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Nagoya Math. J.
Vol. 41 (1971), 89–96

SOLVABILITY OF A CLASS OF RANK 3 PERMUTATION GROUPS[1)]

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1. Introduction. Let G be a rank 3 permutation group of even order on a finite set X, $|X| = n$, and let Δ and Γ be the two nontrivial orbits of G in $X \times X$ under componentwise action. As pointed out by Sims [6], results in [2] can be interpreted as implying that the graph $\mathscr{S} = (X, \Delta)$ is a strongly regular graph, the graph theoretical interpretation of the parameters k, l, λ and μ of [2] being as follows: k is the degree of $\mathscr{S}$, λ is the number of triangles containing a given edge, and μ is the number of paths of length 2 joining a given vertex P to each of the l vertices $\neq P$ which are not adjacent to P. The group G acts as an automorphism group on $\mathscr{S}$ and on its complement $\overline{\mathscr{S}} = (X, \Gamma)$.

A family of solutions of the conditions in [2] for the parameters n, k, l, λ, μ is given by

(1) $n = 4t + 1, \; k = l = 2t, \; \mu = \lambda + 1 = t.$

This family includes the only case in which the adjacency matrix A of $\mathscr{S}$ has irrational eigenvalues [2].

Assuming that (1) holds for G, we have by [2] that

(2) G is primitive,

(3) $\overline{\mathscr{S}}$ is a strongly regular graph whose parameters satisfy (1), and

(4) $A^2 + A - tI = tF$, where F has all entries 1.

Here we consider the case in which t is a prime, proving

THEOREM 1. *If G is a rank 3 permutation group with parameters given by (1) with t a prime, then G is solvable.*

Received November 20, 1969.

1) Research supported in part by the National Science Foundation.

As explained in § 2, the groups G of Theorem 1 are actually determined (Theorem 2). Our result implies that for admissable prime values of t the graph $\mathcal{S}$ is unique up to isomorphism. We do not know if strongly regular graphs satisfying (1) but not admitting rank 3 automorphism groups can exist, nor do we have an example of a nonsolvable group of rank 3 whose parameters satisfy (1).

For the most part we follow the notation and terminology of Wielandt's book [7]. But if G is a permutation group on X and $\Phi \subseteq X$ we write G_{Φ} and $G_{[\Phi]}$ respectively for the setwise and pointwise stabilizers of Φ, and if $H \leq G_{\Phi}$, we denote by $H|\Phi$ the image under restriction of H in the symmetric group on Φ. We use the notation and terminology of [2] and [3] for rank 3 permutation groups. For the connection between permutation groups and graphs see the papers [5] and [6] of Sims.

2. Examples of Singer type.

Let p be a prime and ρ an integer >0 such that $p^{\rho} = 4t + 1$. Let M be the additive group of the field $F_{p^{\rho}}$. Identify a primitive element ξ of $F_{p^{\rho}}$ with the automorphism $x \to x\xi$ of M and let τ be an automorphism of $F_{p^{\rho}}$ regarded as an automorphism of M. Then $G = M\langle \xi^2, \tau \rangle$ acts as a rank 3 group of permutations M satisfying (1).[2] A permutation group isomorphic with one of these groups G will be called a rank 3 group of Singer type. The graph $\mathcal{S}$ (for suitable choice of Δ) is isomorphic with the graph whose vertices are the elements of $F_{p^{\rho}}$, two being adjacent if and only if their difference is a nonzero square. Of course if t is a prime >2 then either $\rho = 1$ or $p = s$ and ρ is an odd prime.

In proving Theorem 1 we actually prove

THEOREM 2. *Under the hypotheses of Theorem 1, G must be of Singer type.* The remainder of this paper is devoted to the proof of this result.

3. The case in which t is a prime.

From now on G will be a rank 3 group satisfying (1) and the additional condition that is a prime. If G has degree 9 then it is of Singer type, so we assume that $t > 2$. If $n = 4t + 1$ is a prime then G is of Singer type by a theorem of Burnside [7; Th. 11.7]. Hence we assume that

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[2] The values for λ and μ follow at once from the existence of an isomorphism of $\mathcal{S}$ onto $\bar{\mathcal{S}}$, namely $x \to nx$, $x \in F_q$, n a fixed nonsquare.

(5) *t is an odd prime and $4t+1$ is not a prime.*

Choose $P \in X$ and put $H = G_p$. The H-orbits $\neq \{P\}$ are

$\Delta(P) =$ the set of all points of X adjacent to P and

$\Gamma(P) =$ the set of all points $\neq \{P\}$ of X not adjacent to P in the graph $\mathcal{S}$.

