i(z)|$ for each z and set

$$1-\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{S_i}\right)}{k\mu(r,S)}=\delta_i(\mathfrak{S})\qquad (i=0,1,\cdots,k).$$

Particularly, when $\lim_{r\to\infty}\frac{N\left(r,\frac{1}{S_i}\right)}{k\mu(r,S)}$ exists, we set

$$1-\lim_{r\to\infty}\frac{N\left(r,\frac{1}{S_i}\right)}{k\mu(r,S_i)}=\bar{\delta}_j(\mathfrak{S}).$$

Then we have $0\leq\delta_i(\mathfrak{S})\leq 1$ $(i=0,1,\cdots,k)$, since by Jensen's formula

$$N\left(r,\frac{1}{S_i}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log|S_i(re^{i\theta})|\,d\theta-\log|S_i(0)|^{2)}$$

$$\leq\frac{k}{2k\pi}\int_0^{2\pi}\log S(re^{i\theta})d\theta+O(1)=k\mu(r,S)+O(1).$$

LEMMA 1. *For a system* $\mathfrak{S}(z)=(S_0(z),S_1(z),\cdots,S_k(z))$, *if* $\delta_j(\mathfrak{S})>0$ *for some* $j(0\leq j\leq k)$, *then*

$$\underline{\lim_{r\to\infty}}\frac{-\log\dfrac{|S_i(z)|^2}{\sum_0^k|S_i(z)|^2}}{2k\mu(r,S)}\geq\delta_j(\mathfrak{S})>0,$$

uniformly in θ *as* $z=re^{i\theta}\to\infty$ *outside an* $\mathscr{E}$*-set.*

Proof. From our hypothesis, we have

$$N\left(r,\frac{1}{S_i}\right)<(1-\delta_j(\mathfrak{S})+o(1))k\mu(r,S).$$

Since $\mathfrak{S}(z)$ satisfies (6), we can apply (4) and (5) to $S_j(z)$ and have

$$\log|S_i(z)|<(1-\delta_j(\mathfrak{S})+o(1))k\mu(r,S), \tag{7}$$

uniformly in θ as $z=re^{i\theta}\to\infty$ outside an $\mathscr{E}$-set.

By Cauchy's inequality, we have for all ν $(\nu=0,1,\cdots,k)$

$$\log\left(\sum_{i=0}^k|S_i(z)|^2\right)\geq\log\left\{\frac{1}{k+1}\left(\sum_{i=0}^k|S_i(z)|\right)^2\right\}=2\log\left(\sum_{i=0}^k|S_i(z)|\right)+\log\frac{1}{k+1}$$

$$\geq 2\log|S_\nu(z)|+\log\frac{1}{k+1}.$$

2) We assume that $S_i(0)\neq 0,\infty$.

Applying (5) to $S_\nu(z)$, we have for all $\nu(\nu = 0, 1, \cdots, k)$

$$\log\left(\sum_{i=0}^{k} |S_i(z)|^2\right) \geq 2(1 + o(1))\log M(r, S_\nu) + \log\frac{1}{k+1},$$

and hence

$$\log\left(\sum_{i=0}^{k} |S_i(z)|^2\right) \geq 2(1 + o(1)) \max_{0\leq\nu\leq k} \log M(r, S_\nu) + \log\frac{1}{k+1},$$

uniformly in θ as $z = re^{i\theta} \to \infty$ outside an $\mathscr{E}$-set.

On the other hand, by definition of $S(z)$,

$$S(z) \leq \max_{0\leq\nu\leq k} M(r, S_\nu) \qquad (|z| = r)$$

so that $\mu(r, S) = \frac{1}{2k\pi}\int_0^{2\pi} \log S(re^{i\theta})\, d\theta \leq \frac{1}{k} \max_{0\leq\nu\leq k} \log M(r, S_\nu)$. Thus we have

$$\log\left(\sum_{i=0}^{k} |S_i(z)|^2\right) \geq 2k(1 + o(1))\mu(r, S), \tag{8}$$

uniformly in θ as $z = re^{i\theta} \to \infty$ outside the $\mathscr{E}$-set.

