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NORMAL LIGHT INTERIOR FUNCTIONS DEFINED IN THE UNIT DISK

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1. Preliminaries

Let D be the unit disk, C the unit circle, and f a continuous function from D into the Riemann sphere W. We say that f is *normal* if f is uniformly continuous with respect to the non-Euclidean hyperbolic metric in D and the chordal metric in W. Let $\chi(w_1, w_2)$ denote the chordal distance between the points $w_1, w_2 \in W$; and let $\rho(z_1, z_2)$ denote the non-Euclidean hyperbolic distance between the points $z_1, z_2 \in \in D$ [6]. If $\{z_n\}$ and $\{z'_n\}$ are two sequences of points in D with $\rho(z_n, z'_n) \to 0$, we say that $\{z_n\}$ and $\{z'_n\}$ are *close sequences.*

Let A be an open subarc of C, possibly C itself. A *Koebe sequence of arcs relative to* A is a sequence $\{J_n\}$ of Jordan arcs such that: (a) for every $\varepsilon > 0$,

$$J_n \subset \{z \in D : |z - a| < \varepsilon \text{ for some } a \in A\}$$

for all but finitely many n, and (b) every open sector Δ of D subtending an arc of C that lies strictly interior to A has the property that, for all but finitely many n, the arc J_n contains a subarc L_n lying wholly in Δ except for its two end points which lie on distinct sides of Δ.

We say that the function f has the limit c along the sequence of arcs $\{J_n\}$ (denoted by $f(J_n) \to c$) provided that, for every $\varepsilon > 0$, $\chi(c, f(J_n)) < \varepsilon$ for all but finitely many n.

2. Factorization of light interior functions

Let f be a light interior function from D into W, i.e. f is an open map which does not take any continuum into a single point. Church [4, p. 86] has pointed out that f has the representation $f = g \circ h$ where h is a

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homeomorphism of D onto a Riemann surface R and g is a non-constant meromorphic function defined on R. In view of the uniformization theorem [1, p. 181], there exists a conformal mapping φ of R onto either the unit disk or the finite complex plane. We will be concerned with the case when the range of φ is the unit disk, but remark that similar results hold when the range is the complex plane. Therefore, if f is a light interior function from D into W then f has a *factorization* $f = g \circ h$ where h is a homeomorphism of D onto D and g is a non-constant meromorphic function in D. Conversely, if h is a homeomorphism of D onto D and g is a non-constant meromorphic function in D then the function $f = g \circ h$ is light interior.

DEFINITION 1. *Let h be a homeomorphism of D onto D. If h is uniformly continuous with respect to the non-Euclidean hyperbolic metric in both its domain and range then we say that h is HUC.*

DEFINITION 2. *Let f be a light interior function in D with factorization $f = g \circ h$. If h is HUC then f has a type I factorization; otherwise f has a type II factorization.*

THEOREM 1. *If f is a light interior function in D then f has a unique factorization type.*

Proof. Let f have the factorization $f = g \circ h$. Suppose f also has the factorization $f = G \circ H$. Then as pointed out by Church [4, p. 86] $h \circ H^{-1}$ is a conformal homeomorphism. In view of Pick's theorem [6, Theorem 15. 1.3, p. 239] both $h \circ H^{-1}$ and $h^{-1} \circ H$ are *HUC*. Since the composition of two uniformly continuous functions is uniformly continuous, it follows that h is *HUC* if and only if H is *HUC*; and the proof of the theorem is complete.

3. Necessary conditions for both f and g normal

Noshiro [10, p. 154] has divided the class of normal meromorphic functions in D into two categories which are defined as follows: A normal meromorphic function g in D is of the *first category* if the normal family $\left\{g\left(\frac{a-z}{1-\bar{a}z}\right): a \in D\right\}$ admits no constant limit; otherwise g is of the *second category*.

