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INTRODUCTION OF A BASIC THEORY OF OBJECTS

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Introduction

In constructing various kind of mathematical theories on the basis of a common basic theory, it has been very usual to take up the set theory as the common basic theory. This approach has been already successful to a certain extent and looks like successfully developable in the future not only in constructing mathematical theories standing on the classical logic but also in constructing formal theories standing on weaker logics. In constructing mathematical theories standing on the classical logic, it has been successful in most cases only by interpreting mathematical notions in the set theory without defining any special interpretation of logical notions. In constructing any mathematical theory standing on weaker logics such as the intuitionistic logic, however, we have to give a special interpretation for logical notions, too.

As it has been my opinion that the basic theory should be as simple and natural as possible, I have tried another approach. I have taken up an extremely simple logic called the primitive logic as our basic theory. It was amazing for me to know that any finitely axiomatizable formal theory standing on either the classical logic or the intuitionistic logic can be constructed on the primitive logic without presupposing any assumption such as axioms. (See my paper [1].) We can establish even intermediate logics in the same way if we restrict ourselves to the proposition logics. (See my paper [2].) This looks like to suggest that more vast class of formal theories including almost all the important mathematical theories are reducible to the primitive logic. Simply speaking, our only problem is how to axiomatize each formal theory in a finite number of axioms.

According to my opinion, however, we are trying to construct formal theories in some basic theory because we are seeking after something universal

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behind various kind of formal theories. It is true that the primitive logic can be regarded as a universal basis of all the really important formal theories, but it looks like undesirable to try to axiomatize finitely each formal theories case by case. We should rather proceed as far as possible following a common high-way in axiomatizing them finitely. Can we not proceed a common high-way a little further than the primitive logic before branching into special formal theories? Is not there a big mile-stone theory at the end of this common high-way? I believe, I have found out one, which is a basic theory of objects standing on the minimal logic.

This theory resembles formal set theories, but much weaker than the usual set-theory. The new theory of objects can be regarded as an improvement of the theory which I have developed in my very old paper [3]. The new theory is so formulated that it can give a common basis for finite axiomatizations of so-called axiom-schematic formal theories. As we usually develop formal theories by making use of logical notions "implication", "universal quantification", "conjunction", "disjunction", "negation", and "existential quantification", I have adopted the minimal logic as the weakest logic among logics having all these logical notions.

A typical example of an axiom-schema would be the defining axiom of functions in any theory Φ having equality notion. Namely, if equality is assumed or defined in Φ, we can define function $f(x, \cdots, z)$ by any proposition $\underset{\sim}{A}(w, x, \cdots, z)$ satisfying the conditions

$$(x)\cdots(z)(\exists w)\underset{\sim}{A}(w, x, \cdots, z),$$
$$(x)\cdots(z)(u)(v)(\underset{\sim}{A}(u, x, \cdots, z) \wedge \underset{\sim}{A}(v, x, \cdots, z). \longrightarrow u = v).$$

For any proposition of this kind, we assert

$$(\exists f)(x)\cdots(z)(w)(w = f(x, \cdots, z). \equiv \underset{\sim}{A}(w, x, \cdots, z)).$$

What kind of propositions are admissible in place of $\underset{\sim}{A}(w, x, \cdots, z)$ becomes clear only after we know what kind of notions can be used in Φ. I will call the whole class of primitive notions of a theory the *primitive vocabulary* of the theory. To axiomatize any axiom-schematic theory, it is desirable to introduce such kind of formula $p(x, \cdots, z)$ that satisfies

$$\text{(A)} \quad (\exists p)(x)\cdots(z)(p(x, \cdots, z) \equiv \underset{\sim}{A}(x, \cdots, z))$$

for any proposition $\underset{\sim}{A}(x, \cdots, z)$ expressible in terms of the primitive vocabulary of the theory. Naturally, $p(x, \cdots, z)$ would not be expressible in terms of the primitive vocabulary.

Taking up this situation into our consideration, I will introduce our object theory depending on a certain primitive vocabulary. Namely, I will call here any finite set of relations $R_1, \cdots, R_s$ primitive vocabulary only when the number of places, say $n_i (i = 1, \cdots, s)$, of each relation R_i is fixed. The relations R_i are called ***primitive notions*** of the primitive vocabulary $\{R_1, \cdots, R_s\}$. Now, let $\underset{\sim}{V}$ be any primitive vocabulary. Then, any proposition expressible exclusively in terms of the primitive vocabulary $\underset{\sim}{V}$ is called a ***$\underset{\sim}{V}$-proposition***, and any theory dealing with exclusively $\underset{\sim}{V}$-propositions is called a ***$\underset{\sim}{V}$-theory***.

Now, let $\underset{\sim}{W}$ be a richer primitive vocabulary than $\underset{\sim}{V}$, i.e. $\underset{\sim}{W} \supseteq \underset{\sim}{V}$. Then, any $\underset{\sim}{W}$-theory Φ is called a *literal extension* of a $\underset{\sim}{V}$-theory $\sum$ if and only if the following condition holds: "Any $\underset{\sim}{V}$-proposition is provable in $\sum$ if and only if it is provable in Φ".

Any $\underset{\sim}{V}$-theory Φ is called *axiomatically* stronger than another $\underset{\sim}{V}$-theory $\sum$ if and only if Φ is stronger than $\sum$ by a finite number of axioms. Any $\underset{\sim}{V}$-theory Φ is called *axiom-schematically* stronger than another $\underset{\sim}{V}$-theory $\sum$ if and only if Φ is stronger than $\sum$ by a finite number of axiom-schemata which are expressible in terms of the metanotations of the form $\underset{\sim}{A}(x, \cdots, z)$ for $\underset{\sim}{V}$-propositions together with the usual notations. For expressions of this kind, we assume here that $\underset{\sim}{A}(x, \cdots, z)$ is defined by a $\underset{\sim}{V}$-proposition $\underset{\sim}{A}(u, \cdots, w)$ for a suitable sequence of mutually distinct variables $u, \cdots, w$ which do not occur in $\underset{\sim}{A}(x, \cdots, z)$. (See Remarks 2.9 and 2.13.) Any $\underset{\sim}{V}$-theory is called *purely logical* if and only if it has no axiom. Any $\underset{\sim}{V}$-theory which is *axiomatically* (or *axiom-schematically*) stronger than the purely logical $\underset{\sim}{V}$-theory is called *axiomatic* (or *axiom-schematic*.)

Naturally, the notion "purely logical" depends on what kind of logic we are adopting. In this paper, I adopt the minimal predicate logic **LM**. The logic **LM** is nicely interpreted by taking up a proposition constant λ and by regarding $\sim \underset{\sim}{A}$ as $\underset{\sim}{A} \longrightarrow \lambda$. According to this interpretation, any $\underset{\sim}{V}$-theory standing on the intuitionistic predicate logic **LJ** without having further axioms is stronger than the purely logical $\underset{\sim}{V}$-theory by the axiom-schema

$(\lambda) \quad \lambda \longrightarrow \underset{\sim}{A},$

and any $\underline{V}$-theory standing on the classical predicate logic is stronger than the purely logical $\underline{V}$-theory by the axiom-schemata (λ) and

(P) $((\underline{A} \longrightarrow \underline{B}) \longrightarrow \underline{A}) \longrightarrow \underline{A}$.

The basic theory of objects for any primitive vocabulary $\underline{V}$ to be introduced in this paper is an axiomatic theory for which the followings are expected:

(E1) This basic object theory is an extension of the purely logical $\underline{V}$-theory,

(E2) Any axiom-schematic V-theory has a *literal extention* which is *axiomatically* stronger than this basic object theory.

According to the terminology introduced here, my former result (cf. [1]) can be stated as follows:

Any axiomatic $\underline{V}$-theory can be purely logically constructed on the primitive logic.

If the basic object theory can be introduced just as we have expected, it would be a remarkable milestone on the common high-way at the branching point for various places constructing axiom-schematic theories on the minimal logic **LM.** We can introduce any axiom-schematic theory standing on the classical logic or the intuitionistic logic by taking up the axiom-schema (λ) or, by taking up simultaneously the axiom-schemata (λ) and (**P**), respectively.

In (1), the basic theory of objects depending on a certain vocabulary $\underline{V}$ is introduced. The object theory is denoted by $B(\underline{V})$. To introduce the basic theory $B(\underline{V})$, we take up two more primitive notions S and T from outside of the vocabulary $\underline{V}$, where S represents a binary relation of the form $p(x)$ (read: "x is p") and T represents a three-placed relation of the form $x<y,z>$ (read: "x is the ordered pair of y and z"). Now, let us denote by $\underline{V}^+$ the union set of $\underline{V}$ and $\{S,T\}$. The basic object theory $B(\underline{V})$ is a $\underline{V}^+$-theory, which is expected to be a literal extension of the purely logical $\underline{V}$-theory. $B(\underline{V})$ is also expected to have an axiomatically stronger $\underline{V}^+$-theory for any axiom-schematic $\underline{V}$-theory, the former being a literal extension of the latter.

The relations $p(x)$ and $x\langle y,z\rangle$ are introduced for the purpose of expressing any $\underline{V}$-proposition of the form $\underline{A}(u,\cdots,w)$ by a proposition of the form $p(u,\cdots,w)$, which stands for $p(<u,\cdots,w>)$ as usual. However, we have to interpret further what $<u,\cdots,w>$ means here. Surely, it can be interpreted as $<u, <\cdots, w>>$, but we have to introduce a usage of terms of the form $<\underline{u}, \underline{w}>$, where $\underline{u}$ and $\underline{v}$ stand for some other terms. I use this kind of symbols, for example, $p(\underline{u}, \underline{w})$ as standing for $(\exists z)(p(z) \wedge z<\underline{u}, \underline{w}>)$. We need only ordered pairs of the form $<\underline{u}, \underline{v}>$ as our terms, but, to develop the basic object theory $B(\underline{V})$ nicely, it is far more convenient to use term symbols. In (2), I describe the same basic object theory by making use of terms in introducing even some suitable inference rules for proposition containing terms. I will denote this basic theory by $B[\underline{V}]$.

The object theory $B[\underline{V}]$ deals with generalized $\underline{V}^+$-propositions containing terms. Any generalized $\underline{V}^+$-proposition can be interpreted by a V^+-proposition in such way that any generalized $\underline{V}^+$-proposition is provable in $B[\underline{V}]$ if and only if its interpretation is provable in $B(\underline{V})$. This theorem is proved also in (2). According to this result, we have no need to deal with these theories separatedly. We denote them by the same notation $B(\underline{V})$ thereafter.

