

## EVANS-KURAMOCHI EXHAUSTION FUNCTIONS ON NON-ALGEBROID RIEMANN SURFACES

J.E. SKEATH

1. Let  $\mathfrak{X}$  denote a non-compact parabolic Riemann surface, and let  $D \subset \mathfrak{X}$  be compact and such that each frontier point of  $D$  is contained in a continuum that is also contained in  $D$ . Under these conditions, Kuramochi [2] (see also [3]) has established the existence of a function  $u$  on  $\mathfrak{X} - D$  satisfying

- (a)  $u \geq 0$ , harmonic on  $\mathfrak{X} - D$ ,
- (b)  $u$  vanishes continuously on *fr*  $D$ , the frontier of  $D$ ,
- (c)  $u$  tends to  $+\infty$  at the ideal boundary of  $\mathfrak{X}$ .

Any such function will be called an (Evans-Kuramochi) *exhaustion function* on  $\mathfrak{X} - D$ . An exhaustion function  $u$  on  $\mathfrak{X} - D$  will be said to satisfy the *k-condition* if and only if there exists an integer  $k$  such that the number of components of the level loci  $\{u = s\}$  is bounded above by  $k$  independent of  $s$ ,  $0 < s < +\infty$ .

In [4] it was shown that an extension of the Denjoy-Carleman-Ahlfors theorem in subharmonic form can be obtained for any surface admitting an exhaustion function satisfying the *k-condition* for some  $k$ . Any  $n$ -sheeted algebroid Riemann surface over the finite plane falls under this classification. In this paper we construct a non-algebroid surface admitting an exhaustion function satisfying the *k-condition*, thus answering in the affirmative a question raised in [4]. As we shall see, the desired surface is closely related to that constructed by Heins in [1, pp. 297-299].

2. Following Heins [1], we begin by constructing a non-algebroid surface given as an explicit covering surface of the extended plane. Thus, let  $\{a_n\}_{n=0}^{\infty}$  denote a sequence of positive reals such that  $a_0 > e$  and  $\inf a_n/a_{n-1} > 1$ . Let  $\{b_{4n+2}\}_{n=0}^{\infty}$  be such that  $a_{4n+1} < b_{4n+2} < a_{4n+2}$  and let each segment

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$[a_{4n+1}, a_{4n+2}]$  be subdivided into an odd number ( $> 1 + a_{4n+2}$ ) of subsegments. Define

- $E^1$ , the finite plane less the slits  $[a_{2n}, a_{2n+1}]$  and every alternate subsegment of  $[a_{4n+1}, a_{4n+2}]$  starting with the second (all  $n$ );
- $E^2$ , the region  $E^1$  less the slits  $[-b_{4n+2}, -a_{4n+1}]$  (all  $n$ );
- $\sigma_n$ , the extended plane less the slit  $[-b_{4n+2}, -a_{4n+1}]$ .

Let  $\mathfrak{X}$  denote the Riemann surface formed by joining copies of  $E^1$  and  $E^2$  along their common slits in the usual way, identifying the upper edges of the slits of  $E^1$  with the corresponding lower edges of the slits of  $E^2$  and vice versa. The remaining free edges of  $E^2$  are identified with the opposite edges of the slits of copies of the corresponding  $\sigma_n$ .

It follows as in [1] that every non-constant meromorphic function on any end  $\Omega$  of  $\mathfrak{X}$  (see [1] for terminology) takes on all values infinitely often with the exception of at most two. Thus  $\mathfrak{X}$  is non-algebroid. We wish to establish the following

**THEOREM 1.** *Given  $\{a_n\}_{n=0}^\infty$  satisfying the above conditions, there exists  $\{b_{4n+2}\}_{n=0}^\infty$  satisfying  $a_{4n+1} < b_{4n+2} < a_{4n+2}$  and a subdivision of  $[a_{4n+1}, a_{4n+2}]$  into an odd number ( $> 1 + a_{4n+2}$ ) of subsegments (all  $n$ ) such that the surface  $\mathfrak{X}$  constructed as above from these quantities admits an exhaustion function satisfying the  $k$ -condition for  $k \leq 2$  on  $\mathfrak{X} - D$  where  $D$  denotes the set of points over  $|z| \leq 1$  in the copies of  $E^1$  and  $E^2$ .*

