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INVARIANTS OF CERTAIN GROUPS I[1]

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Let G be a group and let k be a field. *A* k-representation ρ of G is a homomorphism of G into the group of non-singular linear transformations of some finite-dimensional vector space V over k. Let K be the field of fractions of the symmetric algebra $S(V)$ of V, then G acts naturally on K as k-automorphisms. There is a natural inclusion map $V \to K$, so we view V as a k-subvector space of K. Let $v_1, v_2, \cdots, v_n$ be a basis for V, then K is generated by $v_1, v_2, \cdots, v_n$ over k as a field and these are algebraically independent over k, that is, K is a rational field over k with the transcendence degree n. All elements of K fixed by G form a subfield of K. We denote this subfield by K^G.

We say that ρ *has the property* $[R]$ *if* K^G *is a rational field over* k.

Kuniyoshi proved that if G is a finite p-group and if k is a field of characteristic p, the regular representation has the property $[R]$ ([3]). Gaschütz generalized this result to an arbitrary representations ([2]). We shall give other generalizations of their results.

Let G be a group and let ρ be a k-representation of G. Let V be the underlying space of this representation. ρ is called triangularizable if there exisrs a G-invariant flag[3] in V.

Followings are examples of triangularizable representations:

(1) G is a finite commutative group of exponent m and k is a field whose characteristic does not divide m and which contains a primitive m-th root of unity. Then every k-representation of G is triangulariazble.

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3) *A* flag F in V is a sequence of subspaces of V F: $V=V_n \supset V_{n-2} \supset \cdots V_1 \supset V_0=(0)$ such that $\dim V_i = i$ ($n=\dim V$). F is G-invariant if $\rho(g)(V_i) \subset V_i$ for all $g \in G$ and all i.

(2) G is a finite p-group where p is a prime number and k is an arbirtary field of characteristic p. Then every k-representation of G is triangulariazble.

Since there is no adequate reference, we give a sketch of a proof. Let V be a representation space of G. It suffices to show that there exists a non-zero G-invariant element in V. Since G is a p-group, there exists an element g of order p in the center of G. It is immediate that $(\rho(g))-1)^p=0$. Therefore, there exists an integer i $(0\leq i<p)$ such that $V'=(\rho(g)-1)^iV\neq 0$ and $(\rho(g))-1)V'=(\rho(g)-1)^{i+1}V=0$. An element in V' is G-invariant. Let V_0 be the subspace consisting of all G-invariant elements in V_0. Since g is in the center of G, $G/\langle g\rangle$ acts on V_0 naturally. By mathematical induction on the order of G, V_0 has a non-zero $G/\langle g\rangle$-invariant (hence, G-invariant) element.

(3) (Lie-Kolchin) G is a connected solvable algebraic group over an algebraically closed field. Then any rational representation of G is triangularizable. ([1], Theorem 10.4).

(4) G is a connected solvabletopological group. Then every continuous representation on a finite dimensional vector space over the complex number field is triangularizable. ([6], Theorem 5.1*, Lemma 5.11).

THEOREM 1. *Let G be a group and let k be a field. Then every triangularizable k-representation of G has the property* (R).

By the triangularizability, the problem reduces by induction to proving

LEMMA. *Let G be a group acting on a field K. If G acts also on a polynomial ring of one variable $K[t]$ in the following way:*

$$g(t)=\lambda(g)t+\mu(g),\quad g\in G$$

where $\lambda(g)$ $(\neq 0)$ and $\mu(g)$ belong to K, then there exists an element x in $K[t]$ such that $K(K(t)^G)=K(x)$.

