

NOTE ON THE CHARACTERS OF SOLVABLE GROUPS

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§ 1.

Let \mathcal{G} be a solvable group of order g . Let p be a prime and let $g = p^a g'$ with $(p, g') = 1$. In [4] we have tried to find sufficient conditions for \mathcal{G} to possess an irreducible character of p -defect 0, that is, a character whose degree is divisible by p^a .

The following theorem (for arbitrary finite groups) is well-known ([1, (9F)]).

I. If \mathcal{G} possesses an irreducible character of p -defect 0, then \mathcal{G} contains no non-trivial normal p -subgroup.

Now what actually was proved in the proof of the main theorem in [4] (Theorem 1) is the following theorem (cf. [5]).

II. Let \mathcal{G} contain no non-trivial normal p -subgroup. (1) If p is odd and is not a Mersenne prime, then there exist two Sylow p -subgroups \mathfrak{P}_1 and \mathfrak{P}_2 such that $\mathfrak{P}_1 \cap \mathfrak{P}_2 = \mathcal{G}$. (2) (1) also holds for a Mersenne prime p , provided that the order of \mathcal{G} is odd. (3) (1) also holds for $p = 2$, provided that every odd prime divisor q of the order of \mathcal{G} is not a Fermat prime and is congruent to 1 mod 4. (4) (1) also holds for any prime p , provided that elements of order p of a Sylow p -subgroup together with the identity forms a subgroup.

Since then Green ([2]) has proved the following theorem (for arbitrary finite groups).

III. If \mathcal{G} possesses an irreducible character of p -defect 0, then there exist two Sylow p -subgroups \mathfrak{P}_1 and \mathfrak{P}_2 such that $\mathfrak{P}_1 \cap \mathfrak{P}_2 = \mathcal{G}$. Namely, the conclusion of II, (1) holds without any restriction on p .

The following theorem (for arbitrary finite groups) is also well-known ([1, (6G)]).

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IV. If \mathfrak{G} possesses an irreducible character of p -defect 0, then \mathfrak{G} contains an element G of p -defect 0, that is, an element G the order of whose centralizer is prime to p .

On the other hand, the following fact is noticed in [4] (Lemma 1).

V. Let \mathfrak{N} be a normal subgroup of \mathfrak{G} , whose order is prime to p . If \mathfrak{N} contains an element of p -defect 0 (in \mathfrak{G}), then \mathfrak{G} possesses an irreducible character of p -defect 0.

Now the primary concern of [4] was the following proposition.

(#) Under certain circumstances the non-existence of non-trivial normal p -subgroups implies the existence of an irreducible character of p -defect 0.

Unfortunately, in the formulation of Theorem 1 in [4] a strong condition, which is the main driving power of the induction argument in the proof of Theorem 1 in [4] is carelessly not stated.** That is the following condition.

($\mathfrak{P} \rightarrow p$). Let \mathfrak{P} be a fixed Sylow p -subgroup of \mathfrak{G} . Then every element of \mathfrak{G} , which is commutative with no element ($\neq E$) of \mathfrak{P} , has p -defect 0.

Now the purpose of this note is (1) to show that under the condition ($\mathfrak{P} \rightarrow p$), together with the condition in II securing the conclusion of II, (1), (#) always is true, (ii) to state some conditions on the group structure under which (#) is true without assuming the condition ($\mathfrak{P} \rightarrow p$), (iii) to discuss some examples which show the necessity of the condition ($\mathfrak{P} \rightarrow p$), and (iv) to discuss the proof of a theorem to which Theorem 1 of [4] has been applied ([6]).

§ 2.

PROPOSITION 1. Assume that \mathfrak{G} satisfies the condition ($\mathfrak{P} \rightarrow p$). Let G be an element of \mathfrak{G} such that $\mathfrak{P} \cap G^{-1}\mathfrak{P}G = \mathfrak{G}$. Then G is an element of p -defect 0.

Proof. Assume that G is not an element of p -defect 0. Then there exists an element $H (\neq E)$ of \mathfrak{G} which is commutative with G and has order a power of p . By the condition ($\mathfrak{P} \rightarrow p$) we may assume that H belongs to \mathfrak{P} . Then $\mathfrak{P} \cap G^{-1}\mathfrak{P}G$ contains H . This is a contradiction.