Before proceeding to the proof of Theorem 1, we establish the following notation. If  $b_2, \dots, b_{4m+2}$  are given such that  $a_{4n+1} < b_{4n+2} < a_{4n+2}$ ,  $0 \leq n \leq m$ , and if an odd number ( $> 1 + a_{4n+2}$ ) of subsegments subdividing  $[a_{4n+1}, a_{4n+2}]$ ,  $0 \leq n \leq m$ , are given, let

- $E_m^1$  denote the finite plane less the slits  $[a_{2n}, a_{2n+1}]$  for  $0 \leq n \leq 2m + 1$ , the slits  $[a_{4n}, a_{4n+2}]$  for  $n \geq m + 1$ , and every alternate subsegment starting with the second of  $[a_{4n+1}, a_{4n+2}]$  for  $0 \leq n \leq m$ ;
- $E_m^2$  denote the region  $E_m^1$  less the slits  $[-b_{4n+2}, -a_{4n+1}]$  for  $0 \leq n \leq m$ ;
- $\sigma_n$  as before,  $0 \leq n \leq m$ .

Let  $\mathfrak{X}_m$ ,  $m \geq 0$  denote the surface formed from the above quantities as in the construction of  $\mathfrak{X}$ . Finally, let  $\mathfrak{X}_{-1}$  denote the surface constructed by copies of  $E_{-1}^1$  and  $E_{-1}^2$  by identifying opposite edges in the usual way where  $E_{-1}^1 = E_{-1}^2 =$  the finite plane less the slits  $[a_{4n}, a_{4n+2}]$  (all  $n$ ).

Note that, since  $\inf a_n/a_{n-1} > 1$ , any such  $\mathfrak{X}_m$ ,  $m \geq -1$ , has harmonic dimension one in the sense of Heins [1]. If  $\mathfrak{X}_m$ ,  $m \geq -1$ , is given, let  $D_m$

denote the set of points in  $\mathfrak{A}_m$  lying over  $|z| \leq 1$  in the copies of  $E_m^1$  and  $E_m^2$ . Let  $p_m$  denote the point in  $\mathfrak{A}_m$  lying over  $z = e$  in the copy of  $E_m^1$ , and let  $u_m$  denote the unique exhaustion function on  $\mathfrak{A}_m - D_m$  normalized such that  $u_m(p_m) = 1$ .

We assert that Theorem 1 is a consequence of

**THEOREM 2.** *Given  $\{a_n\}_{n=0}^\infty$  satisfying the conditions of Theorem 1, there exists a sequence  $\{b_{4n+2}\}_{n=0}^\infty$  and a subdivision of  $[a_{4n+1}, a_{4n+2}]$  (all  $n$ ) satisfying the conditions of Theorem 1 such that for all  $m$ , the normalized exhaustion function  $u_m$  on  $\mathfrak{A}_m - D_m$  satisfies the following conditions.*

**I.**  $u_m(p) = A_m \log |c(p)| + B_m + H_m(p)$ , if  $p$  lies in the joining of  $E_m^1$  and  $E_m^2$  over  $|z| \geq r_m = a_{4m+3}$  where  $c$  is the natural projection map and

- a)  $A_m, B_m$  are constants,  $A_m > \frac{1}{2}$ ;
- b)  $H_m$  is harmonic,  $|H_m| < \frac{\log r}{4}$ ,  $r = \inf a_n / a_{n-1}$ ;
- c)  $|H_m(p)| < \frac{c_m}{|c(p)|} < \frac{(1 - 1/r) \log 2}{4|c(p)|}$  if  $c(p) \in [a_{4n}, a_{4n+3}]$  and  $n \geq m + 1$ .

