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ON HOLOMORPHIC FAMILIES OF HOLOMORPHIC MAPS

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Let D be the unit disk $\{z : |z| < 1\}$ in the complex plane $\boldsymbol{C}$ with boundary ∂D and closure $\bar{D}$, and denote by $\boldsymbol{R}$ the image of the canonical embedding $r \to r + iO$ of the real line into $\boldsymbol{C}$. The symbol ε will be used throughout to denote a complex parameter; the unit disk in the complex ε-plane will be denoted by D_p. A C^{1+a} map $\mathscr{C} : \partial D \times D_p \to \boldsymbol{D}$ $(0 < a < 1)$ is called a *holomorphic family of* C^{1+a} *curves* if

1° $\mathscr{C}_\varepsilon = \mathscr{C} | \partial D \times \{\varepsilon\}$ is a C^{1+a} Jordan curve in $\boldsymbol{C}$ for every $\varepsilon \in D_p$;

2° $\mathscr{C}_t = \mathscr{C} | \{t\} \times D_p$ is a holomorphic function for every $t \in \partial D$;

3° $\dfrac{\partial \mathscr{C}(t, \varepsilon)}{\partial t}$ is continuous in t and ε.

Denote by Ω_ε the simply-connected region in $\boldsymbol{C}$ bounded by $\mathscr{C}(\partial D \times \{\varepsilon\})$.

We are interested in the existence of holomorphic maps $f : D \times D_p \to \boldsymbol{C}$ which map $D \times \{\varepsilon\}$ conformally onto Ω_ε for every $\varepsilon \in D_p$ (f is then said to be *associated with* $\mathscr{C}$). The following theorem will be proved.

THEOREM 1. *Let* $\mathscr{C} : \partial D \times D_p \to \boldsymbol{C}$ *be a holomorphic family of* C^{1+a} *curves. If* f *is a holomorphic map associated with* $\mathscr{C}$, *then there exists a* C^{1+a} *homeomorphism* $g : \partial D \to \partial D$ *for which*

$$\mathscr{C}(t, \varepsilon) = f(g(t), \varepsilon) \qquad (*)$$

for all $(t, \varepsilon) \in \partial D \times D_p$, *where* f *on the right hand side denotes the continuous extension of* f *to* $\bar{D} \times D_p$.

Now $\mathscr{C}$ can always be normalized by the condition that for some $\varepsilon_0 \in D_p$, $\mathscr{C}_{\varepsilon_0}$ is the boundary value of a conformal map of $D \times \{\varepsilon_0\}$ onto Ω_{ε_0} (for let g_{ε_0} be such a conformal map, the existence of which is ensured by

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the Riemann mapping theorem, and replace $\mathscr{C}(t,\varepsilon)$ with $\mathscr{C}(\pi\circ(\mathscr{C}_{\varepsilon_0})^{-1}\circ g_{\varepsilon_0}(t),\varepsilon)$ with projection $\pi:\partial D\times D_p\rightarrow\partial D$). If $\mathscr{C}$ is normalized in this sense, setting $\varepsilon=\varepsilon_0$ in (*) shows that $g:\partial D\rightarrow\partial D$ is the boundary value of a conformal map cf D onto itself. Consequently, we have

COROLLARY 1. *Let $\mathscr{C}:\partial D\times D_p\rightarrow \boldsymbol{C}$ be a normalized holomorphic family of $C^{1+\alpha}$ curves. Then there is a holomorphic map $f:D\times D_p\rightarrow\boldsymbol{C}$ associated with $\mathscr{C}$ if and only if $\mathscr{C}$ itself is the boundary value of a holomorphic map associated with $\mathscr{C}$.*

We may write

$$\mathscr{C}(t,\varepsilon)=\sum_{k=0}^{\infty}c_k(t)\varepsilon^k,$$

where if $\mathscr{C}$ is normalized at $\varepsilon=0$, $c_0(t)$ is the boundary value of a conformal map of D onto Ω_0.