In (3), we describe a theory which makes it clear in $B(\underline{V})$ that there exists such p that

$$(u)\cdots(w)(p(u, \cdots, w) \equiv \underline{A}(u, \cdots, w))$$

for any $\underline{V}$-proposition $\underline{A}(u, \cdots, w)$.

According to this result, we can embed any axiom-schematic $\underline{V}$-theory in a theory which is axiomatically stronger than $B(\underline{V})$. This is explained in (4) by some examples. However, our description of (4) and (5) of this paper is merely suggestive. Indeed, we have introduced $B(\underline{V})$ expecting the followings:

(A) $B(\underline{V})$ is an extension of the purely logical $\underline{V}^+$-theory,

(B) Any axiom-schematic $\underline{V}$-theory has a literal extension which is axiomatically stronger than $B(\underline{V})$.

We do not prove these completely in this paper. In fact, in $\underline{V}$-theories, we might have the equality notion and the notion of ordered pairs already. In such cases, it is desirable that the newly defined equality notion and the newly introduced ordered pair notion would match very well with the original ones. These are problems to be discussed later.

In (5), I will give some concluding remarks, especially in connection with tabooistic construction of mathematical theories. According to the conclusion of my paper [1], any theory can be constructed on the primitive logic tabooistically if the theory can be introduced by a finite number of axioms standing on the intuitionistic logic, the classical logic, or the minimal logic. However, I believe that there is a good reason to proceed axiomatically in a unified way as far as possible in constructing mathematical theories. I will discuss these matters in the concluding remarks.

(1) The theory $B(\underline{V})$

Let $\underline{V}$ be any finite set of predicates which is regarded as a primitive vocabulary. I will denote it by

$$\underline{V} = \{R_1, \cdots, R_s\},$$

where R_i is an n_i-place predicate for each i $(n_i \geqq 0)$, and a 0-ary predicate λ is assumed to belong to $\underline{V}$. Let us further assume that a binary predicate S and a three-place predicate T are taken from outside of the vocabulary $\underline{V}$. The notation $p(x)$ is used mostly in place of $S(p, x)$ and the notation $x<y, z>$ in place of $T(x, y, z)$. I will define the vocabulary by

$$\underline{V}^+ = \{R_1, \cdots, R_n, S, T\}.$$

Any proposition expressible in terms of the primitive notions of a vocabulary $\underline{W}$ is called a $\underline{W}$*-proposition.*

The theory we are going to introduce here is an axiomatic $\underline{V}^+$-theory standing on the intuitionistic predicate logic of the first order without assuming the negation notion. I will denote this logic hereafter by **LP** (*positive logic*). Before introducing the axioms of the theory $B(\underline{V})$, however, I will define the *term* and *term expression* notions (terms and term expressions are denoted by lower case underlined letters) and the *equality* notion (notation: =) used in this theory to make the description simpler.

DEFINITION 1.1. Any finite sequence of symbols is called a *term* if and only if it can be confirmed to be a term by the following two rules:

T1. Any variable is a term.

T2. For any pair of terms $\underline{x}$ and $\underline{y}$, $<\underline{x}, \underline{y}>$ is also a term.

DEFINITION 1.2. Any sequence of symbols is called a *term expression* if and only if it can be confirmed to be a term expression by the following rules:

TE1. Any term is a term expression.

TE2. For any sequence of term expressions $\underline{t}_1, \cdots, \underline{t}_n (n \geq 1)$, $<\underline{t}_1, \cdots, \underline{t}_n>$ is a term expression.

Definition 1.3. Any term expression $\underline{t}$ is called reducible to a sequence $\underline{s}$ of symbols if and only if either of the following conditions holds for $\underline{s}$ and $\underline{t}$:

RTE1. The sequence $\underline{s}$ of symbols is obtained from the term expression $\underline{t}$ on replacing its term expression part of the form $<\underline{u}>$ for any term expression $\underline{u}$ by the term expression $\underline{u}$ itself.

RTE2. The sequence of symbols is obtained from the term expression $\underline{t}$ on replacing its term expression $<\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_n>$ for any sequence $\underline{u}_1, \cdots, \underline{u}_n$ of term expressions by the term expression $<\underline{u}_1, <\underline{u}_2, \cdots, \underline{u}_n>>$ $(n \geq 3)$.

Theorem 1.4. *If any term expression* $\underline{t}$ *is reducible to a sequence* $\underline{s}$ *of symbols,* $\underline{s}$ *is also a term expression.*

Definition 1.5. Any (finite or infinite) sequence $\underline{t}_1, \underline{t}_2, \cdots$ of term expressions is called a *reduction sequence* if and only if $\underline{t}_{i+1}$ is a reduced term expression of the term expression $\underline{t}_i$ for every i.

Theorem 1.6. *The chain condition holds for reduction sequences of term expressions. In other words, every reduction sequence of term expressions is finite.*

Definition 1.7. Any reduction sequence is called *complete* if and only if it ends with a term.

Theorem 1.8. *For any term expression* $\underline{t}$, *there is a complete reduction sequence beginning with* $\underline{t}$.

Definition 1.9. Any pair of term expressions $\underline{s}$ and $\underline{t}$ are called *cofinal* if and only if there is a pair of complete reduction sequences beginning with $\underline{s}$ and $\underline{t}$, respectively, and ending with a common term.

Theorem 1.10. *Any term expression is confinal with one and only one term.*

Definition 1.11. The number of pairs of the brackets “$<\ \ >$” occurring in a term is called the rank of the term. By the rank of a term expression, we understand the rank of the term which is cofinal with the term expression.

It is convenient to use terms as well as term expressions in describing the basic object theory $B(\underline{V})$. Accordingly, I will define here the notion $[\underline{V}^+]$-propositions containing term expressions.

DEFINITION 1.12. Any finite sequence of symbols is called a $[\underline{V}^+]$-proposition if and only if it can be confirmed to be a $[\underline{V}^+]$-proposition by the following two rules:

$[\underline{V}^+]1$. For any sequence of term expressions $\underline{t}_1, \cdots, \underline{t}_{n_i}$, the expression $R_i(\underline{t}_1, \cdots, \underline{t}_{n_i})$ is a $[\underline{V}^+]$-proposition. For any variable p and a term expression $p(\underline{t})$ is a $[\underline{V}^+]$-proposition. For any variable x and a pair of term expressions $\underline{y}$ and $\underline{z}$, the expression $x<\underline{y}, \underline{z}>$ is a $[\underline{V}^+]$-proposition.

$[\underline{V}^+]2$. For any $[\underline{V}^+]$-proposition $\underline{F}$, any expression of the forms $(x)(\underline{F})$ and $(\exists x)(\underline{F})$ is a $[\underline{V}^+]$-proposition, x being any variable. For any pair of $[\underline{V}^+]$-propositions $\underline{F}$ and $\underline{G}$, any expression of the forms $(\underline{F}) \longrightarrow (\underline{G})$, $(\underline{F}) \wedge (\underline{G})$, $(\underline{F}) \vee (\underline{G})$, and $(\underline{F}) \equiv (\underline{G})$ is a $[\underline{V}^+]$-proposition. Parentheses can be spared if there is no fear of ambiguity.

DEFINITION 1.13. For any pair of term expressions $\underline{x}$ and $\underline{y}$, $\underline{x} = \underline{y}$ is defined by

$$\underline{x} = \underline{y} \equiv (p)(p(\underline{x}) \equiv p(\underline{y})).$$

DEFINITION 1.14. Any formula of the form $p(\underline{x}, \cdots, \underline{z})$ for any variable p and any sequence of term expressions $\underline{x}, \cdots, \underline{z}$ stands for $p(<\underline{x}, \cdots, \underline{z}>)$.

As we deal with $[\underline{V}^+]$-propositions having term expressions in the theory $B[\underline{V}]$, we can not say that we deal with exclusively $\underline{V}^+$-propositions in $B[\underline{V}]$. To every $[\underline{V}^+]$-proposition $\underline{F}$ in $B[\underline{V}]$, however, we can associate a $\underline{V}^+$-proposition $\underline{F}'$. Accordingly, we can regard $\underline{F}$ as standing for $\underline{F}'$, and we can regard $B[\underline{V}]$ as a $\underline{V}^+$-theory by this interpretation. To introduce the association of $[\underline{V}^+]$-propositions to $\underline{V}^+$-propositions, we introduce here the notation of the form $(u, \underline{t})$ for any variable u and any term $\underline{t}$.

DEFINITION 1.15. For any variable u and any term $\underline{w}$, the notation $(u, \underline{w})$ is defined recursively by

$$(u, v) \equiv . \; u = v \text{ for any variable } v,$$

and

$$(u, <\underline{s}, \underline{t}>) \equiv . \; (\exists s)(\exists t)(u<s, t> \wedge (s, \underline{s}) \wedge (t, \underline{t}))$$

for any pair of terms $\underline{s}$ and $\underline{t}$, assuming that s and t do not occur free in the term $\underline{s}$ nor in the term $\underline{t}$.

Now, we define our association of $[\underline{V}^+]$-propositions $\underline{F}$ having term expressions to $\underline{V}^+$-propositions. At first, we replace each term expression of $\underline{F}$ by a term which is cofinal with the term expression, so we can assume without loss of generality that $\underline{F}$ does not contain any term expression other than terms.

DEFINITION 1. 16. The interpretation of $[\underline{V}^+]$-propositions by $\underline{V}^+$-propositions is word-for-word. Namely,

I1. Any proposition of the form $R_i(\underline{x}_1, \cdots, \underline{x}_{n_i})$ for any sequence of terms $\underline{x}_1, \cdots, \underline{x}_{n_i}$ is interpreted by

$$(\exists x_1)\cdots(\exists x_{n_i})(R_i(x_1, \cdots, x_{n_i}) \wedge (x_1, \underline{x}_1) \wedge \cdots \wedge (x_{n_i}, \underline{x}_{n_i})),$$

where the variables $x_1, \cdots, x_{n_i}$ are assumed to have no occurrence in $R_i(\underline{x}_1, \cdots, \underline{x}_{n_i})$.

I2. Any proposition of the form $p(\underline{x})$ for any variable p and any term $\underline{x}$ is interpreted by

$$(\exists x)(p(x) \wedge (x, \underline{x})),$$

where the variable x is assumed to have no occurence in $p(\underline{x})$.

I3. Any proposition of the form $x<\underline{y}, \underline{z}>$ for any variable x and any pair of terms $\underline{y}$ and $\underline{z}$ is interpreted by

$$(\exists y)(\exists z)(x<y, z> \wedge (y, \underline{y}) \wedge (z, \underline{z})),$$

where the variables y and z are assumed to have no occurrence in $x<\underline{y}, \underline{z}>$.