**II.** For each  $s$ ,  $0 < s < A_m \log r_m + B_m + \frac{\log r}{4}$ , the level locus  $\{u_m = s\}$  is contained in a relatively compact open subset  $\Omega$  of  $\mathfrak{A}_m - D_m$  less the points in  $E_m^1 \cup E_m^2$  over  $|z| \geq a_{4m+4}$  where  $\Omega$  is either

- a) a region having genus one and connectivity two,
- b) a plane region having connectivity three, or
- c) the union of two disjoint doubly connected plane regions.

**III.**  $u_m$  satisfies the  $k$ -condition for  $k \leq 2$ .

We remark that conditions I and II of Theorem 2 imply condition III. In fact, if there exists  $s < A_m \log r_m + B_m + \frac{\log r}{4}$  such that  $\{u_m = s\}$  has three or more components, then by condition II some subset of these components forms the boundary of a relatively compact subregion of  $\mathfrak{A}_m - D_m$ , and thus  $u_m$  must be identically constant. If  $s \geq A_m \log r_m + B_m + \frac{\log r}{4}$ , then by condition I  $s > \max u_m$  in the joining of  $E_m^1$  and  $E_m^2$  over  $|z| = r_m$ . Therefore,  $\{u_m = s\}$  lies in the joining of  $E_m^1$  and  $E_m^2$  over  $|z| > r_m$ , and the representation given by condition I is valid. Now if  $p_1$  and  $p_2$  are points in the joining of  $E_m^1$  and  $E_m^2$  over  $|z| > r_m$  such that  $|c(p_2)/c(p_1)| \geq r$ , then

by condition I we have

$$\begin{aligned} u_m(p_2) - u_m(p_1) &= A_m \log |c(p_2)/c(p_1)| + H_m(p_2) - H_m(p_1) \\ &\geq A_m \log |c(p_2)/c(p_1)| - \frac{1}{2} \log r \\ &> 0. \end{aligned}$$

It follows that  $\{u_m = s\}$  lies over an annular region of the form  $R_1 < |z| < R_2$  with  $R_2/R_1 < r$ . Since  $r = \inf a_n/a_{n-1}$ ,  $\{u_m = s\}$  is contained in a set  $\Omega$  satisfying either condition IIb) or condition IIc). Thus  $\{u_m = s\}$  consists of at most two components. Condition III is established.

*Proof that Theorem 2 implies Theorem 1:* Let  $\mathfrak{X}$  denote the Riemann surface constructed as in Theorem 1 from the quantities given in Theorem 2. Observe that, for all  $m$ ,  $u_m$  on  $\mathfrak{X}_m - D_m$  can be considered as a function defined on  $\mathfrak{X} - D$  less the points in the joining of  $E^1$  and  $E^2$  over  $[a_{4m+1}, a_{4m+2}]$  for  $n \geq m+1$ , the points in the copy of  $E^2$  over  $[-b_{4m+2}, -a_{4m+1}]$  for  $n \geq m+1$ , and the points of  $\sigma_n$  for  $n \geq m+1$ . In particular, for any compact subset  $K$  of  $\mathfrak{X} - D$ ,  $u_m$  is defined on  $K$  if  $m$  is sufficiently large. Moreover,  $u_m > 0$  and  $u_m(p^0) = 1$  (all  $m$ ) where  $p^0$  denotes the point over  $z = e$  in  $E^1$ . It follows that  $\{u_m | m \geq -1\}$  is normal on  $\mathfrak{X} - D$ , and thus there exists a subsequence almost uniformly convergent to a harmonic function  $u > 0$  on  $\mathfrak{X} - D$ ,  $u(p^0) = 1$ . It is easily seen that  $u$  vanishes continuously at the frontier of  $D$ . Since  $\mathfrak{X}$  has harmonic dimension one and since  $\mathfrak{X}$  has at least one exhaustion function on  $\mathfrak{X} - D$ , it follows that there is (up to constant multiples) exactly one such exhaustion function, and that function must be  $u$ . (It follows from this that the original sequence  $\{u_m\}_{m=-1}^\infty$  converges almost uniformly to  $u$  although this result will not be needed in what follows.)

