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Nagoya Math. J.
Vol. 39 (1970), 7–21

SAMPLE PROPERTIES OF WEAKLY STATIONARY PROCESSES

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1. Introduction. Let $X(t) = X(t, \omega)$, $-\infty < t < \infty$, be a stationary stochastic process with

(1. 1) $$EX(t) = 0, \quad E|X(t)|^2 < \infty, \quad -\infty < t < \infty$$

and the continuous covariance function

(1. 2) $$\rho(u) = \int_{-\infty}^{\infty} e^{ixu} dF(x),$$

where $F(x)$ is the spectral distribution function. $X(t)$ then admits the harmonic representation

(1. 3) $$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\xi(\lambda),$$

where $\xi(\lambda)$ is a stochastic process with orthogonal increments and the property that

(1. 4) $$Ed\xi(\lambda) = 0, \quad E|d\xi(\lambda)|^2 = dF(\lambda).$$

Two stochastic processes $X(t)$ and $X_1(t)$ are said to be equivalent to each other, if

$$P(X(t) = X_1(t)) = 1, \quad \text{for each } t.$$

When $X(t)$ is equivalent to a process continuous almost surely or differentiable almost surley, $X(t)$ is called sample continuous or sample differentiable respectively.

One of the authors has shown the following theorem [3].

THEOREM A. *Suppose that for a given weakly stationary process $X(t)$ there is a function $g(x)$ which is even, non-negative and non-decreasing for $x>0$ and is such that*

(1. 5) $$\sum_{n=1}^{\infty} \frac{1}{g(n)} < \infty,$$

Received May 28, 1969. This work partly supported by *NSF* Grant 9396.

(1. 6) $$\int_{-\infty}^{\infty} g(x)dF(x) < \infty.$$

Then $X(t)$ is sample continuous.

The condition (1. 6) with $g(x) = |x|(\log^+|x|)^\beta$, $\beta > 1$, implies the condition

(1. 7) $$\varphi(h) = O(h/|\log|h||^r) \quad \text{for} \quad r > 2,$$

as $h \to 0$, where $\varphi(h) = 2\rho(o) - \rho(h) - \rho(-h)$ ([3] (3. 8) and (3. 9)). This generalizes the Cramér-Leadbetter's result on sample continuity of a weakly stationary process ([1], p. 125).

In **2**, we shall give the conditions which assure the sample differentiability of a process. We can adopt the method for the proof similar to what we did proving Theorem A, namely we make use of the approximate Fourier series [3] [6] associated with a given weakly stationary process. In **3**, we shall show that the same reasoning still applies to get the "sample Hölder property".

In the paper of one of the authors [2], Theorem A was motivated by a theorem on the absolute convergence of the Fourier series of a given process truncated at $-T$ and T. But it involved some erroneous argument although the theorem itself is right, and the different method using the approximate Fourier series was employed to prove Theorem A in [3]. In **4**, it is shown that the original way of proving is effective if some modifications are made with a slight additional condition on $g(x)$.

Finally we mention that the conditions on the existence of $g(x)$ in Theorem A are also necessary for all the weakly stationary processes with a given spectral distribution $F(x)$ to be sample continuous. This has been shown by I. Kubo [4] and will be given in a separate forthcoming paper.

2. Sample differentiability of a weakly stationary process.

M. Loève [5] studied the sample differentiability of a weakly stationary process and proved among others the following theorem.

THEOREM B. *If the covariance function $\rho(u)$ of a weakly stationary process $X(t)$ with* (1. 1) *and* (1. 2), *is $(2n+2)$-times differentiable, then $X(t)$ is sample n-times differentiable.*

Cramér and Leadbetter [1] generalized this result to obtain Theorem C below.

Write

$$\Delta_u^{2k}\rho(-ku)=\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\rho((k-j)u), \tag{2. 1}$$

where k is a non-negative integer.

THEOREM C. *If the covariance function* $\rho(u)$ *of a weakly stationary process* $X(t)$ *with* (1. 1) *and* (1. 2) *satisfies*

$$\Delta_u^{2n+1}\rho(-ku)=O(|u|^{2n+1}/|\log|u||^q), \quad as \quad u\to 0 \quad for \quad q>3, \tag{2. 2}$$

then $X(t)$ *is sample n-times differentiable.*

This is a slight completion of the Cramér-Leadbetter's result. They actually have shown Theorem C for the case $n=0, 1$.

The aim of this section is to generalize Theorem C further.

In association with a given weakly stationary process $X(t)$ with (1. 1), (1. 2) and the representation (1. 3), we define a sequence of uncorrelated random variables

$$\xi_n=\xi_n(T)=\int_{2n\pi/T}^{2(n+1)\pi/T}d\xi(\lambda), \qquad n=0, \pm 1, \cdots, \tag{2. 3}$$

where T is any positive number. We also define

$$\hat{X}(t,T)=\hat{X}(t)=\sum_{n=-\infty}^{\infty}e^{2\pi it/T}\xi_n. \tag{2. 4}$$

Actually ξ_n's are uncorrelated because of the orthogonality of the increments of $\xi(\lambda)$ and (2. 4) is well-defined, the series being interpreted to converge in L^2-norm.