I4. Let us denote our interpretation by supplying a prime. Then,

$(\underline{A} \longrightarrow \underline{B})'$ is $\underline{A}' \longrightarrow \underline{B}'$, $(\underline{A} \wedge \underline{B})'$ is $\underline{A}' \wedge \underline{B}'$,

$(\underline{A} \vee \underline{B})'$ is $\underline{A}' \vee \underline{B}'$, $(\underline{A} \equiv \underline{B})'$ is $\underline{A}' \equiv \underline{B}'$,

$((x)\underline{A})'$ is $(x)(\underline{A}')$, and $((\exists x)\underline{A})'$ is $(\exists x)(\underline{A}')$.

Now, we state the axiom system of $B(\underline{V})$.

AXIOM SYSTEM 1. 17. The axioms of the theory $B(\underline{V})$ are the following $15 + s$ propositions.

OP1. $(x)(y)(\exists z)z<x, y>$.

OP2. $(x)(y)(z)(u)(v)(w)(z<x, y> \wedge w<u, v>.$
$$\longrightarrow (z = w. \equiv. x = u \wedge y = v)).$$

PNR$_i$. $(\exists p)(x_1)\cdots(x_{n_i})(p(x_1, \cdots, x_{n_i}) \equiv R_i(x_1, \cdots, x_{n_i}))$
$(i = 1, \cdots, s)$.

OPR. $(\exists p)(x)(y)(z)(p(x, y, z) \equiv x<y, z>)$.

ER. $(\exists p)(x)(y)(p(x, y) \equiv. \ x = y)$.

GP1. $(x)(p)(\exists q)(y)(q(y) \equiv p(x, y))$.

GP2. $(p)(\exists q)(x)(y)(q(x, y) \equiv p(y))$.

GP3. $(p)(\exists q)(x)(y)(q(x, y) \equiv p(y, x))$.

GP4. $(p)(\exists q)(x)(y)(q(x, y) \equiv p(x, x, y))$.

GP5. $(p)(\exists q)(x)(y)(z)(q(<x, y>, z) \equiv p(x, y, z))$.

GP6. $(p)(\exists q)(x)(y)(z)(q(x, y, z) \equiv p(<x, y>, z))$.

GP7. $(p)(\exists q)(x)(s)(t)(y)(q(x, s, t, y) \equiv p(x, t, s, y))$.

GPI. $(p)(q)(\exists r)(x)(r(x) \equiv. \ p(x) \rightarrow q(x))$.

GPC. $(p)(q)(\exists r)(x)(r(x) \equiv. \ p(x) \wedge q(x))$.

GPD. $(p)(q)(\exists r)(x)(r(x) \equiv. \ p(x) \vee q(x))$.

GPU. $(p)(\exists q)(x)(q(x) \equiv (y)p(y, x))$.

GPE. $(p)(\exists q)(x)(q(x) \equiv (\exists y)p(y, x))$.

Hereafter, I will refer to this axiom system simply by $\sum(\underline{V})$.

(2) **On the theories $B(\underline{V})$ and $B[\underline{V}]$.**

Every axiom of $B(\underline{V})$ can be interpreted as a $\underline{V}^+$-proposition, so we can develop a $\underline{V}^+$-theory starting from the axiom system $\sum(\underline{V})$ and standing on the usual positive logic **LP**, or on the minimal logic **LM** regarding $\underline{A} \longrightarrow \lambda$ as $\sim\underline{A}$. However, we can also choose another way in developing a theory starting from the same axiom system $\sum(\underline{V})$ but standing on a modified logic so as to match with our system which has adopted term expressions. Namely, we adopt in the modified logic the following two inference rules together with the usual inference rules of **LP** or **LM.**

(UT) Any $\underline{V}^+$-proposition of the form $\underline{A}(\underline{t})$ for any term expression $\underline{t}$ can be deduced from $(x)\underline{A}(x)$.

(E*T) $(\exists x)\underline{A}(x)$ is deducible from any $\underline{V}^+$-proposition of the form $\underline{A}(\underline{t})$ for any t rm expression $\underline{t}$.

I will refer to this logic hereafter by **LPT** (intuitionistic positive logic admitting term expressions) or LMT, and I will refer to the theory starting from the same axiom system $\sum(\underline{V})$ but developed in the logic LPT or LMT by $B[\underline{V}]$.

I will show in the following that the theories $B(\underline{V})$ and $B[\underline{V}]$ are essentially the same theory.

THEOREM 2.1. *$(\exists t)(t, \underline{t})$ for any term expression $\underline{t}$ is deducible in $B(\underline{V})$, assuming that the variable t does not occur in the term expression $\underline{t}$.*

Proof. According to Definitions 1.13 and 1.15, $(\exists t)(t, s)$ is provable in **LP** (or **LM**) for any variable s. Next, let us assume $(\exists s)(s, \underline{s})$ and $(\exists t)(t, \underline{t})$. So, let s and t satisfy $(s, \underline{s})$ and $(t, \underline{t})$, respectively. Then, we can find out such u satisfying $u<s, t>$ according to OP1. Consequently, we have

$$(\exists u)(\exists s)(\exists t)(u<s, t> \wedge (s, \underline{s}) \wedge (t, \underline{t})),$$

which is nothing but $(\exists u)(u, <\underline{s}, \underline{t}>)$ by Definition 1.15. According to the recursive Definition 1.1 of terms, we have thus proved that $(\exists t)(t, \underline{t})$ holds for every term $\underline{t}$.

THEOREM 2.2. *Reflexivity, symmetricity, and transitivity of equality among term expressions are deducible in $B(\underline{V})$.*

THEOREM 2.3. *$(u)(v)((u, \underline{w}) \wedge (v, \underline{w}). \longrightarrow u = v)$ is deducible in $B(\underline{V})$ for any term expression $\underline{w}$.*

Proof. Without loss of generality, we can assume that $\underline{w}$ is a term. I will prove this theorem by complete induction with respect to the rank ρ of the term $\underline{w}$. In the case $\rho = 0$, we can prove this theorem by Theorem 2.2.

In the case $0 < \rho = r$, the term $\underline{w}$ must be of the form $<\underline{s}, \underline{t}>$, and we have to prove

$$(u)(v)((\exists s)(\exists t)(u<s, t> \wedge (s, \underline{s}) \wedge (t, \underline{t})) \wedge \\ (\exists s')(\exists t')(v<s', t'> \wedge (s', \underline{s}) \wedge (t', \underline{t})). \longrightarrow u = v).$$

For any s, s', t, t' satisfying $(s, \underline{s}), (s', \underline{s}), (t, \underline{t})$, and $(t', \underline{t})$, we have $s = s'$ and $t = t'$ by our induction assumption. Hence, we have $u = v$ from $u<s, t>$ and $v<s, t>$ according to Axiom OP2.

THEOREM 2.4. *$u<\underline{s}, \underline{t}>$ is identically equivalent to $(u, <\underline{s}, \underline{t}>)$ for any variable u and any pair of term expressions $\underline{s}$ and $\underline{t}$, in $B(\underline{V})$.*

Proof. By Definitions 1.15 and 1.16.

THEOREM 2.5. *$(u, \underline{w})$ is equivalent to $u = \underline{w}$ for any variable u and any term expression $\underline{w}$, in $B(\underline{V})$.*

Proof. Without loss of generality, we can assume that $\underline{w}$ is a term. We will prove this theorem by complete induction with respect to the rank ρ of the term $\underline{w}$.

In the case $\rho = 0$, the term $\underline{w}$ is a variable, say w. (u, w) is surely equivalent to $u = w$ by Definition 1.15.

Now I will prove the equivalence of $(u, \underline{w})$ and $u = \underline{w}$ for the case $0 < \rho = r$. Then, $\underline{w}$ is a term of the form $<\underline{s}, \underline{t}>$ for two terms $\underline{s}$ and $\underline{t}$, whose ranks are both less than r.

At first, let us assume $(u, \underline{w})$ for the case $0 < \rho = r$. Then, $u = \underline{w}$, i.e. $(p)(p(u) \equiv p(\underline{w}))$ by Definition 1.13, holds. For: At first, take any p satisfying $p(u)$, then, $p(u) \wedge (u, \underline{w})$ holds. Hence,

$$(\exists w)(p(w) \wedge (w, \underline{w})), \text{ i.e. } p(\underline{w})$$

holds by Definition 1.16. Next, take any p satisfying $p(\underline{w})$. Then, by Definition 1.16, we can find out such w that satisfies $p(w)$ and $(w, \underline{w})$. Because we have assumed $(u, \underline{w})$, so $u = w$ holds, according to Theorem 2.3. Because $p(w)$ holds, so $p(u)$ holds too according to Definition 1.12.

Next, let us assume $u = \underline{w}$. Then,

$$(p)(p(u) \equiv p(\underline{w})), \quad \text{i.e.} \quad (p)(p(u) \equiv (\exists w)(p(w) \wedge (w, \underline{w}))$$

holds by Definitions 1.12 and 1.15. According to Theorem 2.1, there is such s and t that satisfy $(s, \underline{s})$ and $(t, \underline{t})$. On the other hand, according to OPR, there is such h that satisfies

$$(x)(y)(z)(h(x, y, z) \equiv x<y, z>).$$

By successive applications of the axiom group GP1—GP7, we can see that there is such p that satisfies

$$(v)(p(v) \equiv v<s, t>).$$

According to OP1, the e is such w that satisfies $w<s,t>$. Accordingly, we have

$$(\exists s)(\exists t)(w<s,t> \wedge (s,\underline{s}) \wedge (t,\underline{t})), \qquad \text{i.e.} \qquad (w,\underline{w}).$$

Also, $p(w)$ holds by definition of p. Hence, $(\exists w)(p(w) \wedge (w,\underline{w}))$ holds. Accordingly, by our assumption. $p(u)$ holds too, which means that $u<s,t>$ holds by definition of p. Therefore, we have

$$(\exists s)(\exists t)(u<s,t> \wedge (s,\underline{s}) \wedge (t,\underline{t})), \qquad \text{i.e.} \qquad (u,\underline{w}).$$

THEOREM 2.6. *For any term expressions* $\underline{s}, \underline{t}, \underline{u}$, *and* $\underline{v}$, *the equality* $<\underline{s},\underline{t}> = <\underline{u},\underline{v}>$ *is equivalent to the pair of equalities* $\underline{s} = \underline{u}$ *and* $\underline{t} = \underline{v}$, *in* $B(\underline{V})$.

