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THE BEHAVIOUR OF MEROMORPHIC FUNCTIONS WITH A SET OF SINGULARITIES OF THE CLASS N_B^0

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1. Introduction. Let E be a totally disconnected compact set in the complex z-plane and let G be the complementary domain of E with respect to the extended z-plane. Consider a domain in G whose relative boundary consists of at most a countable number of analytic curves clustering nowhere in G. Such a domain is called a subregion in G. If for any subregion in G there exists no non-constant single-valued bounded analytic function whose real part vanishes continuously on its relative boundary, then the set E is said to be in the class N_B^0. It should be noted that the class N_B^0 is a subclass of the class N_B in the sense of Ahlfors-Beurling [1]. Further it is known that any compact subset of a set belonging to N_B^0 is also in N_B^0 (cf. [6]).

The boundary behaviour of meromorphic functions with a set of logarithmic capacity zero of essential singularities was discussed from the view point of cluster sets by many authors. The purpose of this paper is to generalize some of these theorems by replacing "a set of logarithmic capacity zero of essential singularities" by "a set of essential singularities belonging to N_B^0". In § 2, we state some properties of the set of N_B^0 which are similar to those of the set of N_B. § 3 and § 4 are devoted to treat the boundary behaviour of meromorphic functions with the set of singularities of the class N_B^0.

2. Let E be a compact set in the complex z-plane. Denote by E^* the set of points $z \in E$ such that for any neighbourhood U of z, the closure of the intersection $U \cap E$ does not belong to N_B. The set E^* is called the B-kernel of E. This notion was introduced by Kuroda [4].

In the following, we shall introduce the notion of the B_0-kernel of E and we shall show that the B_0-kernel has analogous properties to those of the B-kernel. We consider the set $E^{(*)}$ of points $z \in E$ such that for any

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neighbourhood U of z, the closure $\overline{E \cap U}$ does not belong to N_B^0. We shall call the set $E^{(*)}$ the B_0-kernel of E. It is easy to prove that $E^{(*)} \subset E^*$ and $E^{(*)}$ is closed. Further, it is obvious that, if the set E belongs to N_B^0, then the B_0-kernel $E^{(*)}$ of E is empty.

First we can prove the following theorem.

THEOREM 1. *If the set E does not belong to N_B^0, then the B_0-kernel $E^{(*)}$ of E is not empty and the B_0-kernel of $E^{(*)}$ is identical with $E^{(*)}$.*

Proof. To prove the first assertion we may assume that E is totally disconnected. Because, in the case where E contains a continuum κ, clearly $\kappa \subset E^{(*)}$ and the assertion is obvious. Suppose that E does not belong to N_B^0. Then there exists a subregion $\varDelta$ in the complementary domain G of E, whose boundary consists of a compact subset E_0 of E and at most a countable number of analytic curves γ in G, and a non-constant single-valued bounded analytic function $f(z)$ in $\varDelta$ whose real part $u(z)$ vanishes continuously on γ. We may assume, without loss of generality, that $u(z)$ is positive throughout $\varDelta$.

Putting $M = \sup\limits_{\varDelta} u(z)$, we can find a sequence of points $\{z_n\}$ $(n=1, 2, \cdots)$ in $\varDelta$ such that $\lim\limits_{n\to\infty} u(z_n) = M$. Denote by an accumlating point of $\{z_n\}$. It is obvious that the point ζ belongs to E_0. For any neighbourhood U of ζ, we describe a simple closed analytic curve C inside U which surrounds the point ζ and does not pass through any point of E_0. The intersection $\varDelta \cap C$ is a loop-cut of $\varDelta$ or consists of a finite number of cross-cuts of $\varDelta$.

We put $m = \sup\limits_{\varDelta \cap C} u(z)$ and we consider a subregion $\varDelta_1$ in which $u(z) > m$ and which contains an infinite subsequence of $\{z_n\}$ tending to ζ. Obviously the subregion $\varDelta_1$ is contained in the interior of C and the boundary of $\varDelta_1$ consists of a compact subset E_1 of E_0 and a relative boundary γ_1. The function $f(z) - m$ is non-constant, single-valued, bounded and analytic in $\varDelta_1$ and its real part vanishes continuously on γ_1. Hence the set E_1 does not belong to N_B^0. Since E_1 is contained in the closure $\overline{U \cap E}$, we see that the closure $\overline{U \cap E}$ does not belong to N_B^0. Therefore the B_0-kernel $E^{(*)}$ of E contains the point ζ. Thus the first assertion of the theorem is proved.