However, we have shown in [3] and [4] that under the conditions either in Theorem A or in Lemma 3 below, the series in (2. 4) is absolutely convergent almost surely and hence $X(t)$ may be identified to be the sum of the series. Also it was shown that *in this case* $\hat{X}_k(t)=\hat{X}(t,2^k)$ *converges uniformly for every finite interval* $|t|\leq A$ *as* $k\to\infty$ *almost surely to a weakly stationary process* $X_0(t)$, *which is sample continuous, and is equivalent to* $X(t)$.

LEMMA 1. *If*

$$\sum_{n=-\infty}^{\infty}|n|^r|\xi_n|<\infty \tag{2. 5}$$

almost surely, where r is a positive integer, then $X(t)$ is equivalent to a weakly stationary process with the almost sure continuous r-th derivative.

Proof. Since the series on the right of (2. 4) is absolutely and uniformly convergent almost surely, because of (2. 5), we may suppose that $X(t)$ itself is represented by the series in (2. 4) for every t almost surely, and has the continuous r-th derivative almost surely. We shall, however, prove Lemma 1 when $r=1$. r repetitions of the same argument give us the required.

$$\frac{\hat{X}(t+h)-\hat{X}(t)}{h}=\sum_{n=-\infty}^{\infty}\left[\frac{e^{2n\pi ih/T}-1}{h}e^{2n\pi it/T}\xi_n\right]. \tag{2. 6}$$

The series on the right is dominated in absolute value by $(2\pi/T)\sum|n||\xi_n|$ almost surely and since each term converges as $h\to 0$, the limit of (2. 6) as $h\to 0$ should exist and $\hat{X}'(t)$ is given by $\frac{2\pi i}{T}\sum_{n=-\infty}^{\infty}e^{2n\pi it/T}n\xi_n$, which is continuous almost surely. Generally $\hat{X}^{(r)}(t)$ is given by $\left(\frac{2\pi i}{T}\right)^r\sum_{n=-\infty}^{\infty}e^{2n\pi it/T}n^r\xi_n$.

LEMMA 2. *Let $h(x)$ be non-negative and non-decreasing over $[0,\infty)$ and let $F(x)$ be a spectral distribution. Then the inequalities*

$$\begin{aligned}\frac{1}{2}\sum_{n\neq 0}h\Big(\frac{|n|-1}{a}\Big)(F(n+1)-F(n))^{1/2}+\frac{1}{2}h(0)(F(1)-F(0))^{1/2}\le\\ \le\sum_n h(|n|)(F(a(n+1))-F(an))^{1/2}\le\\ \le\Big(\frac{1}{a}+1\Big)^{1/2}\sum_n h\Big(\frac{|n|+1}{a}\Big)(F(n+1)-F(n))^{1/2}\end{aligned} \tag{2. 7}$$

hold for $0<a<1$.

Proof. Since $\sqrt{x+y}\le\sqrt{x}+\sqrt{y}$ for $x,y\ge 0$, we have

$$\begin{aligned}I&=\sum_n h(|n|)(F(a(n+1))-F(an))^{1/2}\\ &=\sum_k\Big[h\Big(\Big|\Big[\frac{k}{a}\Big]\Big|\Big)\Big(F\Big(a\Big[\frac{k}{a}\Big]+a\Big)-F\Big(a\Big[\frac{k}{a}\Big]\Big)\Big)^{1/2}+\sum_{n=[k/a]+1}^{[(k+1)/a]-1}h(|n|)(F(a(n+1))-F(an))^{1/2}\Big]\\ &\le\sum_k\Big[h\Big(\Big|\Big[\frac{k}{a}\Big]\Big|\Big)\Big(F(k)-F\Big(a\Big[\frac{k}{a}\Big]\Big)\Big)^{1/2}+h\Big(\Big|\Big[\frac{k}{a}\Big]\Big|\Big)\Big(F\Big(a\Big[\frac{k}{a}\Big]+a\Big)-F(k)\Big)^{1/2}+\\ &\quad+\sum_{n=[k/a]+1}^{[(k+1)/a]-1}h\Big(\frac{|k|+1}{a}\Big)\Big(F(a(n+1))-F(an)\Big)^{1/2}\Big]\\ &\le\sum_k h\Big(\frac{|k|+1}{a}\Big)\Big(\frac{1}{a}+1\Big)^{1/2}\Big(F(k+1)-F(k)\Big)^{1/2}\end{aligned}$$

The last inequality is obtained by Schwarz inequality. Since $2\sqrt{x+y} \geqq \sqrt{x} + \sqrt{y}$ for $x, y \geqq 0$, we have similarly

$$2I \geqq \sum_k \Big[h\Big(\Big[\frac{k+1}{a}\Big]\Big)\Big(F(k+1)-F\Big(a\Big[\frac{k+1}{a}\Big]\Big)\Big)^{1/2} + h\Big(\Big[\frac{k}{a}\Big]\Big)\Big(F\Big(a\Big[\frac{k}{a}\Big]+a\Big)-F(k)\Big)^{1/2} +$$

$$+ \sum_{n=[k/a]+1}^{[(k+1)/a]-1} h\Big(\frac{|k|-1}{a}\Big)(F(a(n+1)-F(an))^{1/2}\Big] \geqq$$

$$\geqq \sum_k h\Big(\frac{|k|-1}{a}\Big)(F(k+1)-F(k))^{1/2},$$

with the agreement that $h(u) = h(0)$ for $u \leqq 0$.