Proof. Assume at first $<\underline{s},\underline{t}> = <\underline{u},\underline{v}>$. This means

$$\begin{aligned}
&(p)(p(<\underline{s},\underline{t}>) \equiv p(<\underline{u},\underline{v}>)),\\
\text{i.e.}\quad &(p)((\exists z)(p(z) \wedge (z,<\underline{s},\underline{t}>)) \equiv (\exists w)(p(w) \wedge (w,<\underline{u},\underline{v}>)),\\
\text{i.e.}\quad &(p)((\exists z)(p(z) \wedge (\exists s)(\exists t)(z<s,t> \wedge (s,\underline{s}) \wedge (t,\underline{t})))\\
&\qquad \equiv (\exists w)(p(w) \wedge (\exists u)(\exists v)(w<u,v> \wedge (u,\underline{u}) \wedge (v,\underline{v}))))
\end{aligned}$$

by Definition 1.15 and Definition 1.16. Now, by Theorem 2.1 and Axiom OP1, we can take su h s, t, z, u, v, and w that satisfy $(s,\underline{s})$, $(t,\underline{t})$, $z<s,t>$, $(u,\underline{u})$, $(v,\underline{v})$, and $w<u,v>$. According to Theorem 2.5, $s = \underline{s}$, $t = \underline{t}$, $u = \underline{v}$, and $v = \underline{v}$ hold.

We assert now $z = w$. Namely, for any variable p, we assert $p(z) \longrightarrow p(w)$. To prove this, let us assume $p(z)$. Then

$$(\exists z)(p(z) \wedge (\exists s)(\exists t)(z<s,t> \wedge (s,\underline{s}) \wedge (t,\underline{t})))$$

holds. Hence, according to our assumption, there are such u', v', and w' that satisfy $p(w')$, $(u',\underline{u})$, and $(v',\underline{v})$. According to Theorem 2.3, $u' = u$ and $v' = v$. Therefore $w' = w$ holds according to Axiom OP2. Because $p(w')$ holds, $p(w)$ holds too. Similarly, we can prove $p(w) \longrightarrow p(z)$. So, $p(z) \equiv p(w)$ for any p. This means that $z = w$ holds.

Now, according to Axiom OP2, the equalities $s = u$ and $t = v$ holds. Consequently, we have $\underline{s} = \underline{u}$ and $\underline{t} = \underline{v}$ according to Theorem 2.2.

Next, conversely, let us assume $\underline{s} = \underline{u}$ and $\underline{t} = \underline{v}$, and we prove $<\underline{s},\underline{t}> = <\underline{u},\underline{v}>$, which can be expressed by the formula

$$(p)((\exists z)(p(z) \wedge (\exists s)(\exists t)(z{<}s, t{>} \wedge s = \underline{s} \wedge t = \underline{t})) \equiv (\exists w)(p(w) \wedge (\exists u)(\exists v)(w{<}u, v{>} \wedge u = \underline{u} \wedge v = \underline{v})))$$

by Definition 1.14, Definition 1.15, and Theorem 2.5.

To show for any variable p that

(A) $(\exists z)(p(z) \wedge (\exists s)(\exists t)(z{<}s, t{>} \wedge s = \underline{s} \wedge t = \underline{t}))$

implies

(B) $(\exists w)(p(w) \wedge (\exists u)(\exists v)(w{<}u, v{>} \wedge u = \underline{u} \wedge v = \underline{v}))$,

let us assume the proposition (A). Then, we can take such s, t, and z that satisfy $p(z)$, $z{<}s, t{>}$, $(s, \underline{s})$, and $(t, \underline{t})$. According to Theorem 2.5, $s = \underline{u}$ and $t = \underline{v}$. Therefore, (A) implies (B). Similarly, (B) implies (A).

THEOREM 2.7. *In $B(\underline{V})$, for any pair of term expression sequences $\underline{x}_1, \cdots, \underline{x}_n$ and $\underline{y}_1, \cdots, \underline{y}_n$, the equality*

$$<\underline{x}_1, \cdots, \underline{x}_n> = <\underline{y}_1, \cdots, \underline{y}_n>$$

is equivalent to the sequence of equalities

$$\underline{x}_1 = \underline{y}_1, \cdots, \underline{x}_n = \underline{y}_n.$$

THEOREM 2.8. *Let $\underline{t}(w)$ be a term expression, possibly containing free variable w, and $\underline{t}(\underline{u})$ be the expression obtained by substituting the term expression $\underline{u}$ in place of w in $\underline{t}(w)$. Then, $\underline{t}(\underline{u})$ is also a term expression.*

Proof. It is enough to prove this theorem for any pair of terms $\underline{t}(w)$ and $\underline{u}$. We prove this theorem by complete induction with respect to the rank ρ of the term $\underline{t}(w)$.

In the case $\rho = 0$, $\underline{t}(w)$ is a variable v which may coincide with w. If $\underline{t}(w)$ is w, $\underline{t}(\underline{u})$ turns out to be the term $\underline{u}$. If $\underline{t}(w)$ is the variable v distinct from w, $\underline{t}(\underline{u})$ is the term v.

Next, let us assume our theorem for the case $0 \leq \rho < r$ for every natural number ρ, and we will prove our theorem for the case $\rho = r$. Namely, let us take any term $\underline{t}(w)$ of the rank r. Then, $\underline{t}(w)$ is a term of the form $<\underline{x}(w), \underline{y}(w)>$, and the term $\underline{x}(w)$ as well as the term $\underline{y}(w)$ is a term of a rank lower than r. Hence, the expression $\underline{x}(\underline{u})$ as well as the expression $\underline{y}(\underline{u})$ is a term expression according to our induction assumption. On the other hand, $\underline{t}(\underline{u})$ is an expression of the form $<\underline{x}(\underline{u}),\ \underline{y}(\underline{u})>$. Hence, the expression $\underline{t}(\underline{u})$ is a term according to Definition 1.1.

Remark 2.9. Let $\underline{t}(w)$ be a term expression, and $\underline{u}$ be another term expression. Then the variable w does not occur in $\underline{t}(\underline{u})$ unless w occurs in $\underline{u}$. However, in the case where $\underline{u}$ is a variable u which occurs already in $\underline{t}(w)$, we might express $\underline{t}(\underline{u})$ by $\underline{t}(u)$, which would cause confusion. Consequently, we will call any term expression of the form $\underline{t}(w)$ a term expression having its proper variable w (or, w is proper in $\underline{t}(w)$) if and only if w does not occur in $\underline{t}(u)$ for a variable u.

We can generalize the notion easily for the many variable cases. Namely, for any sequence of mutually distinct variables $x, \cdots, z$, we call any term expression $\underline{t}(x, \cdots, z)$ a term expression having $x, \cdots, z$ as its proper variables (or; $x, \cdots, z$ are proper in $\underline{t}(x, \cdots, z)$) if and only if the variables $x, \cdots, z$ do not occur in $\underline{t}(x', \cdots, z')$ for some suitable $x', \cdots, z'$.

THEOREM 2.10. *For any term expression* $\underline{t}(x, \cdots, z)$ *having the variables* $x, \cdots, z$ *as its proper variables and for any pair of term expression sequences* $\underline{x}, \cdots, \underline{z}$ *and* $\underline{x}', \cdots, \underline{z}'$, *the set of equalities*

$$\underline{x} = \underline{x}', \cdots, \text{ and } \underline{z} = \underline{z}'$$

implies the equality $\underline{t}(\underline{x}, \cdots, \underline{z}) = \underline{t}(\underline{x}', \cdots, \underline{z}')$ *in* $B(\underline{V})$.

Proof. For simplicity's sake, I will prove this theorem in the single variable case for any term $\underline{t}(x)$ by complete induction with respect to the rank ρ of the term $\underline{t}(x)$.

Let us assume $\underline{x} = \underline{x}'$, and we shall prove $\underline{t}(\underline{x}) = \underline{t}(\underline{x}')$.

In the case $\rho = 0$, the term $\underline{t}(x)$ is a variable u. If x and u coincide, $\underline{t}(\underline{x}) = \underline{t}(\underline{x}')$ holds because the equality is nothing but the assumption $\underline{x} = \underline{x}'$. If x and u do not coincide, the equality $\underline{t}(\underline{x}) = \underline{t}(\underline{x}')$ turns out to be the equality $u = u$, which surely holds according to Theorem 2.2.

Let us assume now that our theorem holds for the case $0 \leq \rho < r$ for every natural number ρ, and we will prove our theorem for the case $\rho = r$. Namely, let us take any term $\underline{t}(x)$ of the rank r. Then, $\underline{t}(x)$ is a term of the form $<\underline{u}(x), \underline{v}(x)>$, and $\underline{u}(x)$ as well as $\underline{v}(x)$ is a term of a rank lower than r and having x as its proper variable. Hence,

$$\underline{u}(\underline{x}) = \underline{u}(\underline{x}') \quad \text{and} \quad \underline{v}(\underline{x}) = \underline{v}(\underline{x}')$$

hold according to our induction assumption. Therefore,

$$<\underline{u}(\underset{\sim}{x}), \underline{v}(\underset{\sim}{x})> \;=\; <\underline{u}(\underset{\sim}{x}'), \underline{v}(\underset{\sim}{x}')>$$

holds according to Theorem 2.6.

On the other hand, $\underline{t}(\underset{\sim}{x})$ is nothing but $<\underline{u}(\underset{\sim}{x}), \underline{v}(\underset{\sim}{x})>$ and $\underline{t}(\underset{\sim}{x}')$ is nothing but $<\underline{u}(\underset{\sim}{x}'), \underline{v}(\underset{\sim}{x}')>$. Hence, we have

$$\underline{t}(\underset{\sim}{x}) = \underline{t}(\underset{\sim}{x}').$$

Remark 2.11. Mostly, we need not prove theorems generally in many variable cases, because we can prove easily the following:

Let $\underline{t}(x, y)$ be any term expression having x and y as its proper variables, and let $\underset{\sim}{x}$ and $\underset{\sim}{y}$ be any term expressions which do not contain the variables x and y. Then, $\underline{t}(\underset{\sim}{x}, y)$ and $\underline{t}(x, \underset{\sim}{y})$ are term expressions having y and x as their proper variables, respectively. Let us further denote $\underline{t}(\underset{\sim}{x}, y)$ and $\underline{t}(x, \underset{\sim}{y})$ by $\underline{u}(y)$ and $\underline{v}(x)$. Then, $\underline{u}(\underset{\sim}{y})$, $\underline{v}(\underset{\sim}{x})$, and $\underline{t}(\underset{\sim}{x}, \underset{\sim}{y})$ are the same term expression.