THEOREM 2. *Let f be a normal light interior function with factorization $f = g \circ h$. If g is a normal meromorphic function then h is normal. Furthermore, if g is a normal meromorphic function of the first category then h is HUC.*

Proof. Let f have the factorization $f = g \circ h$. If h is not normal there exists close sequences $\{z_n\}$ and $\{z_n'\}$ such that $h(z_n) \to e^{i\alpha}$ and $h(z_n') \to e^{i\beta}$ with $0 < \beta - \alpha < 2\pi$ [7]. For each integer n, let J_n be the non-Euclidean geodesic joining z_n to z_n'. Then $\{h(J_n)\}$ is a sequence of Jordan arcs such that for every $\varepsilon > 0$,

$$h(J_n) \subset \{z \in D : 1 - \varepsilon < |z| < 1\}$$

for all but finitely many n, and the end points of $h(J_n)$ tend to $e^{i\alpha}$ and $e^{i\beta}$. Choosing a subsequence of $\{h(J_n)\}$ if necessary, we may assume that there exists a Koebe sequence of arcs $\{L_n\}$ relative to either the open arc (α, β) or the open arc $(\beta, \alpha + 2\pi)$ with $L_n \subset h(J_n)$, and a constant c such that $f(z_n) \to c$.

From the normality of f we have $f(J_n) \to c$, and it follows that $g(L_n) \to c$. By a theorem of Bagemihl and Seidel [2, Theorem 1, p. 10], $g \equiv c$ in violation of our hypothesis. Therefore h is normal and the proof of the first part is complete.

Now assume that g is a normal meromorphic function of the first category. If h is not *HUC* there exists close sequences $\{z_n\}$ and $\{z_n'\}$ and a $\delta > 0$ with $\rho(h(z_n), h(z_n')) \geq \delta$, and a constant c such that $f(z_n) \to c$.

Let $S_n(z) = (h(z_n) - z)/(1 - \overline{h(z_n)}z)$ and let $G_n(z) = g(S_n(z))$. Then the normal family $\{G_n\}$ has a subsequence which converges uniformly on each compact subset of D to a meromorphic function G [8, p. 53]. Let J_n be the non-Euclidean geodesic joining z_n to z_n' and let $L_n = h(J_n)$. Then $d(L_n) = d(S_n^{-1}(L_n)) \geq \delta$, where $d(E)$ is the hyperbolic diameter of the set $E \subset D$. From the normality of f we have $f(J_n) \to c$, so that $g(L_n) \to c$, and hence $G_n(S_n^{-1}(L_n)) \to c$. For r $(0 \leq r \leq \delta)$ fixed, there exists a point $Z_n \in S_n^{-1}(L_n)$ such that $\rho(0, Z_n) = r$. Let Z_0 be a cluster point of the sequence $\{Z_n\}$ on the circle $\{z : \rho(0, z) = r\}$.

Choosing a subsequence of $\{G_n\}$ if necessary, we may assume that $Z_n \to Z_0$ and $G_n(Z_n) \to c$. A familiar argument (see e.g. [3, p. 179]) in the theory of continuous convergence shows that $G(Z_0) = c$. Since r $(0 \leq r \leq \delta)$ was arbitrary, 0 is a limit point of values for which G assumes c and hence $G \equiv c$ in violation our hypothesis. Therefore h is *HUC* and the proof of the theorem is complete.

4. Bounded non-normal light interior functions

Every bounded holomorphic function is normal, but the following result shows that boundedness is not sufficient for a light interior function to be normal.

THEOREM 3. *If a homeomorphism h of D onto D is not HUC, then there exists a Blaschke product B in D such that the bounded light interior function $f = B \circ h$ is not normal.*

Proof. If h is not HUC there exists close sequences $\{z_n\}$ and $\{z_n'\}$ and a $\delta > 0$ such that $\rho(h(z_n), h(z_n')) \geq \delta$. Let $h(z_n) = w_n$ and $h(z_n') = w_n'$. Since h is uniformly continuous on compact subsets we necessarily have that $|z_n| \to 1$, $|z_n'| \to 1$, $|w_n| \to 1$, and $|w_n'| \to 1$. Hence, choosing a subsequence of $\{w_n\}$ if necessary, we may assume that $\{w_n\}$ is a Blaschke sequence, i.e. $\sum_{n=1}^{\infty}(1 - |w_n|) < \infty$. There exists a Blaschke subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a corresponding subsequence $\{w_{n_k}'\}$ of $\{w_n'\}$ for which $\rho(R_{k-1}, r_k) \geq \tanh^{-1}(1-1/k^2)$ where $r_k = \min\{|w_{n_k}|, |w_{n_k}'|\}$ and $R_k = \max\{|w_{n_k}|, |w_{n_k}'|\}$.