We combine (7) and (8) and have from the property a) of $\mathscr{E}$-sets,

$$\log\frac{|S_j(z)|^2}{\sum_{i=0}^{k}|S_i(z)|^2} = 2\log|S_i(z)| - \log\left(\sum_{i=0}^{k} |S_i(z)|^2\right)$$

$$\leq 2k(-\delta_j(\mathfrak{S}) + o(1))\mu(r, S)$$

uniformly in θ as $z = re^{i\theta} \to \infty$ outside an $\mathscr{E}$-set. Thus we obtain the desired result.

By using the property b) of $\mathscr{E}$-sets and the fact that the function $\mu(r, S)$ of r is unbounded, we have that

$$\frac{|S_j(z)|^2}{\sum_{i=0}^{k} |S_i(z)|^2} \to 0$$

as $z = re^{i\theta} \to \infty$ for almost all fixed θ $(0 \leq \theta < 2\pi)$.

3. Before giving the next lemma, we shall state some about the distance between two systems, which was introduced by Dufresnoy [2].

We consider only the systems consisting of $k+1$ complex numbers, all of which are not zero simultaneously. Here if two systems

$$w^{(1)} = (w_0^{(1)}, w_1^{(1)}, \cdots, w_k^{(1)}) \text{ and } w^{(2)} = (w_0^{(2)}, w_1^{(2)}, \cdots, w_k^{(2)})$$

are proportional i.e. $w_i^{(1)} = cw_i^{(2)}$ $(i = 0, 1, \cdots, k)$ for some constant $c (c \neq 0)$, we identify $w^{(1)}$ with $w^{(2)}$.

We set

$$(9) \qquad [[w^{(1)}, w^{(2)}]] = \left\{ \frac{\sum_{i>j} |w_i^{(1)} w_j^{(2)} - w_j^{(1)} w_i^{(2)}|^2}{\sum_{i=0}^{k} |w_i^{(1)}|^2 \sum_{i=0}^{k} |w_i^{(2)}|^2} \right\}^{\frac{1}{2}}$$

Then this satisfies three axioms for distances. According to Dufresnoy [2] we call $[[w^{(1)}, w^{(2)}]]$ the distance between two systems $w^{(1)}$ and $w^{(2)}$. We can easily see that an inequality

$$(10) \qquad [[w^{(1)}, w^{(2)}]]^2 \leq \frac{\sum_{i=0}^{k} |w_i^{(1)} - w_i^{(2)}|^2}{\{\sum_{i=0}^{k} |w_i^{(1)}|^2 \sum_{i=0}^{k} |w_i^{(2)}|^2\}^{1/2}}$$

holds. This shows how our distance relates to the distance in ordinary sense between $w^{(1)}$ and $w^{(2)}$.

Now we consider a non-degenerate, linear and homogeneous substitution of the elements of the system $w = (w_0, w_1, \cdots, w_k)$;

$$(11) \qquad W_i = \sum_{j=0}^{k} a_{ij} w_j \qquad (i = 0, 1, \cdots, k).$$

Then we have a new system $W = (W_0, W_1, \cdots, W_k)$. Let

$$W^{(1)} = (W_0^{(1)}, W_1^{(1)}, \cdots, W_k^{(1)}) \text{ and } W^{(2)} = (W_0^{(2)}, W_1^{(2)}, \cdots, W_k^{(2)})$$

be the systems obtained by the substitution (11) of the elements of systems $w^{(1)}$ and $w^{(2)}$, respectively. Then, using the inequality (10) we have an important property about the distance (9) which is stated as follows;

LEMMA 2. *(Dufresnoy [2]) Under such a substitution, two systems being close to each other correspond to two systems also being close to each other i.e. there exists a constant* c, $0 < c < 1$, *depending only on* a_{ij} $(i, j = 0, 1, \cdots, k)$ *such that*

$$c[[w^{(1)}, w^{(2)}]] < [[W^{(1)}, W^{(2)}]] < c^{-1}[[w^{(1)}, w^{(2)}]].$$

Let
$$p(z) = a_0 z^k + a_1 z^{k+1} + \cdots + a_k = 0$$

$$p^*(z) = a_0^* z^k + a_1^* z^{k-1} + \cdots + a_k^* = 0$$

be two algebraic equations whose coefficients make systems $a = (a_0, a_1, \cdots, a_k)$ and $a^* = (a_0^*, a_1^*, \cdots, a_k^*)$, respectively. By means of distance (9), the well known theorem on continuity of roots of algebraic equations is described as follows;