THEOREM 2.12. *Let $\underline{t}(x)$ be any term expression having x as its proper variable, and let $\underset{\sim}{x}$ be any term expression having no occurrence of x. Then, the propositions*

$$(t, \underline{t}(\underset{\sim}{x})),$$
$$(\exists x)((x, \underset{\sim}{x}) \wedge (t, \underline{t}(x))),$$
$$(x)((x, \underset{\sim}{x}) \longrightarrow (t, \underline{t}(x)))$$

are equivalent to each other, in $B(\underline{V})$.

Proof. Without loss of generality, we can assume that $\underline{t}(x)$ is a term, so I will prove this theorem by complete induction with respect to the rank ρ of the term $\underline{t}(x)$, in $B(\underline{V})$.

In the case $\rho = 0$, $\underline{t}(x)$ is either x itself or any other variable u.

In the first sub-case, where $\underline{t}(x)$ is x itself, $(t, \underline{t}(\underset{\sim}{x}))$ and $(t, \underline{t}(x))$ are $(t, \underset{\sim}{x})$ and (t, x), respectively. If we assume $(t, \underset{\sim}{x})$, then $(t, \underset{\sim}{x}) \wedge (t, t)$ holds because (t, t) is $t = t$ according to Definition 1.15 and $t = t$ holds according to Theorem 2.2. Hence, $(\exists x)((x, \underset{\sim}{x}) \wedge (t, x))$ holds. This is nothing but $(\exists x)((x, \underset{\sim}{x}) \wedge (t, \underline{t}(x)))$. If we assume $(\exists x)((x, \underset{\sim}{x}) \wedge (t, \underline{t}(x)))$, i.e., $(\exists x)((x, \underset{\sim}{x}) \wedge (t, x))$, conversely, we can take such x that satisfies $(x, \underset{\sim}{x})$ and (t, x). According to Theorem 2.5, we have $x = \underset{\sim}{x}$ and $t = x$ in this case. Hence, we have $t = \underline{x}$ by Theorem 2.2. $t = \underline{x}$ is equivalent to $(t, \underline{x})$ again by Theorem 2.5. Hence, $(t, \underline{t}(\underline{x}))$ and $(\exists x)((x, \underline{x}) \wedge (t, \underline{t}(x)))$ are mutually equivalent in this case.

To show the equivalence of $(t, \underline{x})$ and $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$, let us assume again $(t, \underline{x})$, i.e. $t = \underline{x}$ according to Theorem 2.5, at first. To prove $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$, take any x satisfying $(x, \underline{x})$, i.e. $x = \underline{x}$ according to Theorem 2.5. Then, by Theorem 2.2, $t = x$, i.e. $(t, \underline{t}(x))$, holds. Next, conversely, let us assume $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$, i.e. $(x)((x, \underline{x}) \longrightarrow (t, x))$. According to Theorem 2.1, we can take such u that satisfies $(u, \underline{x})$. Hence, by our assumption, $(t, \underline{x})$ holds, according to Theorems 2.2 and 2.5.

In the second sub-case, where $\underline{t}(x)$ is u which is distinct from x. Then, $(t, \underline{t}(x))$ and $(t, \underline{t}(\underline{x}))$ are both $\underline{t} = u$ according to Definition 1.15. If we assume $(t, \underline{t}(\underline{x}))$, i.e. $t = u$. According to Theorem 2.1, we can take such x that satisfies $(x, \underline{x})$. Hence, we have

$$(\exists x)((x, \underline{x}) \wedge t = u), \quad \text{i.e.} \quad (\exists x)((x, \underline{x}) \wedge (t, \underline{t}(x))).$$

Conversely, if we assume

$$(\exists x)((x, \underline{x}) \wedge (t, \underline{t}(x))), \quad \text{i.e.} \quad (\exists x)((x, \underline{x}) \wedge t = u),$$

we have surely $t = u$, i.e. $(t, \underline{t}(\underline{x}))$.

Next, to prove equivalence of $(t, \underline{t}(\underline{x}))$ and $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$, let us assume $(t, \underline{t}(\underline{x}))$, i.e. $t = u$. Then, we surely have

$$(x)((x, \underline{x}) \longrightarrow t = u), \quad \text{i.e.} \quad (x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x))).$$

Next, let us assume conversely

$$(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x))), \quad \text{i.e.} \quad (x)((x, \underline{x}) \longrightarrow t = u).$$

By virtue of Theorem 2.1, we have $(\exists x)(x, \underline{x})$. Hence, we have $t = u$, i.e. $(t, \underline{t}(\underline{x}))$.

Next, I will prove our theorem in the case $0 < \rho = r$ by assuming our theorem in all cases $0 \leq \rho < r$. In the case $\rho = r$, $\underline{t}(x)$ is a term of the form $<\underline{u}(x), \underline{v}(x)>$ for a pair of terms $\underline{u}(x)$ and $\underline{v}(x)$, whose rank are both lower than r, so our theorem holds for these terms.

To prove that $(t, \underline{t}(\underline{x}))$ implies $(\exists x)((x, \underline{x}) \wedge (t, \underline{t}(x)))$ as well as $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$, let us assume $(t, \underline{t}(\underline{x}))$, i.e. $(t, <\underline{u}(\underline{x}), \underline{v}(\underline{x})>)$. According to Definition 1.15, we have

$$(\exists u)(\exists v)((t, <u, v>) \wedge (u, \underline{u}(\underline{x})) \wedge (v, \underline{v}(\underline{x})).$$

Namely, we can take such u and v that satisfies $(t, <u, v>)$, $(u, \underline{u}(\underline{x}))$ and $(v, \underline{v}(\underline{x}))$. Because $(u, \underline{u}(\underline{x}))$ is equivalent to $(\exists x)((x, \underline{x}) \wedge (u, \underline{u}(x)))$ by our induction assumption, we can take such x that satisfies $(x, \underline{x})$ and $(u, \underline{u}(x))$. Because $(v, \underline{v}(\underline{x}))$ is equivalent to $(x)((x, \underline{x}) \longrightarrow (v, \underline{v}(x)))$ by our induction assumption, we have $(v, \underline{v}(x))$. Hence, we have

$$(\exists x)((x, \underline{x}) \wedge (\exists u)(\exists v)((t, <u, v>) \wedge (u, \underline{u}(x)) \wedge (v, \underline{v}(x))),$$
$$\text{i.e. } (\exists x)((x, \underline{x}) \wedge (t, <\underline{u}(x), \underline{v}(x)>)),$$
$$\text{i.e. } (\exists x)((x, \underline{x}) \wedge (t, \underline{t}(x))),$$

according to Definition 1.15.

To prove next $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$, let us take any x satisfying $(x, \underline{x})$. Because $(u, \underline{u}(\underline{x}))$ and $(v, \underline{v}(\underline{x}))$ are equivalent to

$$(x)((x, \underline{x}) \longrightarrow (u, \underline{u}(x))) \quad \text{and} \quad (x)((x, \underline{x}) \longrightarrow (v, \underline{v}(x))),$$

respectively, by our induction assumption, we have $(u, \underline{u}(x))$ and $(v, \underline{v}(x))$. Hence, we have

$$(\exists u)(\exists v)((t, <u, v>) \wedge (u, \underline{u}(x)) \wedge (v, \underline{v}(x))),$$

which is equivalent to

$$(t, <\underline{u}(x), \underline{v}(x)>), \quad \text{i.e.} \quad (t, \underline{t}(x)),$$

according to Definition 1.15. Thus, we have proved $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$.

Next, conversely, let us assume $(\exists x)((x, \underline{x}) \wedge (t, \underline{t}(x)))$. Then, we can take such x that satisfies $(x, \underline{x})$ and $(t, \underline{t}(x))$. $(t, \underline{t}(x))$ is

$$(t, <\underline{u}(x), \underline{v}(x)>), \text{ i.e. } (\exists u)(\exists v)((t, <u, v>) \wedge (u, \underline{u}(x)) \wedge (v, \underline{v}(x))).$$

Hence, we can take such u and v that satisfy $(t, <u, v>)$, $(u, \underline{u}(x))$, and $(v, \underline{v}(x))$. Consequently, we have $(\exists x)((x, \underline{x}) \wedge (u, \underline{u}(x)))$ and $(\exists x)((x, \underline{x}) \wedge (v, \underline{v}(x)))$, which are equivalent to $(u, \underline{u}(\underline{x}))$ and $(v, \underline{v}(\underline{x}))$, respectively, according to our induction assumption. So, we have

$$(\exists u)(\exists v)((t, <u, v>) \wedge (u, \underline{u}(\underline{x})) \wedge (v, \underline{v}(\underline{x}))),$$

which is equivalent to $(t, <\underline{u}(\underline{x}), \underline{v}(\underline{x})>)$, i.e. $(t, \underline{t}(\underline{x}))$ according to Definition 1.15.

To prove that $(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$ implies $(t, \underline{t}(\underline{x}))$, let us assume

$(x)((x, \underline{x}) \longrightarrow (t, \underline{t}(x)))$. According to Theorem 2.1 we can take such x that satisfies $(x, \underline{x})$. Hence, $(t, \underline{t}(x))$, i.e. $(t, <\underline{u}(x), \underline{v}(x)>)$, holds. According to Definition 1.14, we have

$$(\exists u)(\exists v)((t, <u, v>) \wedge (u, \underline{u}(x)) \wedge (v, \underline{v}(x))),$$

so we can take such u and v that satisfy $(t, <u, v>)$, $(u, \underline{u}(x))$, and $(v, \underline{v}(x))$. Consequently, we have

$$(\exists x)((x, \underline{x}) \wedge (u, \underline{u}(x))) \text{ and } (\exists x)((x, \underline{x}) \wedge (v, \underline{v}(x))),$$

which are equivalent to $(u, \underline{u}(\underline{x}))$ and $(v, \underline{v}(\underline{x}))$, respectively, according to our induction assumption. Therefore, we have

$$(\exists u)(\exists v)((t, <u, v>) \wedge (u, \underline{u}(\underline{x})) \wedge (v, \underline{v}(\underline{x}))),$$

which is equivalent to $(t, <\underline{u}(\underline{x}), \underline{v}(\underline{x})>)$, i.e. $(t, \underline{t}(\underline{x}))$, according to Definition 1.14.