If  $u$  does not satisfy the  $k$ -condition for  $k \leq 2$ , then there exists an  $s > 0$ ,  $s$  not a critical level of  $u$ , such that  $\{u = s\}$  consists of  $j (> 2)$  components. Take  $\varepsilon > 0$  such that  $A = \{s - \varepsilon \leq u \leq s + \varepsilon\}$  contains no critical levels of  $u$ . Then  $A$  is compact and consists of  $j$  components, each conformally equivalent to an annulus with  $u = s + \varepsilon$  on one boundary component of each such annulus and  $u = s - \varepsilon$  on the other boundary component (cf. [4]). Moreover, for  $m$  sufficiently large,  $u_m$  is defined on  $A$ , and we can assume  $|u - u_m| < \varepsilon/2$  on  $A$ . But then  $\{u_m = s\}$  has at least one component in each of the  $j (> 2)$  components of  $A$ , a contradiction.

Before turning to the proof of Theorem 2, it will be convenient to have the following two lemmas at our disposal.

LEMMA 1. *If  $\mathfrak{X}_m$  is given,  $m \geq -1$ , and if  $\varphi_m$  denotes the indirectly conformal map from  $\mathfrak{X}_m$  onto itself determined by  $\varphi_m(p) = \bar{p}$ ,  $\bar{p}$  as defined below, then  $u_m \circ \varphi_m = u_m$ . Moreover, in the joining of  $E_m^1$  and  $E_m^2$  over the slits on the positive real axis, the two determinations of  $u_m$  agree.*

DEFINITION: If a point  $p$  in  $\mathfrak{X}_m$  corresponds to a point  $z$  in  $E_m^1$ ,  $E_m^2$  or  $\sigma_n$ ,  $0 \leq n \leq m$ , respectively, then  $\bar{p}$  denotes the point in  $\mathfrak{X}_m$  corresponding to  $\bar{z}$  in  $E_m^1$ ,  $E_m^2$  or  $\sigma_n$ ,  $0 \leq n \leq m$ , respectively. The obvious modifications are made for points of  $\mathfrak{X}_m$  over slits, e.g., if  $p_1$ ,  $p_2$  denote the two points of  $\mathfrak{X}_m$  in the joining of  $E_m^1$  and  $E_m^2$  over a point in  $(a_{4n}, a_{4n+1})$ , then  $\bar{p}_1 = p_2$  and  $\bar{p}_2 = p_1$ .

*Proof of Lemma 1:* Since  $\varphi_m$  is indirectly conformal,  $u_m \circ \varphi_m$  is harmonic. Moreover,  $u_m \circ \varphi_m(p_m) = 1$ . Since  $\mathfrak{X}_m$  has harmonic dimension one, it follows that  $u_m \circ \varphi_m = u_m$ . The remaining assertion of the lemma is an immediate consequence of this property.

LEMMA 2. *Let  $h$ , harmonic on  $|z| > R$ , be such that  $|h| < M$ ,  $\lim_{z \rightarrow \infty} h(z) = 0$ . Then  $|h(z)| \leq \frac{2MR}{|z| + R}$  for  $|z| > R$ .*

*Proof of Lemma 2:* The proof follows by a direct application of Harnack's inequalities to the functions  $M - h\left(\frac{1}{z}\right)$  and  $M + h\left(\frac{1}{z}\right)$  for  $|z| < \frac{1}{R}$ .