LEMMA 3. *If the spectral distribution function F of a given stationary process $X(t)$ satisfies*

$$\sum_n |n|^r (F(n+1)-F(n))^{1/2} < \infty \tag{2. 8}$$

for a non-negative integer r, then (2. 5) *holds almost surely.*

Proof. In order to show (2. 5) it is sufficient to prove

$$E \sum_{n=-\infty}^{\infty} |n|^r |\xi_n| < \infty. \tag{2. 9}$$

By Lemma 2, we have that, for $0 < T \leqq 2\pi$,

$$E\sum |n|^r |\xi_n| \leqq \sum |n|^r \Big[E\Big|\int_{2n\pi/T}^{2(n+1)\pi/T} d\xi(\lambda)\Big|^2\Big]^{1/2} = \tag{2. 10}$$

$$= \sum |n|^r \Big(F\Big(\frac{2(n+1)\pi}{T}\Big) - F\Big(\frac{2n\pi}{T}\Big)\Big)^{1/2} \leqq$$

$$\leqq \Big(\frac{T}{2\pi}+1\Big)^{1/2} \sum \Big[\frac{T(|n|+1)}{2\pi}\Big]^r (F(n+1)-F(n))^{1/2} < \infty.$$

If $T \geqq 2\pi$, then (2. 9) follows from the first inequality of (2. 7).

It is easy to show that $\hat{X}^{(r)}(t)$ is a weakly stationary process, observing $\sum |n|^{2r}(F(2(n+1)\pi/T) - F(2n\pi/T)) < \infty$.

Now we shall prove

THEOREM 1. *If a weakly stationary process $X(t)$ with* (1. 1) *and* (1. 2) *satisfies* (2. 8), *then $X(t)$ is equivalent to a weakly stationary process which has the continuous r-th derivative almost surely.*

Proof. First we prove the theorem for $r = 1$. Denote the differential quotients of $X(t)$ and $\hat{X}(t)$ by

(2. 11) $$D(t,h)=\frac{X(t+h)-X(t)}{h},$$

(2. 12) $$\hat{D}(t,h)=\frac{\hat{X}(t+h)-\hat{X}(t)}{h}$$

respectively. From Lemma 1, the series in (2. 4) is absolutely convergent and $\hat{X}(t)$ may be supposed to be defined by this series. By Lemma 1 and Lemma 3, $\hat{X}(t)$ has the continuous derivative almost surely.

Write $\xi_{n,k}$ for ξ_n with $T=2^k$, $\hat{X}_k(t)$ for the corresponding $\hat{X}(t)$, k being a positive integer. Then

$$\hat{X}_{k+1}(t)-\hat{X}_k(t)=\sum_{n=-\infty}^{\infty}\exp\Big(\frac{2n\pi i t}{2^{k+1}}\Big)\xi_{n,k+1}-\sum_{m=-\infty}^{\infty}\exp\Big(\frac{2m\pi i t}{2^k}\Big)\xi_{m,k}=$$
$$=\sum_{m=-\infty}^{\infty}\Big[\exp\Big(\frac{2\pi i(2m)t}{2^{k+1}}\Big)\xi_{2m,k+1}+$$
$$+\exp\Big(\frac{2\pi i(2m+1)t}{2^{k+1}}\Big)\xi_{2m+1,k+1}-\exp\Big(\frac{2m\pi i t}{2^k}\Big)\xi_{m,k}\Big].$$

Since $\xi_{m,k}=\xi_{2m,k+1}+\xi_{2m+1,k+1}$, we may write

(2. 13) $$\hat{X}_{k+1}(t)-\hat{X}_k(t)=$$
$$=\sum_{m=-\infty}^{\infty}\Big[\exp\Big(\frac{2\pi i(2m+1)t}{2^{k+1}}\Big)-\exp\Big(\frac{2\pi i(2m)t}{2^{k+1}}\Big)\Big]\xi_{2m+1,k+1}.$$

Write $\hat{D}_k(t,h)$ for the differential quotient of $\hat{X}_k(t)$.