It follows easily that

$$\rho(w_{n_k}, w_{n_j}') \geq \begin{cases} \tanh^{-1}(1 - 1/(k+1)^2) & (1 \leq k < j) \\ \tanh^{-1}(1 - 1/k^2) & (1 \leq j < k), \end{cases}$$

and hence

$$\left| \frac{w_{n_k} - w_{n_j}'}{1 - \overline{w_{n_k}} w_{n_j}'} \right| \geq \begin{cases} 1 - 1/(k+1)^2 & (1 \leq k < j) \\ 1 - 1/k^2 & (1 \leq j < k). \end{cases}$$

Recall that $\rho(w_{n_k}, w_{n_k}') \geq \delta > 0$ $(k = 1, 2, \cdots)$ so that

$$\left| \frac{w_{n_k} - w_{n_k}'}{1 - \overline{w_{n_k}} w_{n_k}'} \right| \geq \tanh^{-1}\delta > 0 \quad (k = 1, 2, \cdots).$$

Set $$B(z) = \prod_{k=1}^{\infty} \frac{|w_{n_k}|(w_{n_k} - z)}{w_{n_k}(1 - \overline{w_{n_k}} z)}.$$

Consider $B(w_{n_j}')$ for $j \geq 1$,

$$|B(w_{n_j}')| = \prod_{k=1}^{j-1} \left| \frac{w_{n_k} - w_{n_j}'}{1 - \overline{w_{n_k}} w_{n_j}'} \right| \cdot \left| \frac{w_{n_j} - w_{n_j}'}{1 - \overline{w_{n_j}} w_{n_j}'} \right| \cdot \prod_{k=j+1}^{\infty} \left| \frac{w_{n_k} - w_{n_j}'}{1 - \overline{w_{n_k}} w_{n_j}'} \right|$$

$$\geq (\tanh^{-1}\delta) \prod_{k=1}^{j-1} (1 - 1/(k+1)^2) \prod_{k=j+1}^{\infty} (1 - 1/k^2)$$

$$= (\tanh^{-1}\delta) \prod_{k=2}^{\infty} (1 - 1/k^2) = 1/2 \tan h^{-1}(\delta) > 0.$$

Let $f = B \circ h$. By assumption $\{z_{n_k}\}$ and $\{z'_{n_k}\}$ are necessarily close sequences with

$$\lim f(z_{n_k}) = \lim B(h(z_{n_k})) = \lim B(w_{n_k}) = 0$$

and $|f(z'_{n_k})| = |B(h(z'_{n_k}))| = |B(w'_{n_k})| \geq 1/2 \tanh^{-1}(\delta) > 0$. By a theorem of Lappan [7, Theorem 3, p. 156], f is not normal and the proof is complete.

The previous theorem suggests that the normality of g does not insure the normality of f. An even stronger statement is the following result.

THEOREM 4. *There exists a homeomorphism h of D onto D with the property: If g is a normal meromorphic function in D, which has two distinct asymptotic limits, then the light interior function $f = g \circ h$ is not normal.*

Since a bounded holomorphic function in D is normal and possesses uncountably many distinct radial limits we obtain the following corollary.

COROLLARY. *There exists a homeomorphism h of D onto D with the property: If g is a non-constant bounded holomorphic function in D, then the bounded light interior function $f = g \circ h$ is not normal.*

Proof of Theorem 4. Let $\{R_n\}$ be a strictly increasing sequence of non-negative real numbers with $R_1 = 0$ for which $\rho(R_n, R_{n+1}) = 1/n$. Define the mapping h in D by

$$h(z) = h(re^{i\theta}) = r \exp(i\theta + 2\pi i(r - R_n)/(R_{n+1} - R_n))$$

for $R_n \leq r < R_{n+1}$ $(n = 1, 2, \cdots)$. It is easy to verify that h is a homeomorphism of D onto D.