Let $S(t) \leq H$ be a t-Sylow subgroup of G. By [7; Th. 3.4′] $S(t)$ has two orbits Δ_1 and Δ_2 of length t in $\Delta(P)$ and two orbits Δ_3 and Δ_4 of length t in $\Gamma(P)$. The corresponding martix $\hat{A}$ (cf. [4; Appendix]) has the form

$$\hat{A} = \begin{pmatrix} 0 & t & t & 0 & 0 \\ 1 & x & y & z & w \\ 1 & y & & & \\ 0 & z & & * & \\ 0 & w & & & \end{pmatrix}$$

where $x + y = t - 1$ and $z + w = t$. The rows and columns of $\hat{A}$ are indexed by the $S(t)$-orbits $\Delta_0 = \{P\}$, Δ_1, Δ_2, Δ_3, Δ_4. The entry in the Δ_i-th row and Δ_j-th column is the number of edges from any given vertex in Δ_i to Δ_j. By [4] and (4),

(6) $\hat{A}^2 + \hat{A} - tI = t\hat{F}$ *where* $\hat{F}$ *is the matrix of degree* 5 *having* 1 *in every entry in the first column and all other entries* t.

An essential part of our argument is that the following possibilities for $\hat{A}$ can be ruled out at once by consideration of the (2,2)-entry of (6).

(7) *The cases* (i) $z = t$, $w = 0$, (ii) $x = t - 1$, $y = 0$, (iii) $x = 0$, $y = t - 1$ *and* (iv) $x = y = (t-1)/2$ *are impossible.*

The first application is

(8) *$\Delta(P)$ and $\Gamma(P)$ are faithful H-orbits.*

Proof. Write $T = H_{[\Delta(P)]}$. If $T \neq 1$ then $T|\Gamma(P) \neq 1$ and T is either transitive, has t orbits of length 2 or 2 orbits of length t. Take $Q \in \Delta(P)$, then $T \leq H_Q$ and the set of $k - \lambda - 1 = t$ vertices in $\Gamma(P)$ adjacent to Q is a union of T-orbits. Hence T has 2 orbits Γ_1 and Γ_2 of length t in $\Gamma(P)$, Q is joined to all t points of one of these, say Γ_1, and none of the other. But Γ_1 and Γ_2 are orbits for a t-Sylow subgroup $S(t) \leq H$ and the corresponding matrix $\hat{A}$ has the form

$$\hat{A} = \begin{pmatrix} 0 & t & t & 0 & 0 \\ 1 & x & y & t & 0 \\ 1 & y & & & \\ 0 & t & & * & \\ 0 & 0 & & & \end{pmatrix}$$

contrary to (7).

(9) *If the minimal normal subgroup M of G is regular and if $H = N_G(S(t))$ for some t-Sylow subgroup $S(t)$ of G then G is of Singer type.*

Proof. As a primitive rank 3 group G has a unique minimal normal subgroup M which is elementary abelian if it is regular [3]. Hence, assuming M is regular, we must have $4t+1 = 5^\rho$, ρ an odd prime, under our assumption (5).

We may identify M with the additive group of $\boldsymbol{F}_{5^\rho}$ and regard H as a group of automorphisms of M. Let ξ be a primitive element of $\boldsymbol{F}_{5^\rho}$, identified with the automorphism $x \to x\xi$ of M. Then $S(t) = \langle \xi^4 \rangle$ is t-Sylow subgroup of Aut M so we may assume that $S(t) \leq H$. Since $N_{\mathrm{Aut}\,M}(S(t)) = N_{\mathrm{Aut}\,M}(\langle \xi \rangle) = \langle \xi, \tau \rangle$ where τ is the automorphism $x \to x^5$ of M, and since $\langle \xi \rangle$ is transitive on $M - \{0\}$, we may assume that $H = \langle \xi^2, \tau \rangle$ if $H \neq \langle \xi^2 \rangle$, proving (9).

(10) *$H|\Delta(P)$ and $H|\Gamma(P)$ are imprimitive.*

Proof. By Wielandt's theorem [7; Th. 31.2], if $H|\Delta(P)$ is primitive then either it is doubly transitive or has rank 3 with subdegrees 1, $s(2s+1)$, $(s+1)(2s+1)$. The first case is ruled out because $\lambda \neq 0$, $2t-1$. In the second case the subdegrees of $H|\Delta(P)$ must be 1, $\lambda = t-1$, t, giving $t=1$, contrary to hypothesis.

The rest of our proof of Theorem 2 breaks up into two cases according as $H|\Delta(P)$ has imprimitive blocks of length t or not.

4. **Case A.** Let $\Delta(P) = \Delta_1 + \Delta_2$ be a decomposition of $\Delta(P)$ into imprimitive blocks of length t and let $H_0 = H_{\Delta_1} = H_{\Delta_2}$, so that $H : H_0 = 2$.