PROPOSITION 2. Assume that \mathfrak{G} satisfies the condition ($\mathfrak{P} \rightarrow p$), and that \mathfrak{G} contains no non-trivial normal p -subgroups, and that \mathfrak{G} satisfies the condition in II

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securing the conclusion of II, (1). Let \mathfrak{F} be the Fitting subgroup of \mathfrak{G} . Then \mathfrak{F} contains an element of p -defect 0 (in \mathfrak{G}).

Proof. Consider the subgroup $\mathfrak{F}\mathfrak{P}$. By a theorem of Fitting ([3]) the centralizer of \mathfrak{F} in \mathfrak{G} is contained in \mathfrak{F} . Hence $\mathfrak{F}\mathfrak{P}$ contains no non-trivial normal p -subgroup (of $\mathfrak{F}\mathfrak{P}$). Therefore, by assumption, there exists an element G of $\mathfrak{F}\mathfrak{P}$ such that $\mathfrak{P} \cap \overline{G}\mathfrak{P}G = \mathfrak{E}$. By Proposition 1 G is then an element of p -defect 0.

THEOREM 1. *Assume that \mathfrak{G} satisfies the condition $(\mathfrak{P} \rightarrow p)$, that \mathfrak{G} contains no non-trivial normal p -subgroups, and that \mathfrak{G} satisfies the condition in II securing the conclusion of II, (1). Then \mathfrak{G} possesses an irreducible character of p -defect 0.*

Proof. By Proposition 2 it suffices to apply V to \mathfrak{F} and \mathfrak{G} .

Here a sufficient condition for \mathfrak{G} to secure the property $(\mathfrak{P} \rightarrow p)$ will be noticed.

PROPOSITION 3. *If a Sylow p -complement \mathfrak{H} of \mathfrak{G} is abelian, then \mathfrak{G} satisfies the condition $(\mathfrak{P} \rightarrow p)$.*

Proof. Let G be an element of \mathfrak{G} such that G has order prime to p and commutes with no element ($\neq E$) of \mathfrak{P} . Using P. Hall's theorem we may assume that G belongs to \mathfrak{H} . Let $H \neq E$ be an element of \mathfrak{G} such that H has order a power of p and commutes with G . Then we may write $H = KL$, where K and L are elements of \mathfrak{H} and \mathfrak{P} respectively. Since $GH = HG$ and $GK = KG$, we obtain $GL = LG$. By assumption this implies that $L = E$ and $H = E$. This is a contradiction.

§ 3.

THEOREM 2. *Assume that \mathfrak{G} contains no non-trivial normal p -subgroup, and that \mathfrak{G} satisfies the condition in II securing the conclusion of II, (1). If \mathfrak{G} has nilpotent length 2, then \mathfrak{G} possesses an irreducible character of p -defect 0.*

Proof. By assumption there exists a nilpotent subgroup \mathfrak{N} of \mathfrak{G} such that $\mathfrak{G}/\mathfrak{N}$ is also nilpotent. By assumption the order of \mathfrak{N} is prime to p . Now we apply an induction argument with respect to the order of \mathfrak{G} . If $\mathfrak{G}/\mathfrak{N}$ is not a p -group, \mathfrak{G} contains a proper normal subgroup \mathfrak{H} whose index in \mathfrak{G} is prime to p . By the induction hypothesis \mathfrak{H} possesses an irreducible character ζ of p -defect 0. Let χ be an irreducible component of the

character of \mathcal{G} induced by ζ . Then using Clifford's theorem ([3], p. 565), we see that χ has p -defect 0. Thus we can assume that \mathcal{G}/\mathfrak{N} is a p -group, that is, \mathfrak{N} is a Sylow p -complement of \mathcal{G} .

Let $\mathcal{O}(\mathfrak{N})$ be the Frattini subgroup of \mathfrak{N} . If $\mathcal{O}(\mathfrak{N}) \neq \mathcal{E}$, then consider $\mathcal{G}/\mathcal{O}(\mathfrak{N})$. If $\mathcal{G}/\mathcal{O}(\mathfrak{N})$ contains a non-trivial normal p -subgroup $\mathcal{O}(\mathfrak{N})/\mathcal{O}(\mathfrak{N})$, where $\mathcal{O} \neq \mathcal{E}$ is a p -subgroup of \mathcal{G} , then using Sylow's theorem get $\mathcal{G} = N(\mathcal{O})\mathcal{O}(\mathfrak{N})$, where $N(\mathcal{O})$ is the normalizer of \mathcal{O} in \mathcal{G} . This is a contradiction. Thus we may assume that $\mathcal{O}(\mathfrak{N}) = \mathcal{E}$, which implies that \mathfrak{N} is abelian.