Remark 2.13. Any $[V^+]$-proposition of the form $\underline{A}(\underline{t})$ for any term expression $\underline{t}$ can be obtained by substituting $\underline{t}$ in all places of the variable t of a $[\underline{V}^+]$-proposition of the form $\underline{A}(t)$ for a suitable variable t which does not occur in $\underline{A}(\underline{t})$. Then, the variable t does not occur in $\underline{A}(u)$ for any variable u distinct from t. Moreover, $\underline{A}(t)$ does not have any sub-formula of the forms $t\langle \cdots \rangle$ or $t(\cdots)$. So I will call any $[\underline{V}^+]$-proposition $\underline{A}(t)$ a proposition having t as its ***proper variable*** (or, t is ***proper*** in $\underline{A}(t)$) if and only if t does not occur in $\underline{A}(u)$ for a suitable variable u and $\underline{A}(t)$ does not have any sub-formula of the forms $t\langle \cdots \rangle$ or $t(\cdots)$. Any $[\underline{V}^+]$-proposition of the form $\underline{A}(\underline{t})$ for any term expression $\underline{t}$ can be obtained by substituting $\underline{t}$ in place of t of a $[\underline{V}^+]$-proposition $\underline{A}(t)$ having t as its proper variable.

THEOREM 2.14. *Let $\underline{A}(t)$ be any $[\underline{V}^+]$-proposition having t as its proper variable and let $\underline{t}$ be any term expression having no occurrence of t. Then, the propositions*

$$\underline{A}(\underline{t}), \quad (\exists t)((t, \underline{t}) \wedge \underline{A}(t)), \quad \textit{and} \quad (t)((t, \underline{t}) \longrightarrow \underline{A}(t))$$

are equivalent to each other, in $B(\underline{V})$.

Proof. Without loss of generality, we can assume that all the term expressions occurring in the proposition $\underline{A}(\underline{t})$ are terms. I will prove this theorem by complete induction with respect to the number ν of logical constants in the proposition $\underset{\sim}{A}(t)$.

In the case $\nu = 0$, $\underset{\sim}{A}(t)$ is a proposition having either of the forms

$R_i(\underset{\sim}{x}_1(t), \cdots, \underset{\sim}{x}_{n_i}(t))$ for a sequence of terms $\underset{\sim}{x}_1(t), \cdots, \underset{\sim}{x}_{n_i}(t)$ having t as their proper variable,

$p(\underset{\sim}{x}(t))$ for a variable p and for a term $\underset{\sim}{x}(t)$ having t as its proper variable,

or $z<\underset{\sim}{x}(t), \underset{\sim}{y}(t)>$ for a variable z and for a pair of terms $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ having t as their proper variable.

In the first sub-case, where $\underset{\sim}{A}(t)$ is a proposition of the form $R_i(\underset{\sim}{x}_1(t), \cdots, \underset{\sim}{x}_{n_i}(t))$, the proposition $\underset{\sim}{A}(\underset{\sim}{t})$ is $R_i(\underset{\sim}{x}_1(\underset{\sim}{t}), \cdots, \underset{\sim}{x}_{n_i}(\underset{\sim}{t}))$, which is equivalent to

$$(\exists x_1)\cdots(\exists x_{n_i})(R_i(x_1, \cdots, x_{n_i}) \wedge (x_1, \underset{\sim}{x}_1(t)) \wedge \cdots \wedge (x_{n_i}, \underset{\sim}{x}_{n_i}(t)))$$

according to Definition 1.16. Accordingly, we can take such $x_1, \cdots, x_{n_i}$ that satisfy

$$R_i(x_1, \cdots, x_{n_i}),\ (x_1, \underset{\sim}{x}_1(t)),\ \cdots,\ (x_{n_i}, \underset{\sim}{x}_{n_i}(\underset{\sim}{t})).$$

According to Theorem 2.12, the proposition $(x_1, \underset{\sim}{x}_1(\underset{\sim}{t}))$ is equivalent to $(\exists t)((t, \underset{\sim}{t}) \wedge (x_1, \underset{\sim}{x}_1(t)))$, so we can take such t that satisfies $(t, \underset{\sim}{t})$ and $(x_1, \underset{\sim}{x}_1(t))$, unless $n_i = 0$. According to Theorem 2.12, $(x_k, \underset{\sim}{x}_k(\underset{\sim}{t}))$ for $k \geq 2$, if any, is equivalent to $(t)((t, \underset{\sim}{t}) \longrightarrow (x_k, \underset{\sim}{x}_k(t)))$, so we have $(x_k, \underset{\sim}{x}_k(t))$. Hence, we have

$$(\exists t)((t, \underset{\sim}{t}) \wedge (\exists x_1)\cdots(\exists x_{n_i})(R_i(x_1, \cdots, x_{n_i}) \wedge (x_1, \underset{\sim}{x}_1(t)) \wedge \cdots \wedge (x_{n_i}, \underset{\sim}{x}_{n_i}(t)))),$$

which is nothing but $(\exists t)((t, \underset{\sim}{t}) \wedge \underset{\sim}{A}(t))$.

Next, to prove $(t)((t, \underset{\sim}{t}) \longrightarrow \underset{\sim}{A}(t))$, take any t satisfying $(t, \underset{\sim}{t})$. According to Theorem 2.12, $(x_k, \underset{\sim}{x}_k(t))$ for $k = 1, \cdots, n_i$ is equivalent to $(t)((t, \underset{\sim}{t}) \longrightarrow (x_k, \underset{\sim}{x}_k(t)))$, so we have $(x_k, \underset{\sim}{x}_k(t))$. Consequently, we have

$$(\exists x_1)\cdots(\exists x_{n_i})(R_i(x_1, \cdots, x_{n_i}) \wedge (x_1, \underset{\sim}{x}_1(t)) \wedge \cdots \wedge (x_{n_i}, \underset{\sim}{x}_{n_i}(t))),\ \text{i.e. } \underset{\sim}{A}(t),$$

under the assumption $(t, \underset{\sim}{t})$. Hence, we have $(t)((t, \underset{\sim}{t}) \longrightarrow \underset{\sim}{A}(t))$.

To prove conversely that $(\exists t)((t, \underset{\sim}{t}) \wedge \underset{\sim}{A}(t))$ implies $\underset{\sim}{A}(\underset{\sim}{t})$, let us assume $(\exists t)((t, \underset{\sim}{t}) \wedge \underset{\sim}{A}(t))$ and take such t that satisfies $(t, \underset{\sim}{t})$ and $\underset{\sim}{A}(t)$, i.e.

$$(\exists x_1)\cdots(\exists x_{n_i})(R_i(x_1, \cdots, x_{n_i}) \wedge (x_1, \underset{\sim}{x}_1(t)) \wedge \cdots \wedge (x_{n_i}, \underset{\sim}{x}_{n_i}(t))).$$

Then, we can take such $x_1, \cdots, x_{n_i}$ that satisfy

$$(x_1, \underset{\sim}{x}_1(t)), \cdots, (x_{n_i}, \underset{\sim}{x}_{n_i}(t)).$$

Therefore, we have $(\exists t)((t, \underset{\sim}{t}) \wedge (x_k, \underset{\sim}{x}_k(t)))$, i.e. $(x_k, \underset{\sim}{x}_k(\underset{\sim}{t}))$ according to Theorem 2. 12, for $k = 1, \cdots, n_i$. Consequently,

$$(\exists x_1) \cdots (\exists x_{n_i})(R_i(x_1, \cdots, x_{n_i}) \wedge (x_1, \underset{\sim}{x}_1(\underset{\sim}{t})) \wedge \cdots \wedge (x_{n_i}, \underset{\sim}{x}_{n_i}(\underset{\sim}{t}))), \text{ i.e. } \underset{\sim}{A}(\underset{\sim}{t}),$$

holds.

In the second and the third sub-cases, where $\underset{\sim}{A}(t)$ is a proposition of the forms $p(\underset{\sim}{x}(t))$ or $z<\underset{\sim}{x}(t), \underset{\sim}{y}(t)>$, we can carry out the equivalence proof of $\underset{\sim}{A}(\underset{\sim}{t})$, $(\exists t)((t, \underset{\sim}{t}) \wedge \underset{\sim}{A}(t))$, and $(t)((t, \underset{\sim}{t}) \longrightarrow \underset{\sim}{A}(t))$ quite similarly as the proof of the first sub-case.

Now, we will prove our theorem in the case $0 < \nu = n$, assuming that our theorem holds in every case $0 \leq \nu < n$. In this case, $\underset{\sim}{A}(t)$ is a proposition having either of the forms

$\underset{\sim}{B}(t) \longrightarrow \underset{\sim}{C}(t)$ $\underset{\sim}{B}(t) \wedge \underset{\sim}{C}(t)$ $\underset{\sim}{B}(t) \vee \underset{\sim}{C}(t)$ $\underset{\sim}{B}(t) \equiv \underset{\sim}{C}(t)$	for a pair of $[\underset{\sim}{V}^+]$-propositions $\underset{\sim}{B}(t)$ and $\underset{\sim}{C}(t)$ having t as their proper variable,
or $(u)\underset{\sim}{B}(u, t)$ $(\exists u)\underset{\sim}{B}(u, t)$	for a $\underset{\sim}{V}$-proposition $\underset{\sim}{B}(u, t)$ having t as its proper variable.

However, we need not discuss the case where $\underset{\sim}{A}(t)$ is a proposition of the form $\underset{\sim}{B}(t) \equiv \underset{\sim}{C}(t)$, because $\underset{\sim}{B}(t) \equiv \underset{\sim}{C}(t)$ can be expressed in the form $(\underset{\sim}{B}(t) \longrightarrow \underset{\sim}{C}(t)) \wedge (\underset{\sim}{C}(t) \longrightarrow \underset{\sim}{B}(t))$. Also, in either of these cases, the propositions $\underset{\sim}{B}(t), \underset{\sim}{C}(t)$, and $\underset{\sim}{B}(u, t)$ are $[\underset{\sim}{V}^+]$-propositions having t as their proper variable and containing less number of logical constants than n.

In the first sub-case where $\underset{\sim}{A}(t)$ is a proposition of the form $\underset{\sim}{B}(t) \longrightarrow \underset{\sim}{C}(t)$, the proposition $\underset{\sim}{A}(\underset{\sim}{t})$ is surely the proposition $\underset{\sim}{B}(\underset{\sim}{t}) \longrightarrow \underset{\sim}{C}(\underset{\sim}{t})$. At first, let us assume $\underset{\sim}{A}(\underset{\sim}{t})$, i.e. $\underset{\sim}{B}(\underset{\sim}{t}) \longrightarrow \underset{\sim}{C}(\underset{\sim}{t})$, in the first sub-case. According to Theorem 2.1, we can take such t that satisfies $(t, \underset{\sim}{t})$. Then, I will prove $\underset{\sim}{B}(t) \longrightarrow \underset{\sim}{C}(t)$. To show this, let us assume $\underset{\sim}{B}(t)$. Then, we have $(\exists t)((t, \underset{\sim}{t}) \wedge \underset{\sim}{B}(t))$, which is equivalent to $\underset{\sim}{B}(\underset{\sim}{t})$ by our induction assumption. Hence, we have $\underset{\sim}{C}(\underset{\sim}{t})$, which is equivalent to $(t)((t, \underset{\sim}{t}) \longrightarrow \underset{\sim}{C}(\underset{\sim}{t}))$ by our induction assumption. From this, we can deduce $\underset{\sim}{C}(t)$ easily. Thus, we have $(\exists t)((t, \underset{\sim}{t}) \wedge (\underset{\sim}{B}(t) \longrightarrow \underset{\sim}{C}(t)))$.