COROLLARY 2. *Let $\mathscr{C}:\partial D\times D_p\rightarrow\boldsymbol{C}$ be a holomorphic family of $C^{1+\alpha}$ curves normalized at $\varepsilon=0$. If there is a holomorphic map f associated with $\mathscr{C}$, then necessarily each coefficient $c_k(t)$, $k\geqslant 0$, in the above expansion of $\mathscr{C}$ is the boundary value of a holomorphic function on D.*

EXAMPLE 1. For $|\varepsilon|$ sufficiently small, $\mathscr{C}(t,\varepsilon)=t+\varepsilon\bar{t}$ is a holomorphic family of $C^{1+\alpha}$ curves normalized at $\varepsilon=0$, where $\bar{t}$ is the complex conjugate of t. By corollary 2. there is no holomorphic map associated with $\mathscr{C}$.

S.E. Warschawski [6] has proved a general perturbation theorem which yields the following related result. If we restrict our attention to $\varepsilon\in\boldsymbol{R}$ and replace condition 2° on $\mathscr{C}$ with

> 2'° both $\mathscr{C}(t,\varepsilon)$ and $\dfrac{\partial\mathscr{C}(t,\varepsilon)}{\partial t}$ have "Taylor" expansions at $\varepsilon=0$ of order m,

then there always exists a continuous map $f:D\times(D_p\cap\boldsymbol{R})\rightarrow\boldsymbol{C}$ which maps $D\times\{\varepsilon\}$ conformally onto Ω_ε for every ε and which has a "Taylor" expansion at $\varepsilon=0$ of order m. In particular, if $\mathscr{C}$ depends real analytically on the parameter ε then there exists real analytic f associated with $\mathscr{C}$. This real analytic case was also proved by D. Zeitlin [7] (there are minor differences between these two results of a technical nature). His method involves proving that the solution $F(t,\varepsilon)$ of a certain extension of the well-known

Gershgorin integral equation into the complex domain is a holomorphic function in ε for $\varepsilon \in U$, U being a certain open neighborhood of 0 in D_p. For every $\varepsilon \in U \cap \boldsymbol{R}$, $F(t, \varepsilon)$ gives the mapping function of Ω_ε onto D in the usual manner. An open question is the relationship of F on all of U to, when it exists, a holomorphic map f associated with the holomorphic family of curves whose restriction to $\partial D \times (D_p \cap \boldsymbol{R})$ is the given real analytic family of curves.

The proof of theorem 1 goes as follows. Let $\mathscr{C} : \partial D \times D_p \to \boldsymbol{C}$ be a holomorphic family of $C^{1+\alpha}$ curves and $f : D \times D_p \to \boldsymbol{C}$ a holomorphic map associated with $\mathscr{C}$. Define $\tilde{\mathscr{C}} : \partial D \times D_p \to \boldsymbol{C}^2$ and $\tilde{f} : D \times D_p \to \boldsymbol{C}^2$ by the rules $\tilde{\mathscr{C}}(t, \varepsilon) = (\mathscr{C}(t, \varepsilon), \varepsilon)$ and $f(t, \varepsilon) = (f(t, \varepsilon), \varepsilon)$. Let $\Omega = \{(z, \varepsilon) : z \in \Omega_\varepsilon,\ \varepsilon \in D_p\}$. Then $\tilde{f} : D \times D_p \to \Omega$ is a biholomorphic map and $(\tilde{f}^{-1} | \Omega_\varepsilon \times \{\varepsilon\})(z, \varepsilon) = (f_\varepsilon^{-1}(z), \varepsilon)$, where $f_\varepsilon = f | D \times \{\varepsilon\}$. As is wellknown, $\tilde{f}_\varepsilon = \tilde{f} | D \times \{\varepsilon\}$ has a homeomorphic extension to $\bar{D} \times \{\varepsilon\}$ for every $\varepsilon \in D_p$. It will be shown (lemma 2.) that $\tilde{f}^{-1} \circ \tilde{\mathscr{C}}_t : \{t\} \times D_p \to \partial D \times D_p$ is a holomorphic map for every $t \times \partial D$. This is the central point in the proof of the theorem, for now write