LEMMA 3. *(Dufresnoy* [2]) *Let* $z_1, z_2, \cdots, z_k$ *and* $z_1^*, z_2^*, \cdots, z_k^*$ *be the roots of the equations* $p(z)=0$ *and* $p^*(z)=0$, *respectively. If* $[[a, a^*]]$ *is sufficiently small, then we can associate each* $z_i (i=0,1,\cdots,k)$ *with some* z_j^* $(1 \leq j \leq k)$, *say* z_i *with* $z_{\alpha_i}^*$, *such that*

$$[z_i, z_{\alpha_i}^*] < 8e[[a, a^*]]^{\frac{1}{k}} \qquad (i=1,2,\cdots,k),$$

where [,] *denotes the chordal distance.*

The next lemma is an immediate consequence of Lemma 3.

LEMMA 4. *(Dufresnoy* [2]) *If*

$$\frac{\sum_{i=0}^{p} |a_i|^2}{\sum_{j=0}^{k} |a_j|^2} \qquad (0 \leq p \leq k-1)$$

is sufficiently small, then an algebraic equation

$$p(z) = a_0 z^k + a_1 z^{k-1} + \cdots + a_k = 0$$

has at least $p+1$ *roots whose chordal distances from the point at infinity are less than*

$$8e \left\{ \frac{\sum_{i=0}^{p} |a_i|^2}{\sum_{j=0}^{k} |a_j|^2} \right\}^{\frac{1}{2k}}.$$

For the sake of the later discussion, we shall give a proof following Dufresnoy [2].

Proof. We consider one more equation

$$p^*(z) = a_0^* z^k + a_1^* z^k + \cdots + a_k^* = 0$$

with $a_i^* = 0$ $(i=0,1,\cdots,p)$ and $a_j^* = a_j (j=p+1,\cdots,k)$. Then we have

$$[[a, a^*]] = \left\{ \frac{\sum_{i=0}^{p} |a_i|^2}{\sum_{j=0}^{k} |a_j|^2} \right\}^{\frac{1}{2}}$$

We may consider that the equation $p^*(z)=0$ has k roots, $p+1$ of them lying at the point at infinity. Thus our Lemma is obtained from Lemma 3. Here we note that each of the other $k-p-1$ roots $z_i(i=1,2,\cdots,k-p-1)$ of $p(z)=0$ is associated with one of the $k-p-1$ roots $z_i^*(i=1,2,\cdots,k-p-1)$ of $p^*(z)=0$, say z_l with $z_{\alpha_l}^*$, in such a way that

$$[z_l, z_{\alpha_l}^*] < 8e\left\{\frac{\sum_{i=0}^{p}|a_i|^2}{\sum_{j=0}^{k}|a_j|^2}\right\}^{\frac{1}{2k}} \qquad (l=1,2,\cdots,k-p-1).$$

4. LEMMA 5. *Let* $\mathfrak{S}(z)=(S_0(z), S_1(z),\cdots,S_k(z))$ *be a system such that* $S_i(z)$ $(j=0,1,\cdots,k)$ *have no common zero and satisfy* (6). *If* $\delta_\lambda(\mathfrak{S})=0$ *for only one* $\lambda(0\leq\lambda\leq k)$ *and* $\delta_\nu(\mathfrak{S})>0$ *for other all* $\nu\neq\lambda$ $(0\leq\nu\leq k)$, *then*

$$[[\mathfrak{S}(z_1),\ \mathfrak{S}(z_2)]]\to 0$$

uniformly in θ_m *as* $z_m=r_me^{i\theta_m}\longrightarrow\infty$ *outside an* $\mathscr{E}$*-set* $(m=1,2)$.