To show $(t)((t, \underline{t}) \longrightarrow (\underline{B}(t) \rightarrow \underline{C}(t)))$, let us take any t satisfying $(t, \underline{t})$. Then, we can prove $\underline{B}(t) \longrightarrow \underline{C}(t)$ similarly as the above proof.

To show conversely that $(\exists t)((t, \underline{t}) \wedge (B(t) \longrightarrow \underline{C}(t)))$ implies $\underline{B}(\underline{t}) \longrightarrow \underline{C}(\underline{t})$, assume $(\exists t)((t, \underline{t}) \wedge (\underline{B}(t) \longrightarrow \underline{C}(t))$ and $\underline{B}(t)$. Then, we can take such t that satisfies $(t, \underline{t})$ and $\underline{B}(t) \longrightarrow \underline{C}(t)$. By virtue of our induction assumption, we have $(t)((t, t) \longrightarrow \underline{B}(t))$. Hence, $\underline{B}(t)$ holds, and this implies $\underline{C}(t)$. Accordingly, we have $(\exists t)((t, \underline{t}) \wedge \underline{C}(t))$, which is equivalent to $\underline{C}(\underline{t})$ by our induction assumption.

To show that $(t)((t, \underline{t}) \longrightarrow (\underline{B}(t) \longrightarrow \underline{C}(t)))$ implies $\underline{B}(\underline{t}) \rightarrow \underline{C}(\underline{t})$, let us assume $(t)((t, \underline{t}) \rightarrow (\underline{B}(t) \rightarrow \underline{C}(t)))$ and $\underline{B}(\underline{t})$, which is equivalent to $(\exists t)((t, \underline{t}) \wedge \underline{B}(t))$ according to our induction assumption. Then, we can take such t that satisfies $(t, \underline{t})$ and $\underline{B}(t)$. As we can prove easily $\underline{C}(t)$, we have $(\exists t)((t, \underline{t}) \wedge \underline{C}(t))$ which is equivalent to $\underline{C}(\underline{t})$ according to our induction assumption.

In the second sub-case, where $\underline{A}(t)$ is a proposition of the form $\underline{B}(t) \wedge \underline{C}(t)$, let us at first prove $(\exists t)((t, \underline{t}) \wedge \underline{A}(t))$, i.e. $(\exists t)((t, \underline{t}) \wedge (\underline{B}(t) \wedge \underline{C}(t)))$ by assuming $\underline{A}(\underline{t})$, i.e. $\underline{B}(\underline{t}) \wedge \underline{C}(\underline{t})$. Namely, by induction assumption, we have $(\exists t)((t, \underline{t}) \wedge \underline{B}(t))$ and $(t)((t, \underline{t}) \longrightarrow \underline{C}(t))$, from which we can easily deduce $(\exists t)((t, \underline{t}) \wedge (\underline{B}(t) \wedge \underline{C}(t)))$. $(t)((t, \underline{t}) \longrightarrow \underline{A}(t))$, i.e. $(t)((t, \underline{t}) \longrightarrow (\underline{B}(t) \wedge \underline{C}(t)))$, can be deduced from $(t)((t, \underline{t}) \longrightarrow \underline{B}(t))$ and $(t)((t, \underline{t}) \longrightarrow \underline{C}(t))$ purely logically, which are equivalent to $\underline{B}(\underline{t})$ and $\underline{C}(\underline{t})$ by our induction assumption, respectively. Hence, $\underline{A}(\underline{t})$ implies $(t)((t, \underline{t}) \longrightarrow \underline{A}(t))$.

Also, $(\exists t)((t, \underline{t}) \wedge \underline{A}(t))$, i.e. $(\exists t)((t, \underline{t}) \wedge ((\underline{B}(t) \wedge \underline{C}(t)))$ implies $(\exists t)((t, \underline{t}) \wedge \underline{B}(t)) \wedge (\exists t)((t, \underline{t}) \wedge \underline{C}(t))$ purely logically, and this proposition is equivalent to $\underline{B}(\underline{t}) \wedge \underline{C}(\underline{t})$, i.e. $\underline{A}(\underline{t})$, by our induction assumption.

Further, $(t)((t, \underline{t}) \longrightarrow \underline{A}(t))$, i.e. $(t)((t, \underline{t}) \longrightarrow (\underline{B}(t) \wedge \underline{C}(t)))$ implies $(t)((t, \underline{t}) \longrightarrow \underline{B}(t)) \wedge (t)((t, \underline{t}) \longrightarrow \underline{C}(t))$ purely logically, and this proposition is equivalent to $\underline{B}(\underline{t}) \wedge \underline{C}(\underline{t})$, i.e. $\underline{A}(\underline{t})$, by our induction assumption.

In the third sub-case, where $\underline{A}(t)$ is a proposition of the form $\underline{B}(t) \vee \vee \underline{C}(t)$, I will prove at first

$$(\exists t)((t, \underline{t}) \wedge \underline{A}(t)), \text{ i.e. } (\exists t)((t, \underline{t}) \wedge (\underline{B}(t) \vee \underline{C}(t))),$$

and

$$(t)((t, \underline{t}) \longrightarrow \underline{A}(t)), \text{ i.e. } (t)((t, \underline{t}) \longrightarrow (\underline{B}(t) \vee \underline{C}(t))),$$

by assuming $\underline{A}(\underline{t})$, i.e. $\underline{B}(\underline{t}) \vee \underline{C}(\underline{t})$. These are purely logical consequences of our induction assumption that

$$\underline{B}(\underline{t}), \quad (\exists t)((t, \underline{t}) \wedge \underline{B}(t)), \quad \text{and} \quad (t)((t, \underline{t}) \longrightarrow \underline{B}(t))$$

are mutually equivalent and that

$$\underline{C}(\underline{t}),\quad (\exists t)((t,\underline{t}) \wedge \underline{C}(t)),\quad \text{and}\quad (t)((t,\underline{t}) \longrightarrow \underline{C}(t))$$

are mutually equivalent.

I will prove now $\underline{A}(\underline{t})$ by assuming $(t)((t,\underline{t}) \longrightarrow \underline{A}(t))$. According to Theorem 2. 1, we can take such t that satisfies $(t,\underline{t})$. Hence, $\underline{A}(t)$, i.e. $\underline{B}(t) \vee \underline{C}(t)$, holds for this t. Accordingly,

$$(\exists t)((t,\underline{t}) \wedge \underline{B}(t)) \vee (\exists t)((t,\underline{t}) \wedge \underline{C}(t))$$

holds. According to our induction assumption, however, this is equivalent to $\underline{B}(\underline{t}) \vee \underline{C}(\underline{t})$, i.e. $\underline{A}(\underline{t})$. Similarly, $\underline{A}(\underline{t})$ is deducible from $(\exists t)((t,\underline{t}) \wedge \underline{A}(t))$.

In the fourth sub-case, where $\underline{A}(t)$ is a proposition of the form $(u)\underline{B}(u,t)$, our induction assumption is that

$$\underline{B}(u,\underline{t}),\quad (\exists t)((t,\underline{t}) \wedge \underline{B}(u,t)),\quad \text{and}\quad (t)((t,\underline{t}) \rightarrow \underline{B}(u,t))$$

are mutually equivalent. At first, I will prove $(\exists t)((t,\underline{t}) \wedge \underline{A}(t))$ and $(t)((t,\underline{t}) \longrightarrow \underline{A}(t))$ by assuming $\underline{A}(\underline{t})$, i.e. $(u)\underline{B}(u,\underline{t})$. By virtue of Theorem 2.1, we can take such t that satisfies $(t,\underline{t})$. To show $\underline{A}(t)$, i.e. $(u)\underline{B}(u,t)$ for this t, take any u. Then, $B(u,\underline{t})$ holds, so $(t)((t,\underline{t}) \longrightarrow \underline{B}(u,t))$ holds. Accordingly, we have $\underline{B}(u,t)$. Hence, $(\exists t)((t,\underline{t}) \wedge \underline{A}(t))$. On the other hand, $(t)((t,\underline{t}) \longrightarrow \underline{A}(t))$ is a purely logical consequence of our assumptions. Also, $\underline{A}(\underline{t})$, i.e. $(u)\underline{B}(u,\underline{t})$ is a purely logical consequence of

$$(\exists t)((t,\underline{t}) \wedge \underline{A}(t)),\ \text{i.e.}\ (\exists t)((t,\underline{t}) \wedge (u)B(u,t)),$$

and our induction assumption that $B(u,\underline{t})$ is equivalent to

$$(\exists t)((t,\underline{t}) \wedge \underline{B}(u,t)).$$

To prove $\underline{A}(\underline{t})$, i.e. $(u)\underline{B}(u,\underline{t})$, by assuming $(t)((t,\underline{t}) \longrightarrow \underline{A}(t))$, i.e. $(t)((t,\underline{t}) \longrightarrow (u)\underline{B}(u,t))$, let us take any u. By virtue of Theorem 2.1, we can take such t that satisfies $(t,\underline{t})$. Then, $\underline{B}(u,t)$ holds for these u and t. Accordingly, we have $(\exists t)((t,\underline{t}) \wedge \underline{B}(u,t))$, which is equivalent to $\underline{B}(u,\underline{t})$ according to our induction assumption. Hence, we have $(u)\underline{B}(u,\underline{t})$, i.e. $\underline{A}(\underline{t})$. Similarly, $\underline{A}(\underline{t})$ is deducible from $(\exists t)((t,\underline{t}) \wedge \underline{A}(t))$.