$$\tilde{f}^{-1} \circ \tilde{\mathscr{C}}(t, \varepsilon) = (f'(t, \varepsilon), \varepsilon),$$

where $f' : \partial D \times D_p \to \partial D$ is a continuous map; such an f' clearly exists. According to lemma 2., $(f' | \{t\} \times D_p) : \{t\} \times D_p \to \partial D$ is a holomorphic map for every $t \in \partial D$ and is consequently constant in ε for every $t \in \partial D$. Therefore there is a homeomorphism $g : \partial D \to \partial D$ for which $f'(t, \varepsilon) = g(t)$, which implies that $\tilde{\mathscr{C}}(t, \varepsilon) = \tilde{f}(g(t), \varepsilon)$, and so $\mathscr{C}(t, \varepsilon) = f(g(t), \varepsilon)$. That g is a $C^{1+\alpha}$ map follows from Kellogg's theorem by normalizing $\mathscr{C}$ at some $\varepsilon_0 \in D_p$, and the proof of theorem 1. is complete.

It should be clear from the local nature of lemma 2. that theorem 1. admits readily to generalizations. A few of these are presented after the proofs of lemmas 1. and 2.

§1. Choose any point $\tilde{\mathscr{C}}(t_0, \varepsilon_0) \in bdy\ \Omega$ and let $n = n(t_0, \varepsilon_0)$ be the inward normal to $\mathscr{C}_{\varepsilon_0}$ at $\mathscr{C}(t_0, \varepsilon_0)$, i.e. $n \subseteq \Omega_{\varepsilon_0}$. Denote by $W(\alpha, r) = W_{t_0, \varepsilon_0}(\alpha, r)$ the wedge in Ω_{ε_0} with radius r and interior angle α at the vertex $\mathscr{C}(t_0, \varepsilon)$, and which is symmetric about the normal n, i.e. $W(\alpha, r) = \{z \in \Omega_{\varepsilon_0} : \mathrm{dist}(z, n) \leqslant |z - \mathscr{C}(t_0, \varepsilon_0)| \sin(\alpha/2)$ and $0 < |z - \mathscr{C}(t_0, \varepsilon_0)| < r\}$. Also, for $z \in \Omega_{\varepsilon_0}$ denote by $\tilde{\mathscr{C}}_z = \tilde{\mathscr{C}}_{z, t_0, \varepsilon_0} : D_p \to \boldsymbol{C}^2$ the holomorphic map $(\mathscr{C}_{t_0}(\varepsilon) - \mathscr{C}(t_0, \varepsilon_0) + z, \varepsilon)$. Clearly, for each $z \in \Omega_{\varepsilon_0}$ there is an open neighborhood $U = U_z$ of ε_0 in D_p

such that $\tilde{\mathscr{C}}_z(U)\subseteq\Omega$. Lemma 1. will show that there are wedges $W(\alpha,r)$ for which U_z may be chosen independent of $z\in W(\alpha,r)$.

LEMMA 1. *For every $\alpha(0<\alpha<\pi)$ there is an $r>0$ and an open neighborhood U of ε_0 in D_p such that $\tilde{\mathscr{C}}_z(U)\subseteq\Omega$ for all $z\in W(\alpha,r)$.*

Proof. It is well-known [4] that since $\mathscr{C}_{\varepsilon_0}$ is a $C^{1+\alpha}$ curve, for every β $(0<\beta<\pi/2)$ there is a connected subarc $\Gamma=\Gamma_\beta$ of ∂D containing t_0 in its interior such that the chord joining $\mathscr{C}(t_0,\varepsilon_0)$ and $\mathscr{C}(t,\varepsilon_0)$ makes an angle smaller than β with the tangent line to $\mathscr{C}_{\varepsilon_0}(\partial D)$ at $\mathscr{C}(t_0,\varepsilon_0)$ for every $t\in\Gamma$. It follows from the conditions on the map $\mathscr{C}$ that there is an open neighborhood U_1 of ε_0 such that the same is true for every $\mathscr{C}_\varepsilon$ with $\varepsilon\in U_1$ when β is replaced by 2β. Choose β so that $\pi-4\beta>\alpha$.