Together with the relation for $\hat{X}_k(t+h)$ similar to (2. 13) and noting that, for $|t|\leq A, A$ being a positive number,

(2. 14) $$|e^{iy(t+h)}-e^{iyt}-e^{iz(t+h)}+e^{izt}|\leq$$
$$\leq|(e^{iyt}-e^{izt})(e^{iyh}-1)|+|e^{izt}(e^{iyh}-e^{izh})|\leq$$
$$\leq 4\Big|\sin\frac{y-z}{2}t\ \sin\frac{yh}{2}\Big|+2\Big|\sin\frac{y-z}{2}h\Big|\leq|h||y-z|\cdot(1+A|y|),$$

we obtain

(2. 15) $$|\hat{D}_{k+1}(t,h)-\hat{D}_k(t,h)|\leq$$
$$\leq\Big|\sum_{m=-\infty}^{\infty}\Big[\exp\Big(\frac{\pi i(2m+1)(t+h)}{2^k}\Big)-\exp\Big(\frac{\pi i(2m)(t+h)}{2^k}\Big)-$$
$$-\exp\Big(\frac{\pi i(2m+1)t}{2^k}\Big)+\exp\Big(\frac{\pi i(2m)t}{2^k}\Big)\Big]\xi_{2m+1,k+1}\Big|\leq$$
$$\leq\sum_{m=-\infty}^{\infty}\Big(\frac{\pi}{2^k}+\frac{A\pi^2(2|m|)}{2^k}\Big)|\xi_{2m+1,k+1}|.$$

Therefore we can see by Lemma 2 that for any $\varepsilon_k > 0$

(2. 16)
$$Q_k \equiv P(\sup_{|t|\leq A, h\neq 0} |\hat{D}_{k+1}(t,h) - \hat{D}_k(t,h)| > \varepsilon_k) \leq$$
$$\leq \frac{1}{\varepsilon_k}\frac{\pi}{2^k}\sum_m\Big(1+\frac{2A|m|}{2^k}\Big)E|\xi_{2m+1,k+1}| \leq$$
$$\leq \frac{1}{\varepsilon_k}\frac{\pi}{2^k}\sum_m\Big(1+\frac{2A|m|}{2^k}\Big)\Big(F\Big(\frac{(2m+1)\pi}{2^k}\Big)-F\Big(\frac{2m\pi}{2^k}\Big)\Big)^{1/2}\leq$$
$$\leq \frac{1}{\varepsilon_k}\frac{\pi}{2^k}\sum_n\Big(1+\frac{A|n|}{2^k}\Big)\Big(F\Big(\frac{(n+1)\pi}{2^k}\Big)-F\Big(\frac{n\pi}{2^k}\Big)\Big)^{1/2}\leq$$
$$\leq \frac{1}{\varepsilon_k}\frac{(2^k+\pi)^{1/2}}{2^k\pi^{1/2}}\sum_n(1+A(|n|+1))\,(F(n+1)-(F(n))^{1/2}\leq$$
$$\leq \frac{1}{\varepsilon_k}\frac{C_1}{2^{k/2}}[\sum_n|n|(F(n+1)-F(n))^{1/2}+C_2(F(1)-F(0))^{1/2}],$$

where C_1 and C_2 are constants independent of k. In what follows C_j, $j=3,4,\cdots$, mean some constants independent of k. From (2. 8), it follows that

(2. 17)
$$Q_k \leq \frac{1}{\varepsilon_k}\frac{C_3}{2^{k/2}}\,.$$

If ε_k is chosen to be $2^{-k/4}$, then $\sum\varepsilon_k<\infty$ and $\sum\frac{1}{2^{k/2}\varepsilon_k}<\infty$, so that $\sum Q_k<\infty$. Then Borel-Cantelli lemma gives us that, with probability one

(2. 18)
$$\sup_{h\neq 0,\,|t|\leq A}|\hat{D}_{k+1}(t,h)-\hat{D}_k(t,h)|<\varepsilon_k$$

except for a finite number of k. Hence almost surely $\hat{D}_k(t,h)$ converges as $k\to\infty$ uniformly for $|t|\leq A$ and h.

Now from (2. 18) we have, for k larger than some k_0,

$$\sup_{|t|\leq A}|\hat{D}_k(t,h)-\hat{D}_m(t,h)|\leq\eta_k \qquad \text{(unifomly in } h)$$

almost surely, where $\eta_k=\sum_{j=k}^{\infty}\varepsilon_j$. From the italicized statement before Lemma 1, we have, letting $m\to\infty$

(2. 19)
$$\Big|\hat{D}_k(t,h)-\frac{1}{h}[X_0(t+h)-X_0(t)]\Big|\leq\eta_k \quad \text{(uniformly in } h)$$

almost surely. Let $h\to 0$. Then from Lemma 1 and 2 with $r=1$, $\hat{D}_k(t,h)$ converges almost surely and hence $X_0(t)$ is differentiable almost surely, and is equivalent to $X(t)$.

Finally (2. 19) implies that the derivative $\hat{X}_k'(t)$ of $\hat{X}_k(t)$ converges uniformly to the derivative $X_0'(t)$ of $X_0(t)$. Since Lemmas 1 and 2 give us that $\hat{X}'_k(t)$ is continuous almost surely, $X'_0(t)$ is also sample continuous for every $|t| \leq A$. This proves the theorem for the case $r=1$.

Repeating similar arguments, the general case is shown.