(11) $H_{[\Delta_1]} = H_{[\Delta_2]} = 1$.

Proof. If $H_{[\Delta_1]} \neq 1$ then by (8), its restriction to Δ_2 is $\neq 1$ and hence transitive. Hence $Q \in \Delta_1$ is adjacent to 0 points of Δ_2 and all $t-1$ points of $\Delta_1 - \{Q\}$. Δ_1 and Δ_2 are orbits for a t-Sylow subgroup $S(t) \leq H$ of G and the corresponding matrix $\hat{A}$ has the form

$$\hat{A} = \begin{pmatrix} 0 & t & t & 0 & 0 \\ 1 & t-1 & 0 & z & w \\ 1 & 0 & & & \\ 0 & z & & * & \\ 0 & w & & & \end{pmatrix}$$

contrary to (7).

(12) *$H_0 | \Delta_1$ is not doubly transitive.*

Proof. Suppose that $H_0 | \Delta_1$ is doubly transitive and take $Q \in \Delta_1$. If Q is adjacent to one point of Δ_1 it is adjacent to all $t-1$ points of $\Delta_1 - \{Q\}$ and none of Δ_2, which is impossible as in the proof of (11). Hence Q is adjacent to 0 points of Δ_1 and $t-1$ points of Δ_2 giving an $\hat{A}$ of the form

$$\hat{A} = \begin{pmatrix} 0 & t & t & 0 & 0 \\ 1 & 0 & t-1 & z & w \\ 1 & t-1 & & & \\ 0 & z & & * & \\ 0 & w & & & \end{pmatrix}$$

contrary to (7).

We complete the proof of Theorem 2 in case A by proving

(13) *G is of Singer type.*

Proof. By a Theorem of Burnside [7; Th. 11.7], (12) implies that $H_0 | \Delta_1$ is either regular of Frobenius, and hence $H = N_G(S(t))$ where $S(t)$ is a t-Sylow subgroup of G. Let M be a minimal normal subgroup of G. If M is regular then G is of Singer type by (9). Otherwise $M_P \neq 1$, so that either $|M_P| = 2$ and $2 \| \ |M|$, or $t \| \ |M|$. In either case M is simple. The first case is impossible since there are no such simple groups. In the second case $M : N_M(S(t)) = 1 + 4t$ and we may apply the theorem of Brauer and Reynolds [1]. The single possibility $t=5$ survives the conditions of this theorem, but in this case $|M| = 420$ or 840 which is impossible.

5. Case B. We now assume that neither $H|\Delta(P)$ nor $H|\Gamma(P)$ has imprimitive blocks of length t. Then for each $Q\in\Delta(P)$ there is a unique point $Q^P \neq Q$ in $\Delta(P)$ such that $H_Q = H_Q P$, and for each point $R\in\Gamma(P)$ there is a unique point $R^P \neq R$ in $\Gamma(P)$ such that $H_R = H_R P$. Let Ω be the set of imprimitive blocks $\{Q, Q^P\}$ for $H|\Delta(P)$. We begin the elimination of this situation by proving.

(14) $|H_{[\Omega]}| \leq 2$.

Proof. Put $V = H_{[\Omega]}$, let $S(t) \leq H$ be a t-Sylow subgroup of G and let Δ_1 and Δ_2 be the $S(t)$-orbits in $\Delta(P)$. For $S\in\Delta(P)$, $|\Delta_i \cap \{S, S^P\}| = 1$ $(i=1,2)$. Take $Q\in\Delta_1$ and suppose $V_Q = V_{Q,S}$ for some $S\in\Delta_1 - \{Q\}$. Then $V_Q = V_S$ and hence $V_Q = V_T$ for all $T\in\Delta_1$ since $S(t)$ acts transitively on the set $\{V_Q | Q\in\Delta_1\}$. Hence $V_Q = 1$ and $|V| \leq 2$.

If $V_Q \neq V_{Q,S}$ for all $S\in\Delta_1 - \{Q\}$ then Q adjacent to S implies Q adjacent to S^P, and the matrix $\hat{A}$ determined by $S(t)$ has the form

$$\begin{pmatrix} 0 & t & t & 0 & 0 \\ 1 & \frac{t-1}{2} & \frac{t-1}{2} & z & w \\ 1 & \frac{t-1}{2} & & & \\ 0 & z & & * & \\ 0 & w & & & \end{pmatrix}$$

contrary to (7).

(15) *$H|\Omega$ is doubly transitive.*

Proof. If $H|\Omega$ is not doubly transitive then $S(t) \trianglelefteq H$ by Burnside's Theorem [7; Th. 11.7] and (14). Hence the $S(t)$-orbits are imprimitive blocks for $H|\Delta(P)$, contrary to assumption.