Proof. First of all we show that the field of fractions K' of $K[t]^G$ is $K(t)^G$. Let $F/L\in K(t)^G$, F, $L\in K[t]$. We prove that F/L belongs to K' by the induction on $\deg(F)+\deg(L)$ where deg. means the degree in t. If $\deg(F)$ or $\deg(L)$ is zero, there is nothing to prove. Suppose that $\deg(F)$ and $\deg(L)$ are positive and that F and L are relatively prime. Since $K[t]$ is a unique factorization domain, we have

$$g(F) = \chi(g)F, \quad g(L) = \chi(g)L,$$

where $\chi(g)$ is a character of G with values in K^*. We may assume $\deg(F) \geqq \deg(L)$. Dividing F by L we have

$$F = S \cdot L + R \quad \deg(R) < \deg(L).$$

applying g in G, we get

$$\chi(g)F = \chi(g)\,(g(S))L + g(R).$$

Since $\deg(F) = \deg(g(F))$ and $\deg(L) = \deg(g(L))$, we see that $g(S) = S$ and $g(R) = \chi(g)R$ by the uniqueness of division. By the induction assumption, $R/L \in K'$, hence F/L belongs to K'.

Now this observation shows us that if $K[t]^G \subset K$, then $K(t)^G \subset K$. If $K[t]^G \subset K$, there is nothing to prove. If $K[t]^G \not\subset K$, then choose $x \in K[t]^G - K$ such that $\deg(x)$ is minimal. Then by an argument similar to that in the above observation, we can show that an element in $K[t]^G$ is a polynomial in x with coefficients in K^G, that is, $K[t]^G = K^G[x]$. q.e.d.

Remark 1. This lemma is a generalization of Hilbert's Theorem 90. In fact, let G be a *finite* group of field automorphisms of K and let $\mu(g)$ (resp. $\lambda(g)$) be an additive (resp. multiplicative) cocycle of G with values in K (resp. K^*). Then by defining $g(t) = t + \mu(g)$ (resp. $g(t) = \lambda(g)t$) G acts on the polynomial ring $K[t]$. It is easy to see that $K(K(t)^G) = K(t)$ by the fundamental theorem of Galois theory. By Lemma there is an element x in $K[t]^G$ such that $K(t) = K(x)$. x must be linear in t, say $at + b$, $a, b \in K$. Now $at + b = g(a)g(t) + g(b)$, for all g in G, so $at + b = g(a)\,(t + \mu(g)) + g(b)$ (resp. $at + b = g(a)\lambda(g)t + g(b)$). Hence $\mu(g) = b/a - g(b/a)$ (resp. $\lambda(g) = a\,g(a)^{-1}$). This means $H^1(G, K) = (0)$ (resp. $H^1(G, K^*) = (1)$).

Remark 2. One might be tempted to formulate the lemma in the following way;

Let K_1 be a subfield of a rational field $K(t)$ of one variable (K_1 not necessarily containing K). Then there is an element x in K_1 such that $K(K_1) = K(x)$.

Unfortunately this is not true in general.

Let $K = K(s)$ be a rational field of one variable over a field k. Let $K_1 = k(t^2, t^3 + s)$, where t is an indeterminate. Then this is a counter example.

Proof. We note that $k(s)(t^2, t^3 + s) = k(s, t)$. Suppose that we find an element x in K_1 such that $k(s)(K_1) = k(s)(x)$. Then.

$$x = \frac{\alpha t + \beta}{\gamma t + \delta} \quad \alpha\delta - \beta\gamma \neq 0, \quad \alpha, \beta, \gamma, \delta \in k[s].$$

We may assume that $\alpha \neq 0$. Put $u = t^2$ and $v = t^3 + s$. We can write $F/L = (\alpha t + \beta)/(\gamma t + \delta)$ where F and L belong to $K[u,v]$. Let F_0, L_0 be the constant terms in F, L as polynomials of t then since

$$(\gamma t + \delta)F(u,v) \equiv (\alpha t + \beta)L(u,v) \bmod (t^2),$$

we get that $\delta F_0 = \beta L_0$ and $\gamma F_0 = \alpha L_0$. Therefore $(\alpha\delta - \beta\gamma)F_0 = \alpha(\delta F_0) - \beta(\gamma F_0) = 0$. This is a contradiction, if $F_0 \neq 0$.