By Proposition 3 and Theorem 1 we now get Theorem 2.

THEOREM 3. *Assume that \mathcal{G} contains no non-trivial normal p -subgroup. If \mathcal{G} is metabelian, then \mathcal{G} possesses an irreducible character of p -defect 0.*

Proof. Take the commutator subgroup of \mathcal{G} as \mathfrak{N} in the proof of Theorem 2. Then we can reduce \mathcal{G} to the case where the Sylow p -complements and Sylow p -subgroups are abelian.

THEOREM 4. *Let \mathcal{G} contain no non-trivial normal p -subgroup. If \mathfrak{P} is cyclic, then \mathcal{G} possesses an irreducible character of p -defect 0.*

Proof. Let \mathfrak{R} be a minimal normal subgroup of \mathcal{G} . By assumption \mathfrak{R} has order prime to p . If \mathcal{G}/\mathfrak{R} contains no non-trivial normal p -subgroup, then applying an induction argument to \mathcal{G}/\mathfrak{R} , we see that \mathcal{G}/\mathfrak{R} , and hence \mathcal{G} , possesses an irreducible character of p -defect 0. So let $\mathcal{O}\mathfrak{R}/\mathfrak{R}$ be a normal subgroup of \mathcal{G}/\mathfrak{R} of order p , where \mathcal{O} is a subgroup of \mathcal{G} order p . By Sylow's theorem $\mathcal{O}\mathfrak{R}$ contains all elements of \mathcal{G} of order p . Let G be an element ($\neq E$) of \mathfrak{R} . If G is not of p -defect 0, then the centralizer of G contains some non-trivial, and hence all elements of \mathcal{O} . Thus G belongs to the center of $\mathcal{O}\mathfrak{R}$. Since \mathfrak{R} is minimal, $\mathcal{O}\mathfrak{R} = \mathcal{O} \times \mathfrak{R}$. Therefore \mathcal{O} is normal in \mathcal{G} . This is a contradiction. Thus G has p -defect 0. Now by V we get Theorem 4.

§ 4.

It is not difficult to construct groups which do not satisfy the condition ($\mathfrak{P} \rightarrow p$). The following examples show, however, that Theorems 2, 3 and 4 are also best possible.

Let p , q and r be distinct prime numbers such that $\frac{q^p - 1}{q - 1} = pr$.

Examples of such triplets are $\{p, q, r\} = \{2, 5, 3\}, \{3, 13, 61\}, \{5, 11, 2331\}, \dots$

First we notice some properties of such triplets.

- (i) $q^p \equiv 1 \pmod{r}$ and $q \not\equiv 1 \pmod{r}$.
- (ii) $q \equiv 1 \pmod{p}$.
- (iii) $r \equiv 1 \pmod{p}$.

Proof. (i) If $q \equiv 1 \pmod{r}$, then $\frac{q^p - 1}{q - 1} = q^{p-1} + \dots + q + 1 \equiv p \equiv 0 \pmod{r}$. This is a contradiction. (ii) By Fermat's theorem $q^{p-1} \equiv 1 \pmod{p}$. Since $q^p \equiv 1 \pmod{p}$, we get that $q \equiv 1 \pmod{p}$. (iii) By Fermat's theorem $q^{r-1} \equiv 1 \pmod{r}$. Thus by (i) we obtain $r \equiv 1 \pmod{p}$.

Let $GF(q)$ and $GF(q^p)$ (containing $GF(q)$) denote the fields of q and q^p elements respectively. Let σ be an element of order p in the Galois group of $GF(q^p)$ over $GF(q)$. By (i) $GF(q^p)$ contains a primitive r -th root of unity ε . Then

$$A = \begin{pmatrix} \varepsilon & & & & \\ & \varepsilon^\sigma & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \varepsilon^{\sigma^{p-1}} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

generate a non-cyclic group of order rp . Since the trace of every matrix of $\langle A, B \rangle$ lies in $GF(q)$, there exists a non-singular matrix V with entries in $GF(q^p)$ such that $A^* = V^{-1}AV$ and $B^* = V^{-1}BV$ have entries in $GF(q)$ ([3], p. 545). By (ii) $GF(q)$ contains a primitive p -th root of unity τ . Then A^* , B^* and $C = \begin{pmatrix} \tau & & \\ & \ddots & \\ & & \tau \end{pmatrix}$ generate a group of order p^2r , which is the direct product of $\langle A^*, B^* \rangle$ and $\langle C \rangle$. Let \mathfrak{G} be the split extension of the p -dimensional vector space \mathfrak{B} over $GF(q)$ by $\langle A^*, B^*, C \rangle$.