*Proof of Theorem 2:* The proof is by induction on  $m$ . The case  $m = -1$  is trivial. Here the normalized exhaustion function  $u_{-1}$  on  $\mathfrak{X}_{-1} - D_{-1}$  is given by  $u_{-1}(p) = \log|c(p)|$  where  $c$  is the natural projection. Assume therefore that  $\mathfrak{X}_{m-1}(m \geq 0)$  is given such that  $u_{m-1}$  satisfies conditions I, II and III of Theorem 2 with  $m - 1$  replacing  $m$ . With this assumption, we show there exists  $b_{4m+2}$ ,  $a_{4m+1} < b_{4m+2} < a_{4m+2}$  and a subdivision of  $[a_{4m+1}, a_{4m+2}]$  into an odd number ( $> 1 + a_{4m+2}$ ) of subsegments such that  $u_m$  on  $\mathfrak{X}_m - D_m$  satisfies the conditions of Theorem 2.

Thus, let  $b_{4m+2}(n) = \left(1 - \frac{1}{n}\right)a_{4m+1} + \left(\frac{1}{n}\right)a_{4m+2}$ ,  $n = 1, 2, \dots$ . Let  $\nu$ , an integer, be such that  $a_{4m+2}/2 < \nu < a_{4m+2}$ , and let  $\delta = (a_{4m+2} - a_{4m+1})/\nu$ . Note that  $2\nu + 1 > 1 + a_{4m+2}$  and that  $\delta > 1 - \frac{1}{r}$ ,  $r = \inf a_n/a_{n-1}$ . Let  $\alpha_j = a_{4m+1} + j\delta$ ,  $j = 0, \dots, \nu$ , and introduce  $\alpha_j(n)$ ,  $j = 0, \dots, \nu$ , and  $n = 1, 2, \dots$ , such that

- (i)  $\alpha_0 < \alpha_0(n) < \alpha_1(n) < \alpha_1$ , all  $n$ ;
- (ii)  $\alpha_{j-1} < \alpha_j(n) < \alpha_j$ ,  $j = 2, \dots, \nu$  and all  $n$ ;
- (iii)  $\lim_{n \rightarrow \infty} \alpha_j(n) = \alpha_j$ ,  $j = 0, \dots, \nu$ .

For each  $n$ , the points  $\alpha_0, \alpha_0(n), \alpha_1(n), \alpha_1, \dots, \alpha_\nu(n), \alpha_\nu$  subdivide  $[a_{4m+1}, a_{4m+2}]$  into  $2\nu + 1$  subsegments. Let  $\mathfrak{X}_m^n$  denote the Riemann surface constructed in the usual manner with this choice of subintervals for  $[a_{4m+1}, a_{4m+2}]$ , the subinterval  $[-b_{4m+2}(n), -a_{4m+1}]$ , a copy of the extended plane slit along  $[-b_{4m+2}(u), -a_{4m+1}]$ , and the information given from  $\mathfrak{X}_{m-1}$ . Let  $u_m^n$  denote the normalized exhaustion function on  $\mathfrak{X}_m^n - D_m^n$ .

The functions  $u_m^n$ ,  $n = 1, 2, \dots$ , can be considered as defined on  $\mathfrak{X}_{m-1} - D_{m-1}$  less the appropriate subsets (dependent on  $n$ ). Since  $u_m^n > 0$  and  $u_m^n(p_{m-1}) = 1$  (all  $n$ ), the family  $\{u_m^n\}_{n=1}^\infty$  is normal on  $\mathfrak{X}_{m-1} - D_{m-1}$  less the point of  $E_{m-1}^2$  over  $-a_{4m+1}$  and less the points in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $\alpha_j$ ,  $j = 0, \dots, \nu$ . Hence there exists a subsequence almost uniformly convergent there to a positive harmonic function  $u$  such that  $u(p_{m-1}) = 1$ . Moreover, it is easily seen that  $u$  is bounded in some neighborhood of each of the points deleted from  $\mathfrak{X}_{m-1} - D_{m-1}$ . Thus  $u$  can be extended to a function harmonic on  $\mathfrak{X}_{m-1} - D_{m-1}$ . Also,  $u$  vanishes continuously at  $fr D_{m-1}$ . Since  $\mathfrak{X}_{m-1}$  has harmonic dimension one, it follows that  $u = u_{m-1}$ .