Now it is also known that $r>0$ may be chosen so that $|\mathscr{C}(t,\varepsilon_0)-\mathscr{C}(t_0,\varepsilon_0)|>2r$ for every $t\in\partial D\backslash\Gamma$, and it follows again from the conditions on $\mathscr{C}$ that there is an open neighborhood U_2 of ε_0 such that $|\mathscr{C}(t,\varepsilon)-\mathscr{C}(t_0,\varepsilon)|>r$ for every $t\in\partial D\backslash\Gamma$ and every $\varepsilon\in U_2$.

Let $U=U_1\cap U_2$. If $\tilde{\mathscr{C}}_z(\varepsilon)\in bdy\,\Omega$ for some $\varepsilon\in U$ there must be a $t\in\partial D$ such that $\tilde{\mathscr{C}}_z(\varepsilon)=\tilde{\mathscr{C}}_t(\varepsilon)$, or equivalently $\mathscr{C}_{t_0}(\varepsilon)-\mathscr{C}(t,\varepsilon_0)+z=\mathscr{C}_t(\varepsilon)$. Since $\mathscr{C}(t,\varepsilon)=\mathscr{C}_t(\varepsilon)=\mathscr{C}_\varepsilon(t)$, we have

$$(**)\qquad \tilde{\mathscr{C}}_z(\varepsilon)\in bdy\,\Omega \Longleftrightarrow \mathscr{C}(t_0,\varepsilon)-\mathscr{C}(t_0,\varepsilon_0)+z=\mathscr{C}(t,\varepsilon).$$

Suppose that $t\in\Gamma$. By the choice of Γ, since $\varepsilon\in U$, and since from $(**)$ it follows that $z-\mathscr{C}(t_0,\varepsilon_0)=\mathscr{C}(t,\varepsilon)-\mathscr{C}(t_0,\varepsilon)$, we have $\mathrm{dist}\,(z,n)>|z-\mathscr{C}(t_0,\varepsilon_0)|\sin(\pi/2-2\beta)$. But $\pi/2-2\beta>\alpha/2$, and so $\mathrm{dist}\,(z,n)>|z-\mathscr{C}(t_0,\varepsilon_0)|\sin(\alpha/2)$. Therefore $z\notin W(\alpha,r)$. Now suppose that $t\notin\Gamma$. Then by the choice of r, since $\varepsilon\in U$, and by $(**)$ as above we have $|z-\mathscr{C}(t_0,\varepsilon_0)|>r$, and so $z\notin W(\alpha,r)$. Consequently, $\tilde{\mathscr{C}}_z(\varepsilon)\in bdy\,\Omega$ for some $\varepsilon\in U$ implies that $z\notin W(\alpha,r)$, and the lemma is proved.

LEMMA 2. *$\tilde{f}^{-1}\circ\tilde{\mathscr{C}}_t:\{t\}\times D_p\to\partial D\times D_p$ is a holomorphic map for every $t\in\partial D$.*

Proof. Choose $(t_0,\varepsilon_0)\in\partial D\times D_p$; by lemma 1. there is a sequence of points $\{z_k:k=1,2,\cdots\}\subseteq\Omega_{\varepsilon_0}$ and an open neighborhood U of ε_0 for which $\tilde{\mathscr{C}}_{z_k}(U)\subseteq\Omega$ while $z_k\to\mathscr{C}(t_0,\varepsilon_0)$ as $k\to\infty$. Therefore $\tilde{f}^{-1}\circ\tilde{\mathscr{C}}_{z_k}:U\to D\times D_p$ is a well-defined holomorphic map for every $k=1,2,\cdots$. Since $\tilde{f}|\bar{D}\times\{\varepsilon\}$ is a homeomorphism for every $\varepsilon\in D_p$, the map $f^{-1}\circ\tilde{\mathscr{C}}_{t_0}$ is also defined; clearly the sequence $\{\tilde{f}^{-1}\circ\tilde{\mathscr{C}}_{z_k}:k=1,2,\cdots\}$ converges pointwise to $\tilde{f}^{-1}\circ\tilde{\mathscr{C}}_{t_0}$ on U. The lemma now follows from Vitali's theorem.