\mathfrak{G} is an A -group of order p^2q^pr . \mathfrak{G} has the nilpotent length 3, and the second commutator subgroup of \mathfrak{G} equals \mathfrak{B} which is abelian. \mathfrak{G} contains no non-trivial normal p -subgroup.

Now we show that \mathfrak{G} does not possess an irreducible character of p -defect 0. By IV it is enough to show that \mathfrak{G} does not contain an element of p -defect 0.

Since the Sylow p -subgroups of \mathfrak{G} are not cyclic, there exists an element $V \neq E$ of \mathfrak{B} which is commutative with an element of order p ([3],

p. 502). Since C belongs to the normalizer of $\langle V \rangle$ and does not commute with V , the normalizer of $\langle V \rangle$ has order p^2q^p . If there exists an element V_0 of \mathfrak{B} of p -defect 0, then the normalizer of $\langle V_0 \rangle$ has order pq^p . Since the number of subgroups of order q of \mathfrak{B} equals $\frac{q^p-1}{q-1} = pr$, every subgroup of order q of \mathfrak{B} must be conjugate to $\langle V_0 \rangle$. But certainly $\langle V \rangle$ is not conjugate to $\langle V_0 \rangle$. This shows that there exists no element of \mathfrak{B} of p -defect 0.

§ 5.

Theorem 1 of [4] has been applied to prove the following fact ([6, Proposition 2]). If \mathfrak{G} is an A -group and if G is an element of \mathfrak{G} not belonging to the Fitting subgroup \mathfrak{F} of \mathfrak{G} , then there exists an irreducible character χ of \mathfrak{G} such that $\chi(G) = 0$. We can prove this as follows.

We use an induction argument with respect to the order of the group. Let \mathfrak{M} be a minimal normal subgroup of \mathfrak{G} and let $\mathbf{F}(\mathfrak{M})/\mathfrak{M}$ be the Fitting subgroup of $\mathfrak{G}/\mathfrak{M}$. If G does not belong to $\mathbf{F}(\mathfrak{M})$, then we can apply the induction hypothesis to $G\mathfrak{M}$ and $\mathfrak{G}/\mathfrak{M}$. Hence we may assume that G belongs to $\mathbf{F}(\mathfrak{M})$, which implies that $\mathbf{F}(\mathfrak{M}) \neq \mathfrak{F}$. Thus $\mathbf{F}(\mathfrak{M})$ has nilpotent length 2.

Now let p be a prime divisor of the order of $G\mathfrak{F}$, and let \mathfrak{P} be a Sylow p -subgroup of \mathfrak{F} . Then $\frac{\mathbf{F}(\mathfrak{M})}{\mathfrak{P}}$ contains no non-trivial p -normal subgroup. By Theorem 2 $\frac{\mathbf{F}(\mathfrak{M})}{\mathfrak{P}}$ possesses an irreducible character ζ of p -defect 0. Let χ be an irreducible component of the character of $\frac{\mathfrak{G}}{\mathfrak{P}}$ induced by ζ . Then by a theorem of Clifford ([3, p. 565]) we see that $\chi|_{\mathbf{F}(\mathfrak{M})/\mathfrak{P}}$ decomposes into irreducible characters of $\mathbf{F}(\mathfrak{M})/\mathfrak{P}$ of p -defect 0. Then we get $\chi(G) = 0$ ([1, 6E]).

BIBLIOGRAPHY

- [1] R. Brauer, Zur Darstellungstheorie der Gruppen endlichen Ordnung, I. Math. Zeitschr. **63** (1956), 406–444.
- [2] è.A. Green, Blocks of modular representation, Math. Zeitschr. **79** (1962), 100–115.
- [3] B. Huppert, *Endliche Gruppen*, I. Springer, Berlin-Heidelberg-New York, 1967.
- [4] N. Ito, On the characters of soluble groups, Nagoya Math. J. **3** (1951), 31–48.
- [5] N. Ito, Über den kleinsten p -Durchschnitt auflösbarer Gruppen, Arch. Math. **9** (1958), 27–32.
- [6] N. Ito, Note on A -groups, Nagoya Math J. **4** (1952), 79–81.

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