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THE CENTERS OF SEMI-SIMPLE ALGEBRAS OVER A COMMUTATIVE RING, II

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Dedicated to Professor Keizo Asano on his 60th birthday

In this note we assume that all rings have identities and denote by R a commutative ring. All R-algebras considered are assumed to be finitely generated faithful R-modules. An R-algebra Λ is said to be semi-simple ([5]), if any finitely generated Λ-module is (Λ, R)-projective. Further Λ is said to be weakly semi-simple, if for any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is semi-simple over $R/\mathfrak{m}$. Especially a (weakly) semi-simple R-algebra is called (weakly) simple if it is indecomposable as a ring. It was shown in [5] that any semi-simple R-algebra is weakly semi-simple. Formal properties of (weakly) semi-simple R-algebras were studied in [5], [6], [3], etc. The purpose of this note is, as a continuation to [3], to solve negatively the following basic problems on semi-simple R-algebras:

(I) *Is any central semi-simple R-algebra, which is a projective R-module, separable?*

(II) *Let Λ be a semi-simple R-algebra which is a projective R-module. Is the center of Λ also semi-simple over R?*

(III) *Is any weakly semi-simple R-algebra, which is a projective R-module, semi-simple over R?*

1. We have proved in [3], (2. 1) that the answer to (I) is affirmative for any Artinian ring R. Now we give a negative answer to (I) in case R is a discrete (rank-one) valuation ring in the following

THEOREM 1. *Let R be a discrete (rank-one) valuation ring with a maximal ideal $\mathfrak{p}$. We assume that the characteristic of $R/\mathfrak{p}$ is $p > 0$. If the characteristic of R is zero, we further assume that R contains the primitive p-th root ζ_p of 1. Then the following conditions are equivalent for R:*

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(1) $[R/\mathfrak{p} : (R/\mathfrak{p})^p] \geqq p^2$.

(2) *There exists a central simple, non-separable R-algebra which is a free R-module.*

In [2], (8. 1) we gave the affirmative answer to [III] for any Dedekind domain R. Hence Theorem 1 is an immediate consequence of the following more general

THEOREM 2. *Let R be a local integral domain with a maximal ideal $\mathfrak{m} \neq 0$ and K the quotient field of R. We assume that the characteristic of $R/\mathfrak{m}$ is $p > 0$. If the characteristic of R is 0, we further assume that R contains the primitive p-th root ζ_p of 1. Then the following conditions are equivalent for R:*

(1) $[R/\mathfrak{m} : (R/\mathfrak{m})^p] \geqq p^2$.

(2) *There exists a central weakly semi-simple, non-separable R-algebra Λ, which is a free R-module, such that $K \otimes_R \Lambda$ is separable over K.*

We can observe in the proof of [3], (2. 3) that, if the answer to (II) is affirmative for any Artinian ring R, then the answer to (I) is also affirmative for any Noetherian ring R. Therefore, by Theorem 1, the answer to (II) is negative for an Artinian ring R.

Hence we have only to give the proof of Theorem 2 and a counter-example to (III).

2. We shall give here the proof of Theorem 2. The implication (1) $\Longrightarrow$ (2). Suppose that $[R/\mathfrak{m} : (R/\mathfrak{m})^p] \geqq p^2$. Then there exist elements $\bar{\alpha}, \bar{\beta} \in (R/\mathfrak{m})^{1/p}$ such that $[R/\mathfrak{m}(\bar{\alpha}, \bar{\beta}) : R/\mathfrak{m}] = p^2$. We put $\bar{a} = \bar{\alpha}^p$ and $\bar{b} = \bar{\beta}^p$ and denote by a, b the representatives of $\bar{a}$, $\bar{b}$ in R, respectively. Now it suffices to construct a central weakly simple, non-separable R-algebra Λ, which is a free R-module, such that $K \otimes_R \Lambda$ is separable over K, in each of the following cases:

(i) R is of characteristic p.

(ii) R is of characteristic 0 and contains the primitive p-th root, ζ_p, of 1.