THEOREM 2. *If, for a given weakly stationary process* $X(t)$ *with* (1. 1) *and* (1. 2), *there is a function* $g(x)$, $-\infty < x < \infty$, *which is non-negative, even and non-decreasing for* $x \geqq 0$ *and satisfies*

$$\sum_{n=1}^{\infty} \frac{n^{2r}}{g(n)} < \infty, \tag{2. 20}$$

$$\int g(x)\, dF(x) < \infty, \tag{2. 21}$$

then $X(t)$ *is equivalent to a weakly stationary process which has the continuous r-th derivative almost surely.*

Proof. By Schwarz inequality,

$$\Big[\sum_{n=0}^{\infty} |n|^r (F(n+1)-F(n))^{1/2}\Big]^2 \leq \sum_{n=0}^{\infty} \frac{n^{2r}}{g(n)} \sum_{n=0}^{\infty} g(n)(F(n+1)-F(n))$$
$$\leq \sum_{n=0}^{\infty} \frac{n^{2r}}{g(n)} \int_0^{\infty} g(x)\, dF(x) < \infty.$$

Similarly, we have $\sum_{n=-1}^{-\infty} |n|^r (F(n+1)-F(n))^{1/2} < \infty$. By Theorem 1, the proof is completed.

EXAMPLE 1. If, for some $\varepsilon > 0$ and $B > 0$

$$\int_{B<|x|} |x|^{2r+1} \log|x| \cdot \log_{(2)}|x| \cdots \log_{(n)}|x| \cdot (\log_{(n+1)}|x|)^{1+\varepsilon} dF(x) < \infty \tag{2. 22}$$

holds, then $X(t)$ is sample r-times differentiable, where $\log_{(1)} x = \log x$ and $\log_{(n+1)} x = \log(\log_{(n)} x)$ for $n \geqq 1$.

EXAMPLE 2. Suppose that $F(x)$ is absolutely continuous with the density $f(x)$. If

$$|f(x)| \leq [|x|^{r+1} \log|x| \cdot \log_{(2)}|x| \cdots \log_{(n)}|x| \cdot (\log_{(n+1)}|x|)^{1+\varepsilon}]^{-2} \tag{2. 23}$$

holds for sufficiently large $|x|$ with some $\varepsilon > 0$, then $X(t)$ is sample r-times differentiable.

EXAMPLE 3. Besides the same assumption in Example 2, further suppose that $f(x)$ is non-decreasing as $x \to \pm\infty$. If

$$\int |x|^r f^{1/2}(x)dx < \infty \tag{2. 24}$$

holds, then $X(t)$ is sample r-times differentiable.

3. Sample Hölder continuity.

Let $\Psi(h)$ be a non-decreasing function defined over an interval $(0,1]$ such that $\Psi(h)$ decreases to zero as h does. If a function $f(x)$ on (a,b) satisfies

$$|f(t+h) - f(t)| \leq M\Psi(h) \tag{3. 1}$$

for t, $t+h \in (a,b)$, $|h| < 1$ with some M, then it is said to be *Ψ-Hölder continuous.*

We are going to give sufficient conditions which assure the sample Ψ-Hölder continuity of a weakly stationary process. The method similar to the one applied to the proofs of Theorems 1 and 2 is also applicable.

LEMMA 4. *Let $\Psi(h)$ be a non-decreasing function over $(0,1]$ such that $\Psi(h)/h$ is non-increasing. Then for $0 < h \leq 1$*

$$|\sin xh| \leq \Psi(h)/\Psi(x^{-1}), \quad \text{for} \quad x \geqq 1, \tag{3. 2}$$

$$|\sin xh| \leq \Psi(h)/\Psi\left(\frac{1}{x+1}\right), \quad \text{for} \quad x \geqq 0. \tag{3. 3}$$

Proof. If $0 < xh < 1$, then

$$|\sin xh| \leq xh = \frac{\Psi(h)}{\Psi(x^{-1})} \cdot \frac{\Psi(x^{-1})}{x^{-1}} \cdot \frac{h}{\Psi(h)} \leq \frac{\Psi(h)}{\Psi(x^{-1})}.$$

If $xh \geqq 1$, then since $\Psi(h)$ is non-decreasing

$$|\sin xh| \leq 1 \leq \Psi(h)/\Psi(x^{-1}).$$

Similarly, we can prove (3. 3) observing $xh \leq h(x+1)$.

LEMMA 5. *If $\Psi(h)$ is non-decreasing and $\Psi(h)/h$ is non-increasing over $(0,1]$, and*

$$\sum_{n \neq 0} \Psi^{-1}\left(\frac{1}{|n|}\right)|\xi_n| < \infty \tag{3. 4}$$

almost surely, then $\hat{X}(t)=X(t,T)$, $T>\pi$, *is sample* Ψ*-Hölder continuous where* ξ_n *is defined by* (2. 3) *and* $\hat{X}(t)$ *is defined by* (2. 4).*

Proof. Using Lemma 4, we have

$$|X(t+h)-X(t)| \le \sum_{n\neq 0} |\sin nh\pi/T|\,|\xi_n| \le \tag{3. 5}$$

$$\le \Psi\Big(\Big|\frac{h\pi}{T}\Big|\Big)\sum_{n\neq 0}\Psi^{-1}\Big(\frac{1}{|n|}\Big)\ |\xi_n| \le \Psi(|h|)\sum_{n\neq 0}\Psi^{-1}\Big(\frac{1}{|n|}\Big)|\xi_n|.$$

LEMMA 6. *If the spectral distribution function* $F(x)$ *satisfies*

$$\sum_{n\neq 0}\Psi^{-1}\Big(\frac{1}{|n|}\Big)(F(n+1)-F(n))^{1/2}<\infty. \tag{3. 6}$$

Then (3. 4) *holds almost surely, where* $\Psi(h)$ *is the function in Lemma* 4. *Hence* $X(t)$ *is* Ψ*-Hölder continuous almost surely.*

The proof is carried out as in that of Lemma 3.