Note that  $r_{m-1} < a_{4m}$ , and thus the representation of  $u_{m-1}$  given by condition I of Theorem 2 is valid in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $[a_{4m}, a_{4m+3}]$ . By Lemma 1, the two determinations of  $u_{m-1}$  agree in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $[a_{4m}, a_{4m+3}]$ , and therefore the same is true for  $H_{m-1}$ . Thus, if  $\alpha \in [a_{4m}, a_{4m+3}]$ , let  $u_{m-1}(\alpha)$ ,  $H_{m-1}(\alpha)$ , respectively, denote the common value of the two determinations of  $u_{m-1}$ ,  $H_{m-1}$ , respectively,

over  $\alpha$ . If  $\alpha, \beta \in [a_{4m}, a_{4m+3}]$  and  $\beta > \alpha + \left(1 - \frac{1}{r}\right)$ , then

$$\begin{aligned}
 (1) \quad u_{m-1}(\beta) - u_{m-1}(\alpha) &= A_{m-1} \log(\beta/\alpha) + H_{m-1}(\beta) - H_{m-1}(\alpha) \\
 &> A_{m-1} \log\left(1 + \frac{1 - \frac{1}{r}}{\alpha}\right) - \frac{\left(1 - \frac{1}{r}\right)\log 2}{2\alpha} \\
 &> M > 0
 \end{aligned}$$

since  $\frac{1}{2} < A_{m-1}$ ,  $1 < r$ , and  $1 < a_{4m} \leq \alpha \leq a_{4m+3}$ . In particular,  $\min\{|u_{m-1}(\alpha_j)|$

$-u_{m-1}(\alpha_k) : k \neq j \} > M > 0$ . We can choose disjoint closed disks  $\Delta_j$  in  $a_{4m+1} \leq |z| \leq a_{4m+2}$  such that  $\alpha_j \in \Delta_j$ ,  $j = 0, \dots, \nu$  and such that, if  $K_j$  denotes the points in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $\Delta_j$ , then

$$(2) \quad \max \{ |u_{m-1}(p) - u_{m-1}(\alpha_j)| : p \in K_j \} < \frac{1}{4} M, \quad j = 0, \dots, \nu.$$

Moreover, we can assume  $\alpha_j(n) \in \Delta_j$ , all  $j$ , all  $n$ . Let  $\Delta$  denote a closed disk in  $E_{m-1}^2$  containing the point of  $E_{m-1}^2$  over  $-a_{4m+1}$ ,  $\Delta$  lying over  $a_{4m+1} < |z| < a_{4m+3}$ . If  $\varepsilon_m > 0$  is given, there exists a  $k$  such that  $u_m^k$  is defined on  $\mathfrak{X}_{m-1} - D_{m-1} - \Delta - \bigcup_{j=1}^{\nu} K_j$  and such that  $|u_m^k - u_{m-1}| < \varepsilon_m$  at the points in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| = r \cdot r_m (r_m = a_{4m+3})$  union the points of the frontier of  $\Delta \cup (\bigcup_{j=1}^{\nu} K_j)$ . We will show that if  $\varepsilon_m$  is taken sufficiently small and  $k$  is chosen as above, then  $u_m^k (= u_m)$  on  $\mathfrak{X}_m^k - D_m^k (= \mathfrak{X}_m - D_m)$  satisfies the conditions of Theorem 2. Henceforth, write  $u_m^k = u_m$ ,  $\mathfrak{X}_m^k = \mathfrak{X}_m$  and  $D_m^k = D_m$ . We assume, in particular, that

$$(3) \quad 0 < \varepsilon_m < \frac{M}{4};$$

$$(4) \quad A_{m-1} \log a_{4m} + B_{m-1} + \frac{\log r}{4} + \varepsilon_m < A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4};$$

$$(5) \quad A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4} < A_{m-1} \log a_{4m+3} + B_{m-1} - \frac{\log r}{4} - \varepsilon_m.$$

Since  $A_{m-1} > \frac{1}{2}$  and  $\inf a_n/a_{n-1} = r$ , the conditions in (4) and (5) can be met. Further restrictions on  $\varepsilon_m$  will be imposed later.