Let X, Y be two indeterminates and u a non-zero element in $\mathfrak{m}$. We put $F(X) = X^p - u^{p-1}X - a$ in Case (i) and $F(X) = X^p - a$ in Case (ii). Then we have $F(X)K[X] \cap R[X] = F(X)R[X]$, since $F(X)$ is monic, and therefore, putting $L = K[X]/F(X)K[X]$ and $S = R[X]/F(X)R[X]$, L is the total quotient ring of S. Let α be the residue of X in L. As is well known, L is a Galois extension of K whose Galois group, G, is a cyclic group of order p which is generated by σ such that $\sigma(\alpha)=\alpha+u$ in Case (i) and $\sigma(\alpha)=\alpha\zeta_p$ in Case (ii). Obviously σ operates on S as an automorphism over R and the subring of S consisting of all elements in S fixed under G coincides with R. Let $S[Y]$ be the non-commutative polynomial ring S such that $s^{\sigma}Y=Ys$ for any $s\in S$ and put $\Lambda=S[Y]/(Y^p-b)S[Y]$ and $\sum=K\otimes_R\Lambda$. Then Λ is a free R-module, and $\sum$ is a central separable K-algebra because it is a crossed product. Since $S/\mathfrak{m}S\cong R[\alpha]/\mathfrak{m}R[\alpha]\cong R/\mathfrak{m}[X]/(X^p-\bar{a})R/\mathfrak{m}[X]\cong R/\mathfrak{m}[\alpha]$, σ induces the identity on $S/\mathfrak{m}S$, so that $\Lambda/\mathfrak{m}\Lambda$ is commutative. Hence we observe $\Lambda/\mathfrak{m}\Lambda\cong(R/\mathfrak{m}[\bar{\alpha}])[Y]/(Y^p-\bar{b})(R/\mathfrak{m}[\alpha])[Y]$. Because, by the assumption, $Y^p-\bar{b}$ is irreducible in $(R/\mathfrak{m}[\bar{\alpha}])[Y]$, we have $\Lambda/\mathfrak{m}\Lambda=R/\mathfrak{m}[\bar{\alpha},\bar{\beta}]$. Thus Λ is a central weakly simple, non-separable R-algebra as required.

The implication (2) $\Longrightarrow$ (1). If $R/\mathfrak{m}$ is perfect, then by [4], (1. 1), any weakly semi-simple R-algebra is separable. Therefore it suffices to prove, under the assumption that $[R/\mathfrak{m}:(R/\mathfrak{m})^p]=p$, that any central weakly simple R-algebra Λ, which is a free R-module, such that $K\otimes_R\Lambda$ is separable over K is separable over R. Let Λ be a central weakly simple R-algebra, which is a free R-module, such that $K\otimes_R\Lambda$ is separable over K. By using the Henselization of R, we may suppose that $\Lambda/\mathfrak{m}\Lambda$ is a division $R/\mathfrak{m}$-algebra. Let $\bar{C}$ be a maximal commutative subfield of $\Lambda/\mathfrak{m}\Lambda$, and put $n^2=\dim_{R/\mathfrak{m}}\Lambda/\mathfrak{m}\Lambda=\mathrm{rank}_R\Lambda$ and $m=\dim_{R/\mathfrak{m}}\bar{C}$. Then it is well known that $n\leqq m$. However, since $[R/\mathfrak{m}:(R/\mathfrak{m})^p]=p$, we have $\bar{C}=R/\mathfrak{m}[\bar{\alpha}]$ for some $\bar{\alpha}\in\bar{C}$. If we denote by α a representative of $\bar{\alpha}$ in Λ, then $\{1,\alpha,\alpha^2,\cdots,\alpha^{n-1}\}$ is a subset of a free R-basis of Λ. But, since the degree of the reduced characteristic polynomial of α in $K\otimes_R\Lambda$ is equal to n by [4], § 3, we find that $\alpha^n\in K+K\alpha+\cdots+K\alpha^{n-1}$. From this it follows that $m\leqq n$, i.e., $m=n$. Thus the center of $\Lambda/\mathfrak{m}\Lambda$ coincides with $R/\mathfrak{m}$, so that $\Lambda/\mathfrak{m}\Lambda$ is a central simple $R/\mathfrak{m}$-algebra. Again by [4], (1. 1), we know that Λ is separable over R. This completes the proof of the theorem.

We note that, in Theorem 2, the hypothesis that R is a local integral domain with a maximal ideal $\mathfrak{m}\neq 0$ can be replaced by the weaker one that R is a local ring whose maximal ideal, $\mathfrak{m}$, contains a non-zero divisor in R and that the implication (2) $\Longrightarrow$ (1) in Theorem 2 was proved without assuming that R contains the primitive p-th root of 1 in case it is of characteristic 0.

Also it should be noted that there exists a discrete valuation ring R which satisfies the condition (1) in Theorem 1 (or 2). In fact let T_1, T_2 and U be three indeterminates and p a prime integer. Now we put $k = Z/pZ(T_1, T_2)$. Then the formal power series ring $k[[U]]$ over k with U is a discrete valuation ring of equi-characteristic $p > 0$ which satisfies the condition (2) in Theorem 1.

3. We shall give a negative answer to (III) by exhibiting an example of a commutative weakly simple, non-simple algebra over a regular local ring R with Krull dimension 2 which is a free R-module.