§ 2. **Generalizations.** (Ahlfors [1] has shown the existence of a holomorphic map f from a bordered Riemann surface with finite genus and a finite number of boundary components onto a full covering surface $S\xrightarrow{\pi}D$ of the unit disk. N. Alling [2] has shown that $\pi\circ f|U$ is a covering map of D near ∂D for some open neighborhood U of ∂X. Theorems 2.-4. can be thought of as concerning holomorphic families of such maps.)

Let X and Y be open Riemann surfaces such that X has a $C^{1+\alpha}$ boundary ∂X, and let V be a connected analytic set in some open set in $\boldsymbol{C}^n$. Let $\mathscr{C}:\partial X\times V\to Y$ be a $C^{1+\alpha}$ map satisfying

1° for every local coordinate t on ∂X for which t^{-1} describes ∂X locally as a $C^{1+\alpha}$ curve, $\mathscr{C}\circ t^{-1}$ is a holomorphic family of $C^{1+\alpha}$ curves on Y ($\mathscr{C}$ is then said to be locally a holomorphic family of $C^{1+\alpha}$ curves on Y);

2° for every $\mathscr{C}_\varepsilon=\mathscr{C}|\partial X\times\{\varepsilon\}$, $\mathscr{C}_\varepsilon(\partial X\times\{\varepsilon\})$ is the boundary of an open Riemann surface Ω_ε.

Theorem 2. is the most straightforward generalization of theorem 1. which can be proved.

THEOREM 2. *Denote the set* $\{(y,\varepsilon):y\in\Omega_\varepsilon,\ \varepsilon\in V\}$ *by* Ω. *There exists a holomorphic map* $f:\Omega\to X$ *which maps* $\partial\Omega_\varepsilon\times\{\varepsilon\}$ *into* ∂X *for every* $\varepsilon\in V$ *if and only if there is a* $C^{1+\alpha}$ *map* $g:\partial X\to\partial X$ *for which*

$$f\circ\tilde{\mathscr{C}}(x,\varepsilon)=g(x)$$

for all $x\in\partial X$ *and all* $\varepsilon\in V$.

More generally, one has

THEOREM 3. *Let* C *be an arc on* ∂X, $\mathscr{C}:C\times V\to Y$ *locally a holomorphic family of* $C^{1+\alpha}$ *curves on* Y, *and* $\Omega_\varepsilon\subseteq Y$ *a bordered Riemann surface with* $\mathscr{C}(C\times\{\varepsilon\})\subseteq\partial\Omega_\varepsilon$ *for every* $\varepsilon\in V$. *Define* Ω *as in theorem* 2. *There is a holomorphic map* $f:\Omega\to X$ *which maps* $\mathscr{C}(C\times\{\varepsilon\})\times\{\varepsilon\}$ *into* ∂X *for every* $\varepsilon\in V$ *if and only if there is a* $C^{1+\alpha}$ *map* $g:C\to\partial X$ *for which*

$$f\circ\tilde{\mathscr{C}}(x,\varepsilon)=g(x)$$

for all $x \in C$ *and all* $\varepsilon \in V$.

Proof. All that must be shown is that theorem 1. remains true when D_p is replaced with the connected analytic set V. First of all, lemmas 1. and 2. carry over just as they were presented when D_p is replaced by a polydisk in $\boldsymbol{C}^k$. This means that theorem 1. is true when D_p is replaced by a connected component V_i of the set of regular points of V; let g_i be the map of theorem 1. for V_i. In fact (*) holds on $Cl_V V_i$ and the usual continuity argument shows that $g_i = g_j$ when $Cl_V V_i \cap Cl_V V_j \neq \emptyset$. The theorem is therefore proved since V is connected and $V = U\{Cl_V V_i : i \in I\}$.