By the maximum principle, we have  $|u_m - u_{m-1}| < \varepsilon_m$  in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| = r_m$ . Let  $h_m$  denote the unique bounded harmonic function defined in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| > r_m$  with boundary values  $u_m - A_{m-1} \log r_m - B_{m-1} - H_{m-1}$ . Note that  $|h_m| < \varepsilon_m$ , and that  $h_m$  has a limit at the ideal boundary of  $\mathfrak{X}_{m-1}$  since  $\mathfrak{X}_{m-1}$  has harmonic dimension one (cf. [1]). Thus we can write  $h_m = f_m + b_m$  where  $b_m$  is constant,  $|b_m| < \varepsilon_m$ ,  $f_m$  is harmonic,  $|f_m| < 2\varepsilon_m$  and  $f_m$  tends to 0 at the ideal boundary of  $\mathfrak{X}_{m-1}$ . The function

$$u_m(p) - A_{m-1} \log r_m - (B_{m-1} + b_m) - (H_{m-1}(p) + f_m(p))$$

is positive harmonic and tends to 0 at the points in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| = r_m$ , and hence is a multiple of  $\log |c(p)/r_m|$  where  $c$  is the natural projection map. We have

$$\begin{aligned} u_m(p) &= A_m \log |c(p)| - (A_m - A_{m-1}) \log r_m + (B_{m-1} + b_m) + (H_{m-1}(p) + f_m(p)) \\ &= A_m \log |c(p)| + B_m + H_m(p), \quad \text{where} \end{aligned}$$

$$(6) \quad \begin{aligned} B_m &= -(A_m - A_{m-1}) \log r_m + (B_{m-1} + b_m) \quad \text{and} \\ H_m &= H_{m-1}(p) + f_m(p), \quad |c(p)| > r_m. \end{aligned}$$

Since  $|u_m - u_{m-1}| < \varepsilon_m$  in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| = r \cdot r_m$ , we have

$$|(A_m - A_{m-1}) \log r + b_m + f_m| < \varepsilon_m$$

or

$$(7) \quad |(A_m - A_{m-1}) \log r| < 2\varepsilon_m.$$

Moreover, if  $F_m$  denotes the function defined on  $|z| > r_m$  as the sum of the two determinations of  $f_m$ , then  $F_m$  is harmonic,  $|F_m| < 4\varepsilon_m$  and  $\lim_{z \rightarrow \infty} F_m = 0$ .

It follows by Lemma 2 that

$$|F_m(z)| < \frac{8\varepsilon_m r_m}{|z| + r_m} \quad \text{for } |z| > r_m.$$

However, by Lemma 1, the two determinations of  $f_m$  agree in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $[a_{4n}, a_{4n+3}]$ ,  $n \geq m+1$ . Thus

$$|f_m(z)| < \frac{4\varepsilon_m r_m}{|z| + r_m} \quad \text{for } z \in [a_{4n}, a_{4n+3}], \quad n \geq m+1.$$

It now follows directly that if  $\varepsilon_m > 0$  is chosen sufficiently small, then  $u_m$ ,  $A_m$  and  $H_m$  satisfy condition I of Theorem 2.