Let R_0 be a regular local ring of characteristic 3 with Krull dimension 2 and $\mathfrak{m}_0 = uR_0 + vR_0$ the unique maximal ideal of R_0. Let T, X be indeterminates. If we put $R = R_0[T]_{\mathfrak{m}_0 R_0[T]}$ and $\mathfrak{m} = \mathfrak{m}_0 R$, then R is also a regular local ring with a maximal ideal $\mathfrak{m}$. Further put $F(X) = X^3 - uX^2 + 2vX - T \in R[X]$ and $S = R[X]/F(X)R[X]$. Then we have $S/\mathfrak{m}S \cong R/\mathfrak{m}[X]/(X^3 - T)\, R/\mathfrak{m}[X] \cong R/\mathfrak{m}[T^{1/3}]$, and therefore S is weakly simple over R. An element $v^3 + u^2v^2 - u^3T$ of R is prime, because it is prime in $R_0[T]$ and contained in $\mathfrak{m}$. Therefore, putting $\mathfrak{p} = (v^3 + u^2v^2 - u^3T)R$, $\mathfrak{p}$ is a prime ideal of height 1 in R. Now we denote by $\bar{u}$, v the residues of u, v in $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and by $\bar{F}(X)$ the residue of $F(X)$ in $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]$. Then we have $d\bar{F}(X)/dX = -2(\bar{u}X - v)$ and $\bar{F}(v/\bar{u}) = \frac{1}{\bar{u}^3}(v^3 + \bar{u}^2v^2 - \bar{u}^3T) = 0$ and so $\bar{F}(X)$ has a multiple root $v/\bar{u}$. Hence, since $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]/\bar{F}(X)R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]$, $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ is not simple over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. This implies that $S_{\mathfrak{p}}$ is not simple over $R_{\mathfrak{p}}$. Accordingly S is not a simple R-algebra. Thus S is as required.

Let R be a Noetherian integrally closed integral domain with quotient field K and Λ an R-order, which is a projective R-module, in a separable K-algebra. For such Λ we consider the following statements:

(1) Λ is weakly semi-simple over R.

(2) For any prime ideal $\mathfrak{p}$ of height 1 in R, $\Lambda_{\mathfrak{p}}$ is semi-simple over $R_{\mathfrak{p}}$.

The above example means that (1) $\Longrightarrow$ (2) is not always valid. We suppose that Λ satisfies one of the following conditions:

(i) R is a regular domain and Λ is commutative.

(ii) Λ has R as its center.

Then it is well known (cf. [7], (41. 1) and [1], (4. 6)) that Λ is separable over R whenever, for any prime ideal $\mathfrak{p}$ of height 1 in R, $\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}$. Finally we shall show that (2) does not imply (1) generally even in case Λ satisfies (i) or (ii).

Let k be a field of characteristic 2 and T, U, V, X, Y be indeterminates. Now we put $R = k[T, U, V]_{Uk[T,U,V]+Vk[T,U,V]}$ and $\mathfrak{m} = UR + VR$. Then R is a regular local ring with a maximal ideal $\mathfrak{m}$ and with Krull dimension 2, and we have $R/\mathfrak{m} \cong k(T)$. Further put $F(X) = X^2 - UX - V$ and $L = K[X]/F(X)K[X]$. Since $F(X)$ is a monic polynomial of the Artin-Schreier type whose coefficients are contained in R, L is a Galois extension of K and $F(X)K[X] \cap R[X] = F(X)R[X]$. Therefore, putting $S = R[X]/F(X)R[X]$, L can be considered as the quotient field of S. Obviously S is a regular local ring with a maximal ideal $\mathfrak{n} = XS + US$ and $S/\mathfrak{n} \cong R/\mathfrak{m} \cong k(T)$. It can be easily seen that $S/\mathfrak{m}S \cong R/\mathfrak{m}[X]/X^2R/\mathfrak{m}[X]$, and so S is not weakly semi-simple over R. However, for a prime ideal $\mathfrak{p} = UR$ of R, we have $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]/(X^2 - V)R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X] = k(T, V^{1/2})$, so that $S_{\mathfrak{p}}$ is simple over $R_{\mathfrak{p}}$. On the other hand, for any prime ideal $\mathfrak{q} \not\ni U$ of height 1 in R, the residue $\bar{F}(X)$ of $F(X)$ in $R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}[X]$ is a separable polynomial and $S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} \cong R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}[X]/\bar{F}(X)R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}[X]$, so that $S_{\mathfrak{q}}$ is separable over $R_{\mathfrak{q}}$. Thus a commutative R-algebra S satisfies (2) but does not satisfy (1). If we denote the residue of X in L by x, then the Galois group of a Galois extension L of K is generated by σ such that $\sigma(x) = x + 1$. Hence σ operates on S as an automorphism over R and the subring of S consisting of all elements of S fixed under σ coincides with R. Let $S[Y]$ be the non-commutative polynomial ring over S with Y such that $s^{\sigma}Y = Ys$ for any $s \in S$ and put $\Lambda = S[Y]/(Y^2 - T)S[Y]$. Then Λ is a central R-algebra which is a free R-module and $K \underset{R}{\otimes} \Lambda$ is separable over K. It can easily be shown that Λ is not weakly semi-simple over R but, for any prime ideal $\mathfrak{p}$ of height 1 in R, $\Lambda_{\mathfrak{p}}$ is simple over $R_{\mathfrak{p}}$.

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