Next we shall prove the second part of the theorem. Let ζ_0 be a point of the B_0-kernel $E^{(*)}$ of E. Then, for any neighbourhood U of ζ_0, the closure $\overline{U \cap E}$ does not belong to N_B^0, so there exists a subregion $\varDelta$ in the complement of $\overline{U\cap E}$, whose boundary consists of a compact subset E_0 of $\overline{U\cap E}$ and the relative boundary γ, and a non-constant single-valued bounded analytic function $f(z)$ whose real part $u(z)$ vanishes continuously on γ. Applying the same argument as in the proof of the first assertion of our theorem, we may suppose that the closure $\bar{\varDelta}$ of $\varDelta$ is contained in U and that $u(z)$ is positive in $\varDelta$.

Denoting by E_1 the set of point $\zeta_1\in E_0$ such that $\limsup_{z\to\zeta_1} u(z)$ is a positive number, we can see that the set E_1 is contained in $E^{(*)}$. Because, if there exists a point $\zeta_1\in E_1$ not belonging to $E^{(*)}$, the definition of the B_0-kernel implies the existence of a neighbourhood V of ζ_1 such that $\overline{V\cap E}$ belongs to N_B^0. If we describe an analytic closed Jordan curve inside $U\cap V$ which surrounds the point ζ_1 and does not meet E and whose interior contains no point of $E^{(*)}$, then we can find a component D of the open set $(K)\cap\varDelta$ such that D has ζ_1 as a boundary point and such that $\limsup_{z\to\zeta_1} u(z)=m>0$ in D. Here (K) denotes the interior of K. The boundary of D consists of a part of γ, a finite number of arcs on K and a compact subset of E_0 belonging to N_B^0. We consider the function $w=f(z)$ in D. By Noshiro's theorem [5], it is seen that the set $\Omega=C_D(f,\zeta_1)-C_\gamma(f,\zeta_1)$ is open, where $C_D(f,\zeta_1)$ and $C_\gamma(f,\zeta_1)$ are the interior cluster set and the boundary cluster set of $f(z)$ at ζ_1, respectively. Clearly $C_\gamma(f,\zeta_1)$ lies on the imaginary axis of the w-plane and $C_D(f,\zeta_1)$ contains a value whose real part equals to $m(>0)$. Therefore, it follows that $\limsup_{z\to\zeta_1} u(z)>m$ in D which is a contradiction. Thus, we have that $E_1\subset E^{(*)}$.

Taking a point z_1 in $\varDelta$, we consider a subregion $\varDelta_1(\subset\varDelta)$ where $u(z)>u(z_1)$. From the definition the subset of E_1 lying on the boundary of $\varDelta_1$ does not belong to N_B^0. Since $E_1\subset E^{(*)}$, we see that the closure $\overline{U\cap E^{(*)}}$ does not belong to N_B^0. Thus we can conclude that the B_0-kernel of $E^{(*)}$ is identical with $E^{(*)}$.

By using Theorem 1, we can prove the following theorem which was informed by Matsumoto in his letter to Kuroda.

THEOREM 2. *If every set E_n $(n=1,2,\cdots)$ belongs to N_B^0 and if the union $E=\bigcup_{n=1}^{\infty}E_n$ is compact, then E also belongs to N_B^0.*

Proof. Contrary to the assertion, suppose that $E=\bigcup_{n=1}^{\infty}E_n$ does not belong to N_B^0. By Theorem 1, the B_0-kernel $E^{(*)}$ of E is not empty. We show that, for any point z of $E^{(*)}$, any neighbourhood U of z and for any positive integer n, the intersection $\{U \cap E^{(*)}\} \cap (E - E_n)$ is not empty. For, otherwise, there exists a point z_0 of $E^{(*)}$, a neighbourhood V_0 of z_0 and a positive integer m such that $V_0 \cap E^{(*)} \subset E_m$. So, for a neighbourhood V_1 of z_0 whose closure is contained in V_0, it holds that $\overline{V_1 \cap E^{(*)}} \subset E_m$ and hence we have that E_m does not belong to N_B^0. This is a contradiction.