THEOREM 3. *If a given weakly stationary process* $X(t)$ *satisfies* (3. 6), *then* $X(t)$ *is equivalent to a weakly stationary process which is* Ψ*-Hölder continuous almost surely, where* $\Psi(h)$ *is the function in Lemma* 4.

The proof is very similar to that for Theorem 1. Write

$$D_\Psi(t,h)=\frac{X(t+h)-X(t)}{\Psi(|h|)},\quad \hat{D}_{\Psi,k}(t,h)=\frac{\hat{X}_k(t+h)-\hat{X}_k(t)}{\Psi(|h|)},$$

where $\hat{X}_k(t)$ is, as before, defined by (2. 4) with $T=2^k$. Using the same notations as in the proof of Theorem 1, we have, analogously to (2. 15), by Lemma 4 and (2. 14),

$$|\hat{D}_{\Psi,k+1}(t,h)-\hat{D}_{\Psi,k}(t,h)| \le$$

$$\le \frac{1}{\Psi(|h|)}\sum_{m=-\infty}^{\infty}\Big(4\Big|\sin\frac{\pi t}{2^{k+1}}\sin\frac{2m\pi h}{2^{k+1}}\Big|+2\Big|\sin\frac{\pi h}{2^{k+1}}\Big|\Big)|\xi_{2m+1,k+1}|$$

$$\le \sum_{m=-\infty}^{\infty}\Big(\frac{\pi}{2^k}\Psi^{-1}(1)+\frac{2\pi A}{2^k}\Psi^{-1}\Big(\frac{2^{k+1}}{2\pi|m|+2^{k+1}}\Big)\Big)|\xi_{2m+1,k+1}|,$$

for $|t|\le A$. Therefore we obtain by Lemma 2

$$Q'_k \equiv P\Big(\sup_{0<|h|\le 1,\,|t|\le A}|\hat{D}_{\Psi,k+1}(t,h)-\hat{D}_{\Psi,k}(t,h)|>\varepsilon_k\Big)\le$$

$$\le \frac{1}{\varepsilon_k}\frac{\pi}{2^k}\sum_{m=-\infty}^{\infty}\Big(\Psi^{-1}(1)+2A\Psi^{-1}\Big(\frac{2^{k+1}}{2\pi|m|+2^{k+1}}\Big)\Big)\Big(F\Big(\frac{(2m+1)\pi}{2^k}\Big)-F\Big(\frac{2m\pi}{2^k}\Big)\Big)^{1/2}$$

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*) $\Psi^{-n}(x)=(\Psi(x))^{-n}$.

$$\leq \frac{1}{\varepsilon_k} \frac{\pi}{2^k} \sum_{n=-\infty}^{\infty} \Big(\Psi^{-1}(1) + 2A\Psi^{-1}\Big(\frac{2^{k+1}}{\pi|n| + 2^{k+1}}\Big)\Big)\Big(F\Big(\frac{(n+1)\pi}{2^k}\Big) - F\Big(\frac{n\pi}{2^k}\Big)\Big)^{1/2} \leq$$

$$\leq \frac{1}{\varepsilon_k} \frac{\pi}{2^k} \sqrt{\frac{2^k}{\pi} + 1} \sum_{n=-\infty}^{\infty} \Big(\Psi^{-1}(1) + 2A\Psi^{-1}\Big(\frac{2}{|n| + 3}\Big)\Big) (F(n+1) - F(n))^{1/2}.$$

Since $\Psi^{-1}\Big(\frac{2}{|n|+3}\Big) \leq \frac{(|n|+3)}{|n|} \Psi^{-1}\Big(\frac{1}{|n|}\Big)$ and $\Psi\Big(\frac{1}{|n|}\Big) \leq \Psi(1)$ for $n \neq 0$,

we get

$$Q'_k \leq \frac{1}{\varepsilon_k} \frac{C_4}{2^{k/2}} [(F(1) - F(0))^{1/2} + \sum_{n \neq 0} \Psi^{-1}\Big(\frac{1}{|n|}\Big) (F(n+1) - F(n))^{1/2}] \leq \tag{3.7}$$

$$\leq \frac{1}{\varepsilon_k} \frac{C_5}{2^{k/2}}.$$

Choosing ε_k as in the proof of Theorem 1, we see from (3. 7) that $\hat{X}_k(t)$ converges uniformly to a weakly stationary process $X_0(t)$ and

$$\Big|\hat{D}_{\Psi,k}(t,h) - \frac{\hat{X}_0(t+h) - \hat{X}_0(t)}{\Psi(|h|)}\Big| \leq \varepsilon_k \quad \text{for} \quad k \geqq k_0.$$

By Lemma 5, $\sup_{0<|h|\leq 1, |t|\leq A} |\hat{D}_{\Psi,k}(t,h)| < \infty$ almost surely, we conclude that $X_0(t)$ is Ψ-Hölder continuous for $|t| \leq A$ for any $A > 0$, which completes the proof.