THEOREM 4. *Let* X, Y, $\mathscr{C} : C \times V \to Y$ *and* $\{\Omega_\varepsilon : \varepsilon \in V\}$ *be given as in theorem* 3. *If* $f : X \times V \to Y$ *is a holomorphic map satisfying*

a) $f(\partial X \times \{\varepsilon\}) \subseteq \partial \Omega_\varepsilon$ *for every* $\varepsilon \in V$;

b) $f_\varepsilon = f | X \times \{\varepsilon\}$ *is a covering map of* Ω_ε *near* $\partial\Omega_\varepsilon$ *for some open neighborhood of* $\partial X \times \{\varepsilon\}$ *in* $X \times \{\varepsilon\}$, *again for every* $\varepsilon \in V$, *then there exists a* $C^{1+\alpha}$ *map* $g : C \to \partial X$ *for which*

$$\mathscr{C}(x, \varepsilon) = f(g(x), \varepsilon)$$

for all $x \in C$ *and all* $\varepsilon \in V$.

By viewing theorems 1.-4. from another point of view one gets mapping theorems for complex manifolds. Theorem 5. below is one such result, although clearly not the most general one.

Let $\boldsymbol{P}$ be a polydisk in $\boldsymbol{C}^{n-1} (n > 1)$ and let C be a subarc of ∂D. Given a holomorphic family of $C^{1+\alpha}$ curves $\mathscr{C}' : C \times \boldsymbol{P} \to \boldsymbol{C}$ and holomorphic maps $\mathscr{C}_\nu : \boldsymbol{P} \to \boldsymbol{C}$ for each $\nu = 2, \cdots, m (m > n)$, define $\mathscr{C} : C \times \boldsymbol{P} \to \boldsymbol{C}^m$ by the rule

$$\mathscr{C}(t, \varepsilon) = (\mathscr{C}'(t, \varepsilon), \ \mathscr{C}_2(\varepsilon), \ \cdots, \ \mathscr{C}_m(\varepsilon))$$

for all $t \in C$ and all $\varepsilon \in \boldsymbol{P}$. We may assume without loss in generality that $\mathscr{C} | C \times \{0\}$ is the boundary value of a holomorphic function on $U \cap D$ for some open set $U \subseteq \boldsymbol{C}$. Let Ω be a domain in $\boldsymbol{C}^m$ for which $\mathscr{C}(C \times \boldsymbol{P}) \subseteq \partial \Omega$.

THEOREM 5. *If there is a holomorphic map* $f : \Omega \to D \times \boldsymbol{P}$ *for which* $f(\mathscr{C}(C \times \boldsymbol{P})) \subseteq \partial D \times \boldsymbol{P}$, *then necessarily* $\mathscr{C}$ *is the boundary value of a holomorphic map* $\mathscr{C} : U \cap D \times \boldsymbol{P} \to \Omega$ *for some open set* $U \subseteq \boldsymbol{C}^n$.

Proof. This theorem is a straightforward generalization of corollary 1.

In view of this theorem one may ask for conditions on $\partial\Omega$ of a given domain Ω under which there exists a subarc C of ∂D and a map $\mathscr{C}: C\times \boldsymbol{P}\to\partial\Omega$ like the one described above. In this direction we have a Levi-type condition.

PROPOSITION 1. *Let Ω be an open domain in $\boldsymbol{C}^n (n>1)$ and suppose that $(z_0,\varepsilon^0)\in\partial\Omega$, where $\varepsilon^0=(\varepsilon_2^0,\cdots,\varepsilon_m^0)\in\boldsymbol{C}^{n-1}$. In order that there exist an open neighborhood U of (z_0,ε^0), a polydisc $\boldsymbol{P}\subseteq\boldsymbol{C}^{n-1}$, a subarc C of ∂D and an injective $C^{2+\alpha}$ map $\mathscr{C}: C\times\boldsymbol{P}\to\partial\Omega\cap U$ satisfying the conditions in theorem 5. it is necessary that there exist an open neighborhood U' of (z_0,ε^0) and a C^2 map $\varphi: U'\to\boldsymbol{R}$ such that*