This fact permits us the following process: First we take a point z_1 of $E^{(*)} \cap (E - E_1)$ and a neighbourhood U_1 of z_1 whose closure $\bar{U}_1$ is disjoint from E_1 and which has a diameter less than 1. Next we take a point z_2 of $\{U_1 \cap E^{(*)}\} \cap (E - E_2)$ and a neighbourhood U_2 of z_2 such that $\bar{U}_2 \subset U_1$, $\bar{U}_2 \cap E_2 = \phi$ and such that U_2 has a diameter less than 1/2. Generally, we take a point z_n of $\{U_{n-1} \cap E^{(*)}\} \cap (E - E_n)$ and a neighbourhood U_n of z_n such that $\bar{U}_n \subset U_{n-1}$, $\bar{U}_n \cap E_n = \phi$ and such that U_n has a diameter less than $1/2^{n-1}$.

Continuing this procedure infinitely, we get a sequence of points $\{z_n\}$ $(n = 1, 2, \cdots)$ and a sequence of neighbourhoods $\{U_n\}$ $(n = 1, 2, \cdots)$. Since $E^{(*)}$ is closed, the sequence $\{z_n\}$ $(n = 1, 2, \cdots)$ tends to a point ζ of $E^{(*)}$. On the other hand, since $\zeta \in \bar{U}_{n+1} \subset U_n$ and $U_n \cap E_n = \phi$, we see that ζ does not belong to E_n $(n = 1, 2, \cdots)$, so ζ does not belong to $E = \bigcup_{n=1}^{\infty} E_n$, which is a contradiction. Thus we have the theorem.

3. As an application of theorems in § **2**, we shall prove some theorems concerning with the behaviour of meromorphic functions. A generalization of Hällström-Kametani's theorem [2], [3] can be stated as follows.

THEOREM 3. *Let E be a compact set of N_B^0 and let D be a domain containing E. Suppose that $w = f(z)$ is a single-valued meromorphic function in the domain $D - E$ which has each point z_0 of E as an essential singularity. Then any compact subset e of the complement of $R_{D-E}(f, z_0)$ belongs to N_B^0. Here $R_{D-E}(f, z_0)$ is the range of values of $f(z)$ at z_0.*

Proof. We denote by e_n $(n = 1, 2, \cdots)$ the set of values omitted by $w = f(z)$ in $(D - E) \cap (K_n)$, where (K_n) denotes the disc $\{z \mid |z - z_0| < 1/n\}$. Then it is obvious that e_n is closed, $e_n \subset e_{n+1}$ and $e \subset \bigcup_{n=1}^{\infty} e_n$. From Theorem 2, we see that to prove the theorem, it is sufficient to show that $e_n \in N_B^0$ $(n = 1, 2, \cdots)$. Contrary to the assertion, we suppose that there exists a set e_n not belonging to N_B^0. By Theorem 1, we can find a point w_0 of e_n

such that for any positive number ρ, the closure of intersection of the disc $(c)=\{w \mid |w-w_0|<\rho\}$ and e_n does not belong to N_B^0. Noting the fact $e_n \in N_B$ (cf. Kuroda [4]), we see from the proof of Theorem 1 that there exists a subregion Δ_w inside (c) with relative boundary γ_w and a non-constant single-valued bounded analytic function $\varphi(w)$ in Δ_w whose real part vanishes continuously on γ_w.

We describe a simple closed curve C inside $D \cap (K_n)$ which surrounds the point z_0 and does not intersect E. We choose as ρ a positive number less than the distance of w_0 from the image of C by $w=f(z)$. We can take a point z_1 in the interior of C whose image lies on Δ_w and we denote by Δ_z the component of the inverse image of Δ_w by $w=f(z)$ which contains the point z_1. Obviously Δ_z is a subregion in the interior of C whose boundary consists of a compact subset E_0 of E and a relative boundary. Considering the composed function $\varphi(f(z))$ in Δ_z, we can see that $E_0(\subset E)$ does not belong to N_B^0. Hence the set E does not belong to N_B^0. This is a contradiction. Thus we get the assertion from our assumption in the theorem.

4. Let Δ be a subregion in the complex z-plane whose boundary consists of a totally disconnected compact set E and at most a countable number of analytic curves γ. Denote by (c) a disc $\{w \mid |w-w_0|<\rho\}$ or $\{w \mid |w|>1/\rho\}$ in the extended w-plane.