THEOREM 4. *If for a given weakly stationary process $X(t)$, there is an even, non-negative, non-decreasing function $g(x)$ such that*

$$\sum_{n=1}^{\infty} \Psi^{-2}\Big(\frac{1}{n}\Big) \cdot g^{-1}(n) < \infty, \tag{3.8}$$

$$\int g(x) dF(x) < \infty. \tag{3.9}$$

Then $X(t)$ is equivalent to a weakly stationary process which is Ψ-Hölder continuous almost surely, where $\Psi(h)$ is the function in Lemma 4.

Proof. By (3. 8) and (3. 9), we have

$$\Big[\sum_{n=1}^{\infty} \Psi^{-1}\Big(\frac{1}{n}\Big) (F(n+1) - F(n))^{1/2}\Big]^2 \leq \sum_{n=1}^{\infty} \Psi^{-2}\Big(\frac{1}{n}\Big) g^{-1}(n) \cdot \sum_{n=1}^{\infty} g(n)(F(n+1) - F(n))$$

$$\leq \sum_{n=1}^{\infty} \Psi^{-2}\Big(\frac{1}{n}\Big) g^{-1}(n) \cdot \int_1^{\infty} g(x) dF(x) < \infty.$$

Similarly, we can see that $\sum_{n=-1}^{-\infty} \Psi^{-1}\left(\frac{1}{|n|}\right)(F(n+1)-F(n))^{1/2} < \infty$. Hence the assertion follows from Theorem 3.

EXAMPLE 4. Suppose that $F(x)$ is absolutely continuous with the density $f(x)$ and that $f(x)$ is non-increasing as $x \to \pm\infty$. If

$$\int_{|x|>1} \Psi^{-1}\left(\frac{1}{|x|}\right) f^{1/2}(x)dx < \infty, \tag{3.10}$$

then $X(t)$ is sample Ψ-Hölder continuous.

EXAMPLE 5. If a separable stationary process $X(t)$ satisfies

$$\int_{|x|\geq B} \Psi^{-2}\left(\frac{1}{|x|}\right)|x|\cdot\log|x|\cdot\log_{(2)}|x|\cdots\log_{(n)}|x|\cdot(\log_{(n+1)}|x|)^{1+\varepsilon}dF(x) < \infty, \tag{3.11}$$

for sufficiently large $B > 0$ with $\varepsilon > 0$, then

$$\lim_{h\to 0}\sup_{|t|\leq A} \frac{|X(t+h)-X(t)|}{\Psi(h)} = 0 \quad \text{a.s..}$$

Especially if, $F(x)$ is absolutely continuous with the density $f(x)$ which satisfies

$$|f(x)| \leq \Psi^2\left(\frac{1}{|x|}\right)[|x|\cdot\log|x|\cdots\log_{(n)}|x|\cdot(\log_{(n+1)}|x|)^{1+\varepsilon}]^{-2},$$

then (3.11) holds.

4. Absolute convergence of the Fourier series of a weakly stationary process.

Let $X(t)$ be a weakly stationary process described in 1. Let T be any positive number. Define

$$Y(t) = \begin{cases} X(t), & t \geq 0, \\ X(-t), & t \leq 0. \end{cases} \tag{4.1}$$

We consider the Fourier series of $Y(t)$ over $(-T,T)$,

$$A_n = \frac{1}{T}\int_{-T}^{T} Y(t)\cos\frac{n\pi t}{T}\,dt, \tag{4.2}$$

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi t}{T}. \tag{4.3}$$

As in [2],

$$(4.4)\qquad EA_n\bar{A}_m = 8e^{i\pi(n-m)/2}\int_{-\infty}^{\infty}\frac{\sin\dfrac{\lambda T+\pi n}{2}\sin\dfrac{\lambda T+\pi m}{2}\cdot\lambda^2T^2}{(\lambda^2T^2-n^2\pi^2)(\lambda^2T^2-m^2\pi^2)}dF(\lambda),$$

$$(4.5)\qquad E|A_n|^2 = 8\int_{-\infty}^{\infty}\frac{\sin^2\left(\dfrac{\lambda T+n\pi}{2}\right)\cdot\lambda^2T^2}{(\lambda^2T^2-n^2\pi^2)^2}dF(\lambda).$$

THEOREM 5. *Let $g(x)$ be even, non-negative and non-decreasing for $x>0$, such that $g(x)/x^2$ is non-increasing for large x and*

$$(4.6)\qquad \sum_{n=1}^{\infty}\frac{1}{g(n)}<\infty.$$

If

$$(4.7)\qquad \int_{-\infty}^{\infty}g(x)dF(x)<\infty,$$

then $\sum_{0}^{\infty}|A_n|$ converges almost surely.