1° *$\{(z,\varepsilon):\varphi(z,\varepsilon)=0\}=U'\cap\partial\Omega$;*

2° *grad $\varphi\neq 0$ on $U'\cap\partial\Omega$;*

3° *denoting $(z,\varepsilon)=(z,\varepsilon_2,\cdots,\varepsilon_n)$ by $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$, then $\sum_{i=1}^{n}\frac{\partial\varphi}{\partial\varepsilon_i}w_i=0$ at $(\varepsilon_1,\cdots,\varepsilon_n)\in U'\cap\partial\Omega$ implies that*

$$\sum_{i,j=1}^{n}\frac{\partial^2\varphi}{\partial\varepsilon_i\partial\bar{\varepsilon}_j}w_i w_j=0.$$

Proof. Given injective $\mathscr{C}$, let ρ denote the coordinate function of $\mathscr{C}^{-1}$; it is known [5] that there is an open neighborhood V of (z_0,ε^0) and holomorphic functions $f_1: V\cap\Omega\to\boldsymbol{C}$, $f_2: V\cap(\boldsymbol{C}^n\backslash\bar{\Omega})\to\boldsymbol{C}$ with C^2 extensions to $\partial\Omega\cap V$ such that $\rho(z,\varepsilon)=f_1(z,\varepsilon)f_2(z,\varepsilon)$ for all $(z,\varepsilon)\in\partial\Omega\cap V$. The differentiability properties of f_1 and f_2 on $\partial\Omega\cap V$ allow us to choose C^2 functions $\tilde{f}_1$ and $\tilde{f}_2$ on V for which $\tilde{f}_1|V\cap\Omega=f_1$ and $\tilde{f}_2|V\cap(\boldsymbol{C}^n\backslash\bar{\Omega})=f_2$. Define the extension of ρ into V to be $f_1(z,\varepsilon)f_2(z,\varepsilon)$ and $\varphi: V\to\boldsymbol{R}$ by the rule $\varphi(z,\varepsilon)=|\rho(z,\varepsilon)|^2-1$. $\tilde{f}_1$ and $\tilde{f}_2$ can clearly be chosen so that 1° is satisfied, while 2° clearly holds for any choice of $\tilde{f}_1$ and $\tilde{f}_2$; 3° is the result of a straightforward computation.

The next result has to do with "extending" differentiable families of complex manifolds to holomorphic families. It will follow from theorem 1. in a manner similar to that for theorems 1.-4. except that X instead of V is to be viewed as the parameter space.

Let $\mathscr{V}\xrightarrow{\omega}X$ be a differentiable (i.e. C^∞) family of complex structures on the complex manifold V in the sense of Kodaira and Spencer [3], where X is an open Riemann surface. This means that for every point $v\in\mathscr{V}$ there is an open neighborhood U of v and a diffeomorphism $\Psi_U: U\to W\times\omega(U)$ for some open set W in $\boldsymbol{C}^n$ such that

1° $\omega = pr^2 \circ \Psi_U$ (pr^2 is the canonical projection $W \times \omega(U) \rightarrow \omega(U)$);

2° $\Psi_U | \Psi_U^{-1}(W \times \{x\})$ is biholomorphic for every $x \in \omega(U)$. If ω is a holomorphic map $\mathscr{V} \xrightarrow{\omega} X$ is called a holomorphic family of complex structures on V.

Two differentiable families of complex structures on V, say $\omega_1 : \mathscr{V} \rightarrow X$ and $\omega_2 : \mathscr{V}_2 \rightarrow X$, are said to be *equivalent* if there is a diffeomorphism $\varphi : \mathscr{V}_1 \rightarrow \mathscr{V}_2$ satisfying

a) $\omega_1 = \omega_2 \circ \varphi$;

b) $\varphi \circ \Psi_U | \Psi_U^{-1}(W \times \{x\})$ is biholomorphic for every $x \in \omega_1(U)$ and every pair U, Ψ_U of open neighborhoods and diffeomorphisms respectively for $\omega_1 : \mathscr{V}_1 \rightarrow X$ as described above.