If $F_0 = 0$, then $F(u,v) = F'(u,v)u^m$ where F' has non-zero constant term. In fact, write.

$$F(u,v) = F'(u,v)u + F''(v), \quad F' \in K[u,v], \quad F'' \in k[v].$$

Since F has no non-zero cnostant term as a polynomial in t, $0 = F(0,s) = F''(s)$, hence $F'' \equiv 0$. Now by this observation we may assume that $F_0 \neq 0$.

q.e.d.

Remark 3. Let V be an underlying space of a k-representation of a *finite* group G. Suppose that V has a *faithful* sub-G-module W which has the property (R), then V has the property (R).

Proof. Let $w_1, w_2, \cdots, w_m$ be a basis for W. We may identify the symmetric algebra $S(W)$ with the polynomial ring $k[w_1, w_2, \cdots, w_m]$. Let K be the field of fractions of $S(W)$. Let $v_1, v_2, \cdots, v_n$ be vectors in V such that they together with $w_1, w_2, \cdots, w_m$ form a basis for V. Let K' be the field of fractions of $S(V) = k[w_1, \cdots, w_m, v_1, \cdots, v_n]$. Then we show that there exist n elements $x_1, x_2, \cdots, x_n$ in K'^G such that $K(K'^G) = K(x_1, x_2, \cdots, x_n)$ $(=K')$. In fact, the action of an element g in G on K' is

$$g \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ 1 \end{pmatrix} = \begin{pmatrix} & & a_1(g) \\ & A_0(g) & \vdots \\ & & a_n(g) \\ 0 \cdots\cdots 0 & & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \\ v_n \\ 1 \end{pmatrix}$$

where $A_0(g) \in GL(n,k) \subset GL(n,K)$ and $a_i(g) \in K$. Let H be the subgroup of

$GL(n+1,K)$ consisting of elements of the type $\begin{bmatrix} A_0 & * \\ 0 & 1 \end{bmatrix}$. We let G act on $GL(n+1,K)$ coefficientwise, then H is G-stable. If we write $(v)={}^t(v_1,\cdots,v_n,1)$ and $A(g)=\begin{bmatrix} A_0(g) & * \\ 0 & 1 \end{bmatrix}$, then $(hg)(v)=A(hg)(v)=h(g(v))={}^hA(g)A(h)(v)$. Therefore, $g\to A(g)^{-1}$ is a cocycle of G with values in H. There is an exact sequence

$$(1)\to \underbrace{K\times\cdots\times K}_{n\text{-tuples}}\to H\to GL(n,K)\to(1)$$

Since $H^1(G,K)$ and $H'(G,GL(h,K))$ are trivial (by assumption, G is finite and the action of G on K is faithful), $H^1(G,H)=(1)$ ([5], p. 133). This means that there exists $B\in H$ such that $A(g)={}^gB\cdot B^{-1}$. If we set $(x)={}^t(x_1,\cdots,x_n,1)=B^{-1}(v)$, then $g(x)={}^gB^{-1}\cdot g(v)={}^gB^{-1}A(g)(v)=B^{-1}(v)=(x)$. x_i's satisfy the property. q.e.d.

THEOREM 2. *A two dimensional representation has the property* (R). *A three dimensional representation has the property* (R) *if* k *is algebraically closed.*

This theorem is essentially due to Noether ([4], § 2)

Proof. Let V be a representation space of a group G and let $x_1,\cdots,x_n$ be a basis of V.

$$K=k(V)=k(x_2x_1^{-1},\cdots,x_nx_1^{-1})(x_1).$$

Since $K_1=k(x_2x_1^{-1},\cdots,x_nx_1^{-1})$ is G-stable and $g(x_1)=(g(x_1)x_1^{-1})x_1$, there exists an element $z\in K^G$ such that $K^G=K_1^G(z)$ by Lemma. If $\dim V=2$, the theorem follows from Lüroth's theorem and if $\dim V=3$, the theorem follows from Zariski-Castelnuovo's theorem. q.e.d.

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