Proof. We may suppose that $g(x)/x^2$ is non-decreasing over $(0,\infty)$. In fact, if $g(x)/x^2$ is non-increasing for $x\geqq B$, then we may define $g(x)$ as it is for $(x\geqq B)$, and $g(B)(x/B)^2$ for $(x\leqq B)$. By (4. 5),

$$\sum_{n=2}^{\infty}g(n)E|A_n|^2 = 8\int_{-\infty}^{\infty}\sum_{n=2}^{\infty}g(n)\frac{\sin^2\left[\frac{1}{2}(|\lambda|T-n\pi)\right]\lambda^2T^2}{(|\lambda|T+\pi n)^2(|\lambda|T-\pi n)^2}dF(\lambda) =$$

$$= 8\int_{|\lambda|>\pi/T}\Big(\sum_{n\geq[|\lambda|T/\pi]+2}\Big)dF(\lambda) + 8\int_{|\lambda|>\pi/T}\Big(\sum_{n=[|\lambda|T/\pi]}^{[|\lambda|T/\pi]+1}\Big)dF(\lambda)$$

$$+ 8\int_{|\lambda|>2\pi/T}\Big(\sum_{[|\lambda|T/\pi]-1\geq n}\Big)dF(\lambda) + 8\int_{|\lambda|\leq\pi/T}\Big(\sum_{n=2}^{\infty}\Big)dF(\lambda) =$$

$$= I_1+I_2+I_3+I_4,$$

say.

Noting that

$$(4.8)\qquad g(Ax)\leq A^2g(x),$$

for $A>1$ $x\geqq 1$ which follows from the assumption that $g(x)/x^2$ is non-increasing for $x\geqq 1$, we see that

$$I_1 \leq 8 \int_{|\lambda|>\pi/T} \sum_{n \geq [|\lambda|T/\pi]+2} \frac{g(n)\lambda^2 T^2}{\pi^2\left(n - \frac{|\lambda|T}{\pi} - 1\right)^2 (|\lambda|T + \pi n)^2} dF(\lambda) \leq$$

$$\leq 8C_1 \int_{|\lambda|\geq\pi/T} g\left(\frac{\lambda T}{\pi}\right) dF(\lambda) < C_2 T^2 \int_{|\lambda|\geq\pi/T} g(\lambda) dF(\lambda) < \infty,$$

where C_1 and C_2 are constants. Here we have used that

$$g(n)\lambda^2 T^2/(|\lambda|T + \pi n)^2 \leq g(\lambda T/\pi).$$

$$I_2 \leq C_3 \int_{|\lambda|>\pi/T} \left[g\left(\frac{\lambda T}{\pi} + 1\right) + g\left(\frac{\lambda T}{\pi}\right)\right] dF(\lambda) \leq C_4 T^2 \int_{|\lambda|>\pi/T} g(\lambda) dF(\lambda) < \infty,$$

C_3, C_4 being constants.

$$I_3 \leq \int_{|\lambda|>\pi/T} \sum_{n \leq [\lambda T/\pi]-1} \frac{\lambda^2 T^2 g([\lambda T/\pi] - 1)}{(|\lambda|T - \pi n)^2 (|\lambda|T + 2)^2} dF(\lambda) \leq$$

$$\leq C_5 \sum_{m=1}^{\infty} \frac{1}{m^2} \int_{|\lambda|>\pi/T} g\left(\frac{\lambda T}{\pi}\right) dF(\lambda) \leq C_6 T^2 \int_{|\lambda|\geq\pi/T} g(\lambda) dF(\lambda) < \infty,$$

where C_5 and C_6 are constants.

Since $\lambda^2 T^2 (|\lambda|T - \pi n)^2 \leq (n-1)^{-2}$ for $|\lambda|T \leq \pi$,

$$I_4 \leq 8 \sum_{n=2}^{\infty} \frac{g(n)}{\pi^2 n^2 (n-1)^2} \int_{|\lambda|\leq\pi/T} dF(\lambda).$$

Since $g(n) \leq n^2 g(1)$ from (4. 8),

$$I_4 < \infty.$$

Hence we have obtained that

$$\sum_{n=2}^{\infty} g(n) E|A_n|^2 < \infty. \tag{4. 9}$$

From this, our conclusion follows immediately, for

$$E\sum_{n=2}^{\infty} |A_n| = E\sum_{n=2}^{\infty} \frac{1}{g^{1/2}(n)} g^{1/2}(n)|A_n| \leq$$

$$\leq \left[\sum_{n=2}^{\infty} \frac{1}{g(n)}\right]^{1/2} E\left[\sum_{n=2}^{\infty} g(n)|A_n|^2\right]^{1/2} \leq$$

$$\leq \left[\sum_{n=2}^{\infty} \frac{1}{g(n)} \left[E\sum_{n=2}^{\infty} g(n)|A_n|^2\right]^{1/2}\right.$$

which is finite by (4. 6) and (4. 8), and $E\sum|A_n|<\infty$ implies the almost sure convergence of $\sum|A_n|$.

As an implication of the conclusion of Theorem 5 is that $X(t)$ *is sample continuous in* $(0, T)$ *for every* $T>0$ which, of course, implies that $X(t)$ is sample continuous in $(0, \infty)$. However, for this statement we need the unnecessary condition that $g(x)x^2$ is non-decreasing.

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