Suppose that $w=f(z)$ be a single-valued meromorphic function in $\Delta \cup \gamma$ such that $w=f(z)$ takes the values belonging to (c) in Δ and takes the values on the circumference of (c) on γ. Denote by Φ_Δ the Riemann covering surface spread over (c) which is defined by the elements $q=[z, f(z)]$ $(z \in \Delta)$ and denote by $n(w)$ the number of sheets of Φ_Δ above $w \in (c)$. We put $N=\sup_{w \in (c)} n(w)$ and denote by $e_{(c)}$ the set of points $w \in (c)$ satisfying the inequality $n(w)<N$.

As an extension of Kuroda's theorem [4], we can get the following.

THEOREM 4. *If E is a set of N_B^0, then any closed subset e of $e_{(c)}$ belongs to N_B^0.*

Proof. We may assume that (c) is an open disc $\{w \mid |w-w_0|<\rho\}$, $w_0 \neq \infty$. For any integer $n (0 \leqq n < N)$, we denote by e_n the set $\{w \mid w \in (c),\ n(w) \leqq n\}$. It is easily seen that e_n is closed with respect to (c),

$e_n \subset e_{n+1}$ and $e_{(c)} = \bigcup_{0 \leqq n < N} e_n$. By the use of Theorem 2, it is sufficient to show that for each n and for any closed set S inside (c), $S \cap e_n$ is a set of N_B^0.

First we shall show that $S \cap e_0$ is a set of N_B^0. If $S \cap e_0$ does not belong to N_B^0, then there exists a subregion Δ_w with relative boundary γ_w contained in the domain $(c) - S \cap e_0$ (cf. Theorem 1) and a non-constant single-valued bounded analytic function $\varphi(w)$ in Δ_w such that the real part of $\varphi(w)$ vanishes continuously on γ_w. Denote by Δ_z a component of the inverse image of Δ_w by $w = f(z)$. The boundary of Δ_z consists of a closed subset E_0 of E and a relative boundary γ_z. Since the composed function $\varphi(f(z))$ is non-constant, single-valued, bounded and analytic in Δ_z and the real part of $\varphi(f(z))$ vanishes continuously on γ_z, the set E_0 does not belong to N_B^0. This contradicts the fact $E_0 \subset E \in N_B^0$.

Next we suppose that there exists a set $S \cap e_n$ not belonging to N_B^0 and we denote by m the smallest of such indices n. Since $S \cap e_{m-1} \in N_B^0$ and $S \cap e_m \notin N_B^0$, Theorem 1 implies that there exists a point $w_1 \in S \cap (e_m - e_{m-1})$ such that for any positive number r, the closure of the intersection of the disc $\{w \mid |w - w_1| < r\}$ and e_m does not belong to N_B^0. Since w_1 is a cluster value of $f(z)$ at a point of E, we can take a positive number r_1 such that the inverse image of the disc $\{w \mid |w - w_1| < r_1\}$ by $w = f(z)$ consists of at most m relatively compact domains and at least one relatively non-compact subregion Δ_1.

Since, in Δ_1, the function $w = f(z)$ takes no value of the set $\{w \mid |w - w_1| \leqq r\} \cap e_m (r < r_1)$ not belonging to N_B^0, the fact stated already leads us to a contradiction.

Using the argument of Noshiro ([5] p. 287) and Theorem 4, we can easily obtain the following extension of Noshiro's theorem [5].

THEOREM 5. *Let D be a domain, Γ its boundary, E a compact set of N_B^0 contained in Γ and z_0 a point of E. Suppose that $w = f(z)$ is a single-valued meromorphic function in D and $\Omega = C_D(f, z_0) - C_{\Gamma - E}(f, z_0)$ is not empty. Then any compact subset e of $\Omega - R_D(f, z_0)$ belongs to N_B^0.*

Remark. Noshiro's theorem is read as follows: Under the same assumption as in Theorem 5 any compact subset of $\Omega - R_D\ (f, z_0)$ belongs to N_B.

5. Remark. In the previous paper [6], we have shown that there exists a compact set E of positive logarithmic capacity belonging to N_B^0 and there exists a single-valued meromorphic function $f(z)$ in the complementary domain of E such that $f(z)$ has an essential singularity at every point of E and such that the set of exceptional values of $f(z)$ in Picard's sense at each point of E is of positive logarithmic capacity belonging to N_B^0.

From this fact, it is easily seen that in Theorem 3, we cannot replace the phrase "e belongs to N_B^0" by the phrase "e is of logarithmic capacity zero". Further, by the example in § 5 of the previous paper [6], we see that in Theorem 4 and 5, the phrase "e belongs to N_B^0" can not be replaced by the phrase "e is of logarithmic capacity zero".

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