Let $X_0 \subseteq \bar{X}_0 \subseteq X$ be an open Riemann surface with differentiable boundary ∂X_0. $\mathscr{V} \xrightarrow{\omega} X$ induces by way of the canonical injections $X_0 \rightarrow X$, $\bar{X}_0 \rightarrow X$, and $\partial X_0 \rightarrow X$ differentiable families $\omega_0 : \mathscr{V}_0 \rightarrow X_0$, $\bar{\omega}_0 : \mathscr{V}_0 \rightarrow \bar{X}_0$ and $\omega_0^\delta : \mathscr{V}_0^\delta \rightarrow \partial X_0$ of complex structures on V which are called the *restrictions* of the family $\omega : \mathscr{V} \rightarrow X$ to X_0, $\bar{X}_0$, and ∂X_0 respectively.

THEOREM 6. *Let X_0 and X be open Riemann surfaces with $\bar{X}_0 \subseteq X$ and $\omega : \mathscr{V} \rightarrow X$, $\tilde{\omega} : \tilde{\mathscr{V}} \rightarrow X$ differentiable families of complex structures on a complex manifold V for which*

1) *the restriction $\tilde{\omega}_0 : \tilde{\mathscr{V}}_0 \rightarrow X_0$ is a holomorphic family of complex structures on V;*

2) *the restrictions $\omega_0^\delta : \mathscr{V}_0^\delta \rightarrow \partial X_0$ and $\tilde{\omega}_0^\delta : \tilde{\mathscr{V}}_0^\delta \rightarrow \partial X_0$ are equivalent.*

Then there is a differentiable map $g : \partial X_0 \rightarrow \partial X_0$ such that

$$\tilde{\omega}_0^\delta = g \circ \omega_0^\delta .$$

Lemmas 1. and 2. yield two other kinds of results. The first concerns boundary values of holomorphic functions of one variable. For example, every injective $C^{1+\alpha}$ map $h : \partial D \rightarrow \boldsymbol{C}$ can be embedded in a normalized holomorphic family of $C^{1+\alpha}$ curves $\mathscr{C} : \partial D \times D_p \rightarrow \boldsymbol{C}$; then the property that h^{-1} is the boundary value of a holomorphic function on the bounded domain with boundary $h(\partial D)$ is equivalent to the existence of a holomorphic map of Ω onto D, where Ω is defined for $\mathscr{C}$ as before. The second result concerns partial differential equations.

THEOREM 7. *Let* $\mathscr{C}:\partial D\times D_p\to \boldsymbol{C}$ *be a holomorphic family of* $C^{1+\alpha}$ *curves and* $\Omega\subseteq \boldsymbol{C}^2$ *the domain described by* $\mathscr{C}$ *as before. Let* f_1, f_2 *be complexvalued,* C^∞ *functions on* Ω *with compact support in* $\bar{\Omega}\backslash\mathscr{C}(\partial D\times D_p)$. *If* u *is a* C^∞ *solution of the system*

$$\frac{\partial u}{\partial z}=f_1,\qquad \frac{\partial u}{\partial \varepsilon}=f_2$$

(in which case u *will have a continuous extension to* $\Omega\cup\mathscr{C}(\partial D\times D_p)$*) satisfying the boundary condition that the topological dimension of* $u\circ\mathscr{C}(\{t\}\times D_p)$ *is smaller than or equal to* 1 *for every* $t\in\partial D$, *then on* $\mathscr{C}(\partial D\times D_p)$ u *is necessarily of the form*

$$u(w)=g\circ\tilde{\mathscr{C}}^{-1}(w),$$

where $g:\partial D\to\boldsymbol{C}$ *is a* $C^{1+\alpha}$ *function.*

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