Since g has two distinct asymptotic limits, a theorem of Lehto and Virtanen [8, Theorem 2, p. 53] implies that g has two distinct radial limits. Let τ_α and τ_β be the radii which terminate at the points $e^{i\alpha}$ and $e^{i\beta}$, respectively, for which $g(re^{i\alpha}) \to a$ and $g(re^{i\beta}) \to b$ with $b \neq a$.

Now the radii of D are mapped onto spirals by h^{-1}. Let $h^{-1}(\tau_\alpha) \cap [R_n, R_{n+1})$ $= z_n$ and $h^{-1}(\tau_\beta) \cap [R_n, R_{n+1}) = z'_n$. Then $\rho(z_n, z'_n) \leq \rho(R_n, R_{n+1}) = 1/n$ with

$f(z_n) = g(h(z_n)) \to a$ and $f(z'_n) = g(h(z'_n)) \to b$. Hence, by a theorem of Lappan [7], f is not normal and the theorem is proved.

5. Sufficient conditions for f normal

We now determine conditions on h and g which insure the normality of f. Since the composition of two uniformly continuous functions is uniformly continuous the first result in this direction is obvious.

THEOREM 5. *Let h be a homeomorphism of D onto D which is HUC. If g is a non-constant normal meromorphic function, then the light interior function $f = g \circ h$ is normal. Furthermore, if both h and h^{-1} are HUC, then g is normal if and only if f is normal.*

Let f be a light interior function in D with factorization $f = g \circ h$ with h a K-quasiconformal homeomorphism of D onto D. We show that f is normal if and only if g is normal. This result was proved by Väisälä [11, Theorem 5, p. 20] whose proof is considerably different.

THEOREM 6. *If h is a K-quasiconformal homeomorphism of D onto D, then both h and h^{-1} are HUC.*

THEOREM 7. *Let f be a light interior function in D with factorization $f = g \circ h$ with h a K-quasiconformal homeomorphism. Then f is normal if and only if g is normal.*

Proof of theorem 6. Since h is K-quasiconformal, by a theorem of Mori [9] h^{-1} is also K-quasiconformal. Hersch and Pfluger [5] have shown that if h is K-quasiconformal then $\rho(h(z), h(z')) \leq \Psi_K(\rho(z, z'))$ where Ψ_K is continuous and strictly increasing and defined for all $x \geq 0$ with $\Psi_K(0) = 0$. It follows easily that h is HUC. Similarly h^{-1} is HUC and the theorem is proved.

Proof of theorem 7. From Theorem 6 both h and h^{-1} are HUC. By Theorem 5, f is normal if and only if g is normal and the theorem is proved.

DEFINITION 3. *Let h be a homeomorphism of D onto D. Define the set $F(h)$ as follows: $e^{i\theta} \in F(h)$ if there exist close sequences $\{z_n\}$ and $\{z'_n\}$ and a $\delta > 0$ for which $\rho(h(z_n), h(z'_n)) \geq \delta$ and $h(z_n) \to e^{i\theta}$.*

THEOREM 8. *Let h be a normal homeomorphism of D onto D. If g is a non-constant normal meromorphic function which is continuous on $D \cup F(h)$, then the light interior function $f = g \circ h$ is normal.*

Proof. If f is not normal there exist close sequences $\{z_n\}$ and $\{z_n'\}$ such that $f(z_n) \to a$ and $f(z_n') \to b$ with $b \neq a$ [7]. It follows from the normality of g that $\{h(z_n)\}$ and $\{h(z_n')\}$ are not close. Choosing a subsequence of $\{z_n\}$ and a corresponding subsequence of $\{z_n'\}$ if necessary, we may assume that $h(z_n) \to e^{i\theta}$ and $h(z_n') \to e^{i\theta}$ with $e^{i\theta} \in F(h)$. But g is continuous on $D \cup F(h)$ and hence $b = \lim f(z_n') = \lim g(h(z_n')) = \lim g(h(z_n)) = \lim f(z_n) = a$ which is a contradiction. Therefore f is normal and the proof is complete.

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