It remains to establish condition II for  $u_m$ . Thus, let  $s$  be such that  $0 < s < A_m \log r_m + B_m + \frac{\log r}{4}$ .

$$\text{Case 1.} \quad 0 < s < A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4}.$$

Case 2.

$$A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4} \leq s \leq A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4}.$$

Case 3.

$$A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4} < s < A_m \log r_m + B_m + \frac{\log r}{4}.$$



Note by condition I that, if  $\varepsilon_m$  is sufficiently small, then  $u_{m-1} > A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4} + \varepsilon_m$  in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| = a_{4m+1}$ . Thus if  $s$  is in Case 1 and  $\varepsilon_m$  is sufficiently small, then  $\{s - \varepsilon_m < u_{m-1} < s + \varepsilon_m\}$  is contained in  $\mathfrak{X}_{m-1} - D_{m-1}$  less the points in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| \geq a_{4m+1}$ , a subset of  $\mathfrak{X}_{m-1} - D_{m-1}$  which can also be considered as a subset of  $\mathfrak{X}_m - D_m$ . Moreover, for such  $s$  we have  $\{u_m = s\} \subset \{s - \varepsilon_m < u_{m-1} < s + \varepsilon_m\}$ . Thus, the facts given in conditions I and II for  $u_{m-1}$  can be used to assure, if  $\varepsilon_m$  is sufficiently small, that  $\{u_m = s\}$ ,  $s$  in Case 1, satisfies condition II. We omit the details of the proof and turn to Case 3 which, as we shall see, is similar to Case 1. If  $\varepsilon_m$  is sufficiently small, note that by condition I we have  $u_{m-1} < A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4} - \varepsilon_m$  in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| = a_{4m+2}$ , and note that by condition I, (6) and (7) we have  $u_{m-1} > A_m \log r_m + B_m + \frac{\log r}{4} + \varepsilon_m$  in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $|z| = r \cdot r_m (< a_{4m+4})$ . For  $s$  in Case 3, we have then that  $\{s - \varepsilon_m < u_{m-1} < s + \varepsilon_m\}$  lies in the joining of  $E_{m-1}^1$  and  $E_{m-1}^2$  over  $a_{4m+2} < |z| < a_{4m+4}$ , a subset of  $\mathfrak{X}_{m-1} - D_{m-1}$  which can also be considered as a subset of  $\mathfrak{X}_m - D_m$  and which, in addition, is a region satisfying condition II(b). Since

$$\{u_m = s\} \subset \{s - \varepsilon_m < u_{m-1} < s + \varepsilon_m\}$$

for  $s$  in Case 3, it follows by the above that condition II is satisfied for such  $s$ . We turn then to the more interesting Case 2. Note by (4) and (5) that

- (8) if  $s$  is in Case 2, then  $\{u_m = s\}$  lies in the joining of  $E_m^1$  and  $E_m^2$  over  $a_{4m} < |z| < a_{4m+3}$  union  $\sigma_m$ .

Let  $A$  denote the set of points in the joining of  $E_m^1$  and  $E_m^2$  over  $(a_{4m}, a_{4m+3})$ , and let  $a_s = \min c(A \cap \{u_m = s\})$ ,  $b_s = \max c(A \cap \{u_m = s\})$  where  $c$  is the natural projection map and  $s$  is in Case 2. Observe that

- (9) if  $p \in A \cap K_j$  for some  $j$ , then by (2) and (3) we have

$$|u_m(p) - u_{m-1}(\alpha_j)| < \frac{M}{2};$$

- (10) if  $p \in A - \bigcup_{j=0}^{\nu} K_j$ , then by (3) we have

$$|u_m(p) - u_{m-1}(p)| < \frac{M}{4}.$$

Inequalities (9) and (10) together with (1) imply that the interval  $[a_s, b_s]$  intersects at most one  $\Delta_j$ . It follows from this and (8) that, for  $s$  in Case 2,  $\{u_m = s\}$  is contained in a relatively compact subregion  $\Omega$  of  $\mathfrak{X}_m - D_m$  less the points in the joining of  $E_m^1$  and  $E_m^2$  over  $|z| > a_{4m+3}$  where  $\Omega$  has genus one and connectivity two. Thus,  $u_m$  satisfies condition II. The proof of Theorem 2 is complete.

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*Swarthmore College*  
*Swarthmore, Pennsylvania*