Proof. For any pair (i,j) $(i\neq j;\, i,j=0,1,\cdots,k)$,

$$\frac{|S_i(z_1)S_j(z_2)-S_j(z_1)S_i(z_2)|}{\left\{\sum_{h=0}^{k}|S_h(z_1)|^2\sum_{h=0}^{k}|S_h(z_2)|^2\right\}^{\frac{1}{2}}} \leq \frac{|S_i(z_1)S_j(z_2)|}{\left\{\sum_{h=0}^{k}|S_h(z_1)|^2\sum_{h=0}^{k}S_h(z_2)|^2\right\}^{\frac{1}{2}}}$$

$$+\frac{|S_j(z_1)S_i(z_2)|}{\left\{\sum_{h=0}^{k}|S_h(z_1)|^2\sum_{h=0}^{k}|S_h(z_2)|^2\right\}^{\frac{1}{2}}} \leq \min_{l=i,j}\left\{\frac{|S_l(z_1)|}{\left(\sum_{h=0}^{k}|S_h(z_1)|^2\right)^{\frac{1}{2}}}+\frac{|S_l(z_2)|}{\left(\sum_{h=0}^{k}|S_n(z_2)|^2\right)^{\frac{1}{2}}}\right\}.$$

By Lemma 1 and our hypotheses, we have for all $\nu(\neq\lambda)$

$$\frac{|S_\nu(z)|}{\left(\sum_{h=0}^{k}|S_h(z)|^2\right)^{\frac{1}{2}}}\to 0$$

uniformly in θ as $z=re^{i\theta}\to\infty$ outside an $\mathscr{E}$-set, and hence

$$\frac{|S_i(z_1)S_j(z_2)-S_j(z_1)S_i(z_2)|}{\left(\sum_{h=0}^{k}|S_h(z_1)|^2\sum_{h=0}^{k}|S_h(z_2)|^2\right)^{\frac{1}{2}}}\to 0.$$

uniformly in θ_m as $z_m=r_me^{i\theta_m}\to\infty$ outside an $\mathscr{E}$-set $(m=1,2)$. Thus our lemma is obtained.

COROLLARY. *Let $f(z)$ be a k-valued algebroid function in $|z| < \infty$ satisfying* (2). *Suppose that $f(z)$ has k deficient values $\alpha_i \, (i = 1, 2, \cdots, k)$. Then for the system $\mathfrak{A}(z) = (A_0(z), A_1(z), \cdots, A_k(z))$, we have the same assertion as that in the above lemma.*

Proof. We take a value α_0 which is different from $\alpha_i \, (i = 1, 2, \cdots, k)$ and set

$$F(z, \alpha_i) = A_0(z)\alpha_i^k + A_1(z)\alpha_i^{k-1} + \cdots + A_k(z) = B_i(z) \qquad (i = 0, 1, 2, \cdots, k). \tag{12}$$

Now we shall prove that for the system $\mathfrak{B}(z) = (B_0(z), B_1(z), \cdots, B_k(z))$, all the conditions of Lemma 5 are satisfied. At first, entire functions $B_i(z)$ $(i = 0, 1, \cdots, k)$ have no common zero. In fact, suppose that $B_i(z)$ $(i = 0, 1, \cdots, k)$ have a common zero a. We solve the equation (12) with respect to $A_i(z)$ $(i = 0, 1, \cdots, k)$ and have

$$A_i(z) = \beta_{i0}B_0(z) + \beta_{i1}B_1(z) + \cdots + \beta_{ik}B_k(z) \qquad (i = 0, 1, \cdots, k \ \beta_{ij};\ \text{constants}) \tag{13}$$

so that a is also a common zero of $A_i(z)$ $(i = 0, 1, \cdots, k)$, which is absurd. Further, we have from (12) and (13),

$$\mu(r, A) = \mu(r, B) + O(1) \tag{14}$$

so that $B_i(z)$ $(i = 0, 1, \cdots, k)$ satisfy (6) by (2) and (3).

Next, since $N\left(r, \frac{1}{f - \alpha_i}\right) = \frac{1}{k} N\left(r, \frac{1}{B_i}\right)$ $(i = 0, 1, \cdots, k)$ and $\alpha_i (i = 1, 2, \cdots, k)$ are deficient values of $f(z)$, we have by (3)

$$\delta_j(\mathfrak{B}) = 1 - \varlimsup_{r\to\infty} \frac{N\left(r, \frac{1}{B_j}\right)}{kT(r, f)} = \delta(\alpha_j, f) > 0 \qquad (j = 1, 2, \cdots, k). \tag{15}$$

On the other hand, the value α_0 is normal by II in § 2, i.e.

$$\bar{\delta}_0(\mathfrak{B}) = 1 - \lim_{r\to\infty} \frac{N\left(r, \frac{1}{B_0}\right)}{kT(r, f)} = \delta(\alpha_0, f) = 0. \tag{16}$$

Now Lemma 5 applied to the system $\mathfrak{B}(z) = (B_0(z), B_1(z), \cdots, B_k(z))$ shows that

$$[[\mathfrak{B}(z_1),\ \mathfrak{B}(z_2)]] \to 0$$

uniformly in θ_m as $z_m = r_m e^{i\theta_m} \longrightarrow \infty$ outside an $\mathscr{C}$-set $(m = 1,2)$.