(16) *The fixed-point set of H_Q for $Q\in\Delta(P)$ is a 5-element set, and $H_Q = G_{R,S}$ for any two distinct points R and S in it.*[3)]

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3) The proof of (16), considerably simplifying the author's original elimination of case B, was provided by Robert Liebler.

Proof. Suppose that $Q^P \in \Delta(Q)$. Then H_Q has no orbits of length 1 in $\Delta(P) \cap \Gamma(Q)$, and since the nontrivial orbits of H_Q in $\Delta(P)$ have length divisilbe by $\frac{t-1}{2}$ by (15) and since $|\Delta(P) \cap \Gamma(Q)| = t$, we find that $t = 3$, contrary to (5). Hence $Q^P \in \Gamma(Q)$.

Certainly $H_Q = G_{P,Q}$ fixes every point of the set $B = \{P, Q, Q^P, P^Q, P^{QP}\}$, and for R, S distinct points of this set, $G_{P,Q} \leq G_{R,S}$. But for any two distinct points U, V in X, $G : G_{U,V} = (4t+1)2t$. Hence $G_{P,Q} = G_{R,S}$ and we see that B is the full set of fixed points of $G_{P,Q}$ and $|B| = 5$.

(17) *For* $Q \in \Delta(P)$ *and* $R = P^Q$, $H_{\{Q,Q^P\}} = H_{\{R,R^P\}}$.

Proof. The number of 5-element subsets $B = \{P, Q, Q^P, R, R^P\}$, $R = P^Q$, is $\frac{(4t+1)t}{5}$, since any two distinct points lie on exactly one so that each point lies on exactly t. Hence $H_B : H_Q = 2$. But $H_{\{Q,Q^P\}} \leq H_B$ so $H_{\{Q,Q^P\}} = H_B$. Similarly $H_{\{R,R^P\}} = H_B$.

We now complete the proof of Theorem 2 by proving

(18) *Case B is impossible.*

Proof. We assume first that $H_{\{Q,Q^P\}}$ is transitive on $\Delta(P) - \{Q, Q^P\}$. Since $H_{\{Q,Q^P\}}$ fixes the union of $\Delta(Q) \cap \Delta(P)$ and $\Delta(Q^P) \cap \Delta(P)$, these two sets must be disjoint. Put $R = P^Q$, then $H_{\{Q,Q^P\}} = H_{\{R,R^P\}}$ is transitive on $\Gamma(P) - \{R, R^P\}$ and fixes the union of $\Delta(Q) \cap \Gamma(P)$ and $\Delta(Q^P) \cap \Gamma(P)$ so that these two sets must be disjoint. Hence $\Delta(Q) \cap \Delta(Q^P) = \{P\}$, giving $t=1$, a contradiction.

We are left with the case in which $H_{\{Q,Q^P\}}$ has two orbits of length $t-1$ in $\Delta(P) - \{Q, Q^P\}$. In this case we conclude from the fact that $H_{\{Q,Q^P\}}$ fixes the union of $\Delta(Q) \cap \Delta(P)$ and $\Delta(Q^P) \cap \Delta(P)$ that

$$(*) \quad \Delta(Q) \cap \Delta(P) = \Delta(Q^P) \cap \Delta(P).$$

Let Δ_1 and Δ_2 be the $S(t)$-orbits in $\Delta(P)$, where $S(t)$ is a t-Sylow subgroup of G, $S(t) \leq H$, with $Q \in \Delta_1$ so that $Q^P \in \Delta_2$. From $(*)$ we see that the number of edges from Q to Δ_i is equal to the number from Q^P to Δ_i $(i = 1, 2)$. Hence $\hat{A}$ determined by $S(t)$ has the form

$$\begin{pmatrix} 0 & t & t & 0 & 0 \\ 1 & x & y & z & w \\ 1 & x & y & & \\ 0 & & & & \\ & & & * & \\ 0 & & & & \end{pmatrix}$$

But then $x = y = \dfrac{t-1}{2}$, contrary to (7).

References

[1] R. Brauer and W.F. Reynolds: On a problem of E. Artin. Ann. Math. **68** (1958), 713–720.
[2] D.G. Higman: Finite permutation groups of rank 3. Math. Z. **86** (1964), 145–156.
[3] ——————: Primitive rank 3 groups with a prime subdegree. Math. Z. **91** (1966), 70–86.
[4] ——————: Intersection matrices for finite permutation groups. J. Alg. **6** (1967), 22–42.
[5] C.C. Sims Graphs and finite permutation groups. Math. Z. **95** (1967), 76–86.
[6] ————: Graphs with rank 3 automorphism groups. J. Comb. Theory (to appear).
[7] H. Wielandt: Finite permutation groups. New York: Academic Press 1964.

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