Since we can take (12) as a non-degenerate, linear and homogeneous substitution of the elements $A_i(z)$ of the system $\mathfrak{A}(z) = (A_0(z),\ A_1(z), \cdots, A_k(z))$, we obtain the desired result by Lemma 2.

5. THEOREM 2. *Let $f(z)$ be a k-valued algebroed function $|z| < \infty$ of arbitrary order. Suppose that there exists a path L on the plane stretching to the point at infinity such that*

$$\frac{|A_0(z)|}{\left(\sum_{i=0}^{k} |A_i(z)|\right)^{\frac{1}{2}}} \to 0 \tag{17}$$

$$[[\mathfrak{A}(z_1),\ \mathfrak{A}(z_2)]] \to 0 \tag{18}$$

as z, z_1 and z_2 tend to infinity along L. Then the infinity is an asymptotic value of $f(z)$.

Proof. We denote by $K(\delta)$ the spherical disk with center at the point at infinity and with chordal radius $\delta > 0$, and denote by $f_i(z)$ $(i=1,2,\cdots,k)$ k roots of $F(z,f) = 0$ for any z counting with their proper multiplicities. We express the curve L by

$$L : z = z(t)\ \ (0 < t < \infty);\ z(t) \to \infty \ \text{ as } \ t \to \infty.$$

Given a sufficiently small $\varepsilon > 0$, we can find from (17) and (18) $t_0^{(n)}$ $(n = 1,2,\cdots)$ depending on ε such that for any $t \geq t_0^{(n)}$,

$$8e\left\{\frac{|A_0(z)|^2}{\sum_{i=0}^{k} |A_i(z)|^2}\right\}^{\frac{1}{2k}} < \frac{\varepsilon}{2(k+1)^n} \quad (z = z(t)) \tag{19}$$

and for any pair t_1 and t_2; $t_1,\ t_2 \geq t_0^{(n)}$,

$$8e[[\mathfrak{A}(z_1),\ \mathfrak{A}(z_2)]]^{\frac{1}{k}} < \frac{\varepsilon}{2(k+1)^n} \quad (z_i = z(t_i);\ i = 1,2). \tag{20}$$

First we take whole branches f_i $(i = 1,2,\cdots,k)$ as our candidates and let z go to infinity along L. Then we drop from the list of candidates branches f_i, if any, with $f_i\ (z(t_0^{(1)})) \notin K(\varepsilon)$. The disk $K\left(\frac{\varepsilon}{2(k+1)}\right)$ contains at lesat one root of the equation $F(z(t_0^{(1)}), f) = 0$ because of Lemma 4 and (19) and so there remains at least one f_j in our list. Next we drop f_i, if any, with $f_i(z(t_0^{(2)})) \notin K\left(\frac{\varepsilon}{k+1}\right)$ from our 2nd list and still have a list containing at least one f_j by the same reason as above. Then we see that, for any f_j in the list, the curve $f_j(z(t))$, $t_0^{(1)} \leq t \leq t_0^{(2)}$, is contained in $K(\varepsilon)$. In fact, if not, the curve $f_j(z(t))$, $t_0^{(1)} \leq t \leq t_0^{(2)}$, can not be covered by any k disks with radii $\frac{\varepsilon}{2(k+1)}$ and so there exists at least one point $z^* = z(t^*)$, $t_0^{(1)} < t^* < t_0^{(2)}$, such that

$$[f_j(z^*),\ f_i(z(t_0^{(1)}))] > \frac{\varepsilon}{2(k+1)} \qquad (i = 1, 2, \cdots, k),$$

which coutradicts Lemma 3 and (20). We repeat the above procedures and, at the n-th step, we drop f_i, if any, with $f_i(z(t_0^{(n)})) \notin K\left[\frac{\varepsilon}{(k+1)^{n-1}}\right]$ from our n-th list, and have the $(n+1)$-th list containing at least one f_j. For any f_j in this list, the curve $f_j(z(t))$, $t_0^{(n-1)} \leq t \leq t_0^{(n)}$, is contained in $K\left[\frac{\varepsilon}{(k+1)^{n-2}}\right]$. Since we have only a finite number of branches f_i, there is at least one f_j, say f_1, which belongs to the n-th list for $n = 1, 2, \cdots$. Thus f_1 satisfies

$$f_1(z(t)) \in K\left[\frac{\varepsilon}{(k+1)^{n-2}}\right], \qquad t \geq t_0^{(n-1)},$$

so that $f_1(z)$ tends to infinity as z goes to infinity along L. The proof is now complete.

Proof of Theorem 1. When $\alpha_i \neq \infty$, we consider $\frac{1}{f-\alpha_i}$ instead of f. Then $\frac{1}{f-\alpha_i}$ is an algebroid function satisfying (2) and has k deficient values, one of which is the infinity, so that we may assume that $\alpha_i = \infty$. From Lemma 1 and Corollary of Lemma 5, the coefficients $A_0(z)$, $A_1(z)$, $\cdots, A_k(z)$ of the defining equation of $f(z)$ satisfying the conditions (17) and (18) outside an $\mathscr{E}$-set, consequently on any half-line $L = re^{i\theta} (r > 0)$ for almost every θ. Applying Theorem 2, we conclude that α_i is an asymptotic value of f along L.

Remark. As we saw in the above proof, we can take any half-line L for almost every θ as an asymptotic path of α_i and hence an L commonly to all α_i; $i = 1, 2, \cdots, k$.

6. LEMMA 6. (*Dufresnoy* [2]) *Let* $p(z)=a_0z^\nu+a_1z^{\nu-1}+\cdots+a_\nu=0$ *be an algebraic equation with*

$$\frac{|a_0|^2}{\sum_{i=0}^{\nu}|a_i|^2}=\frac{\nu}{1+M^2}\quad(M>0).$$

Then $p(z)=0$ *has no root of modulus larger than* M.

From this, we can see that if

$$\frac{|a_0|^2}{\sum_{i=0}^{\nu}|a_i|^2}=\nu d^2\qquad(d>0),$$

every root of $p(z)=0$ lies outside a spherical disk $K(d)$ with center at the point at infinity and with chordal radius d. Using this lemma, we can prove

THEOREM 3[3]. *Let* $f(z)$ *be a k-valued algebroid function in* $|z|<\infty$ *which is defined by* (1) *and satisfies* (2). *Suppose that, for some* $n(0<n\leq k)$, *the system* $\mathfrak{A}(z)=(A_0(z), A_1(z),\cdots,A_k(z))$ *satisfies*

$$\delta_j(\mathfrak{A})>0\ (j=0,1,\cdots,n-1),\quad \bar{\delta}_n(\mathfrak{A})=0.$$

Then the infinity is an asymptotic value of $f(z)$.

Prooof. From our hypothesis $\bar{\delta}_n(\mathfrak{A})=0$ and (3), we have $\lim_{r\to\infty}\dfrac{N\left(r,\frac{1}{A_n}\right)}{kT(r,f)}=1.$ Hencd we have by (4) and (5)

$$\log|A_n(z)|^2=(1+o(1))2kT(r,f),$$

uniformly in θ as $z=re^{i\theta}\to\infty$ outside an $\mathscr{E}$-set. Further, we have

$$\log\left(\sum_{i=n}^{k}|A_i(z)|^2\right)\leq\log\left(\sum_{i=0}^{k}|A_i(z)|^2\right)\leq 2\log A(z)+\log(k+1)$$

$$\leq 2\max_{0\leq\nu\leq k}\log M(r,A_\nu)+\log(k+1)=2(1+o(1))\max_{0\leq\nu\leq k}N\left(r,\frac{1}{A_\nu}\right)$$

$$\leq(1+o(1))2kT(r,f).$$

Thus

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3) As for notations used in this theorem, see § 2.

$$\log \frac{|A_n(z)|^2}{\sum_{i=n}^{k} |A_i(z)|^2} = o[T(r,f)]$$

and hence for any small $\varepsilon > 0$,

$$e^{-\varepsilon T(r,f)} < \left(\frac{1}{k-n} \frac{|A_n(z)|^2}{\sum_{i=n}^{k} |A_i(z)|^2} \right)^{\frac{1}{2}} < e^{\varepsilon T(r,f)}$$

uniformly in θ as $z = re^{i\theta} \to \infty$ outside the $\mathscr{E}$-set. Since $\delta_j(\mathfrak{A}) > 0$ $(j = 0, 1, \cdots, n-1)$, we see from Lemma 1,

$$\log \frac{|A_j(z)|^2}{\sum_{i=0}^{k} |A_i(z)|^2} < (-\delta_j(\mathfrak{A}) + o(1))2kT(r,f) \quad (j = 0, 1, \cdots, n-1)$$

and hence

$$8e \left\{ \frac{\sum_{j=0}^{n-1} |A_j(z)|^2}{\sum_{i=0}^{k} |A_i(z)|^2} \right\}^{\frac{1}{2k}} < e^{(-\delta+\varepsilon)T(r,f)} \tag{22}$$

uniformly in θ as $z = re^{i\theta} \to \infty$ outside an $\mathscr{E}$-set, where $\delta = \min_{0 \le j \le n-1} \delta_j(\mathfrak{A}) > 0$.

We take $\varepsilon < \delta/3$ and a path L: $z = z(r) = re^{i\theta}$ $(r_0 < r < \infty)$ such that (21) and (22) hold on L[4], and set

$$d_1(r) = e^{(-\delta+\varepsilon)T(r,f)}$$
$$d_2(r) = e^{-\varepsilon T(r,f)}.$$

We onsider on L the following equation

$$A_n(z) f^{*k-n} + A_{n+1}(z) f^{*k-n-1} + \cdots + A_k(z) = 0.$$

Recall (21). Then we see from Lemma 6 that the roots $f_i^*(z)$ $(i = 1, 2, \cdots, k-n)$ lie outside $K(d_2(r))$. The equation $F(z,f) = 0$ has $k-n$ roots, say $f_i(z)$ $(i = 1, 2, \cdots, k-n)$, such that

$$[f_i^*(z),\ f_i(z)] < d_1(r),$$

because of the comment given just after Lemma 4 and (22). Thus the values $f_i(z)$ $(i = 1, 2, \cdots, k-n)$ lie outside $K(d_2(r) - d_1(r))$. On the other

[4] We can find such a path L because (21) and (22) hold as $z \to \infty$ outside an $\mathscr{E}$-set.

hand, we see from Lemma 4 that the remainder $f_j(z)$ $(j = k-n+1, \cdots, k)$ satisfies

$$[f_j(z), \ \infty] < d_1(r).$$

Since $d_1(r)/d_2(r) = e^{(-\delta+2\varepsilon)T(r,f)} \to 0$ as $r \to \infty$, we see that $K(d_1(r))$ is disjoint with the complement of $K(d_2(r) - d_1(r))$ for every sufficiently large $r \geqq r_1$, whence we can conclude that the brancehs $f_j(z)$ $(j = k-n+1, \cdots, k)$ with $f_j(z(r_1)) \in K(d_1(r_1))$ draw a curve $f_j(z(t))$, $t \geqq r \geqq r_1$, in $K(d_1(r))$. In fact, if the curve $f_j(z(t))$, $t \geqq r$ $(\geqq r_1)$, invades the zone; $\{w; d_2(r) - d_1(r) < [w, \infty] < d_1(r)\}$, we have at least one point $z^* = z(t^*)$, $t^* > r$, on the curve such that

$$f_j(z^*) \notin K(d_1(t^*)),$$
$$f_j(z^*) \notin \text{complement of } K(d_2(t^*) - d_1(t^*)),$$

which contradicts the fact that any root of the equation $F(z^*, f) = 0$ must be contained in $K(d_1(t^*))$ or the complement of $K(d_2(t^*) - d_1(t^*))$. Since $d_1(r) \to 0$ $(r \to \infty)$, we see that the branches $f_j(z)$ tend to infinity as $z \to \infty$ along L. Thus our theorem has been established.

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