In the fifth sub-case where $\underline{A}(t)$ is a proposition of the form $(\exists u)\underline{B}(u,t)$, our induction assumption is that

$$\underline{B}(u,\underline{t}),\ (\exists t)((t,\underline{t}) \wedge \underline{B}(u,t)),\quad \text{and}\quad (t)((t,\underline{t}) \longrightarrow \underline{B}(u,t))$$

are mutually equivalent. In this case, $\underline{A}(\underline{t})$, i.e. $(\exists u)\underline{B}(u, \underline{t})$, is purely logically equivalent to

$$(\exists t)((t, \underline{t}) \wedge \underline{A}(t)), \text{ i.e. } (\exists t)((t, \underline{t}) \wedge (\exists u)\underline{B}(u, t)),$$

under the induction assumption that $\underline{B}(u, \underline{t})$ is equivalent to

$$(\exists t)((t, \underline{t}) \wedge \underline{B}(u, t)).$$

Also, $(t)((t, \underline{t}) \longrightarrow \underline{A}(t))$, i.e. $(t)((t, \underline{t}) \longrightarrow (\exists u)B(u, t))$, is a purely logical consequence of $\underline{A}(\underline{t})$, i.e. $(\exists u)B(u, \underline{t})$, under our induction assumption.

Now, I will show $\underline{A}(\underline{t})$ by assuming $(t)((t, \underline{t}) \longrightarrow \underline{A}(t))$, i.e. $(t)((t, \underline{t}) \longrightarrow (\exists u)\underline{B}(u, t))$. By virtue of Theorem 2.1, we can take such t that satisfies $(t, \underline{t})$. For this t, we can take such u that satisfies $\underline{B}(u, t)$. Hence, we have $(\exists t)((t, \underline{t}) \wedge \underline{B}(u, t))$, which is equivalent to $\underline{B}(u, \underline{t})$ according to our induction assumption. Thus, we have $(\exists u)\underline{B}(u, \underline{t})$, i.e. $\underline{A}(\underline{t})$.

Thus, we have proved our theorem in all sub-cases of the case $\nu = n$ by assuming our theorem for cases $0 \leq \nu < n$.

THEOREM 2.15. *Let $\underline{A}(t)$ be any $[\underline{V}^+]$-proposition having t as its proper variable, and $\underline{t}$ be any term expression. Then, in $B(\underline{V})$,*

(UT) *$\underline{A}(\underline{t})$ is deducible from $(x)\underline{A}(x)$.*

(E*T) *$(\exists x)\underline{A}(x)$ is deducible from $\underline{A}(\underline{t})$.*

Proof. $(x)\underline{A}(x)$ surely implies $(t)((t, \underline{t}) \longrightarrow \underline{A}(t))$ in LP (or in LM), and $(t)((t, \underline{t}) \longrightarrow \underline{A}(t))$ is proved to be equivalent to $\underline{A}(\underline{t})$ in $B(\underline{V})$ by Theorem 2.14. Hence, (UT) holds in $B(\underline{V})$.

On the other hand, $(\exists x)\underline{A}(x)$ is deducible from $(\exists t)((t, \underline{t}) \wedge \underline{A}(t))$ in LP (or in LM), and $(\exists t)((t, \underline{t}) \wedge \underline{A}(t))$ is proved to be equivalent to $\underline{A}(\underline{t})$ in $B(\underline{V})$ by Theorem 2.14. Hence, (E*T) holds in $B(\underline{V})$.

Now, I can prove the main theorem of this chapter, which reads:

THEOREM 2.16. *Any $[\underline{V}^+]$-proposition (speaking exactly, its interpretation in $B(\underline{V})$) is provable in $B(\underline{V})$ if and only if it is provable in $B[\underline{V}]$.*

Proof. Because any deduction in the logic LP (or LM) can be regarded as a deduction in LPT (LMT), so we have only to prove that (the interpretation of) any provable $[\underline{V}^+]$-proposition in $B[\underline{V}]$ is also provable in $B(\underline{V})$. Because our interpretation of $[\underline{V}^+]$-propositions in $B(\underline{V})$ is faithful with respect to the logical constants, we have only to check the inference rules (UT) and (E*T) of the logic LPT (or LMT), which are the only inference rules distinct from the corresponding inference rules of the logic LP (LM).

According to Remark 2.13, any $[\underline{V}^{+}]$-proposition of the form $\underline{A}(\underline{t})$ for a term expression $\underline{t}$ can be obtained by substituting $\underline{t}$ in all places of t occurring in $\underline{A}(t)$ having t as its proper variable. So, our problem is to check for every $[\underline{V}^{+}]$-proposition of the form $\underline{A}(t)$ having t as its proper variable and for any term expression $\underline{t}$, whether

(UT) $\underline{A}(\underline{t})$ is deducible from $(x)\underline{A}(x)$

and (E*T) $(\exists x)\underline{A}(x)$ is deducible from $\underline{A}(\underline{t})$

hold in $B(\underline{V})$. However, this has been already proved to hold in Theorem 2.15.

Remark 2.17. By virtue of Theorem 2.16, we have no need to distinguish the theories $B(\underline{V})$ and $B[\underline{V}]$, so I will denote these both simply by $B(\underline{V})$, hereafter.

(3) Abstractions of normal kernels

It was my main purpose in establishing the theory $B(\underline{V})$ to make all the necessary abstractions of the form

$$(\exists p)(x)\cdots(z)(p(x,\cdots,z)\equiv\underline{A}(x,\cdots,z))$$

provable in the theory $B(\underline{V})$. I do not need to make the abstractions of such abnormal kernels as $x(x)$ provable. However, I would like to make all the abstractions of such normal kernels as those expressible by $\underline{V}$-propositions. For this purpose, I will define in this chapter normality of abstraction kernels with respect to the set of variables at first, and thereafter I would like to show that the abstraction

$$(\exists p)(x)\cdots(z)(p(x,\cdots,z)\equiv\underline{A}(x,\cdots,z))$$

is provable in $B(\underline{V})$ for any normal kernel $\underline{A}(x,\cdots,z)$ with respect to variables $x,\cdots,z$. Indeed, we can define normality notion nicely for this purpose.

For practical application, however, it is convenient that we can have such p that satisfy

$$p(\underline{x},\cdots,\underline{z})\equiv\underline{A}(\underline{x},\cdots,\underline{z})$$

for any sequence of term expressions $\underline{x},\cdots,\underline{z}$. This would be meaningful only when $\underline{A}(x, \cdots, z)$ is a $[\underline{V}^+]$-proposition having $x, \cdots, z$ as its proper variables, unless we modify the definition of $\underline{A}(\underline{x}, \cdots, \underline{z})$. In fact, it looks like desirable to call some kernels $\underline{A}(x, \cdots, z)$ normal with respect to $x, \cdots, z$ even though they are not $[\underline{V}^+]$-propositions having $x, \cdots, z$ as their proper variables. Accordingly, we introduce an interpretation of $\underline{A}(\underline{x}, \cdots, \underline{z})$ for some kernels $\underline{A}(x, \cdots, z)$ which are not $[\underline{V}^+]$-propositions having $x, \cdots, z$ as their proper variables. Naturally, we have to define $\underline{A}(\underline{x}, \cdots, \underline{z})$ so as to keep inferences rules (UT) and (E*T) for the newly defined interpretation of $\underline{A}(\underline{x}, \cdots, \underline{z})$.

DEFINITION 3.1. Any $[\underline{V}^+]$-proposition is called *normal* with respect to the set of free variables $x, \cdots, z$ if and only if the proposition can be confirmed to be so by the following rules:

(N1) Any $[\underline{V}^+]$-proposition of the forms

$R_i(\underline{t}, \cdots, \underline{t}_{n_i})$ for any sequence of term expressions $\underline{t}_1, \cdots, \underline{t}_{n_i}$,

$u\langle \underline{t}_1, \cdots, \underline{t}_n \rangle$ for any sequence of term expressions $\underline{t}_1, \cdots, \underline{t}_n$,

or $\underline{t} = \underline{s}$ for any pair of term expressions $\underline{t}$ and $\underline{s}$

is normal with respect to any set of free variables $x, \cdots, z$.

(N2) Any $[\underline{V}^+]$-proposition of the form $p(\underline{x}_1, \cdots, \underline{x}_n)$ is normal with respect to any set of free variables $x, \cdots, z$ except when p appears among $x, \cdots, z$.

(N3) Any $[\underline{V}^+]$-proposition of the forms $\underline{A} \longrightarrow \underline{B}$, $\underline{A} \wedge \underline{B}$, and $\underline{A} \vee \underline{B}$ is normal with respect to the set of free variables $x, \cdots, z$, if $\underline{A}$ as well as $\underline{B}$ is normal with respect to the same set of free variables.

(N4) Any $[\underline{V}^+]$-proposition of the forms $(u)\underline{A}$ and $(\exists u)\underline{A}$ is normal with respect to the set of free variables $x, \cdots, z$, if $\underline{A}$ is normal with respect to the set of free variables $u, x, \cdots, z$.

Remark 3.2. Any $[\underline{V}^+]$-proposition which is normal with respect to $x, \cdots, z$ as well as to $u, \cdots, w$ is normal with respect to $x, \cdots, z, u, \cdots, w$. Any $[\underline{V}^+]$-proposition which is normal with respect to $x, u, \cdots, w$ is normal with respect to $u, \cdots, w$.

THEOREM 3.3. *For any normal $[\underline{V}^+]$-proposition $\underline{A}(x, \cdots, z)$ with respect to the variable set $x, \cdots, z$, there is such a $[\underline{V}^+]$-proposition that satisfies*

1) *$\underset{\sim}{A}'(x, \cdots, z)$ is equivalent to $\underset{\sim}{A}(x, \cdots, z)$ for all free variables in it.*

2) *$\underset{\sim}{A}'(x, \cdots, z)$ is a normal $[\underset{\sim}{V}^+]$-proposition with respect to the variable set $x, \cdots, z$.*

3) *$\underset{\sim}{A}'(x, \cdots, z)$ is a $[\underset{\sim}{V}^+]$-proposition having $x, \cdots, z$ as its proper variables.*

Proof. Let $\underset{\sim}{A}'(x, \cdots, z)$ be the proposition obtained from $\underset{\sim}{A}(x, \cdots, z)$ on replacing every sub-proposition of the form $u<\underset{\sim}{t}_1, \cdots, \underset{\sim}{t}_n>$ for any variable u in the variable set $x, \cdots, z$ by $u = <\underset{\sim}{t}_1, \cdots, \underset{\sim}{t}_n>$. Then:

1) We can prove equivalence of $\underset{\sim}{A}(x, \cdots, z)$ and $A'(x, \cdots, z)$ by Definition 1.3. and Theorems 2.4 and 2.5.

2) We can prove that $\underset{\sim}{A}'(x, \cdots, z)$ is a normal $[\underset{\sim}{V}^+]$-proposition with respect to the variable set $x, \cdots, z$ by Definition 3.1.

3) We can prove that $\underset{\sim}{A}'(x, \cdots, z)$ is a $[\underset{\sim}{V}^+]$-proposition having $x, \cdots, z$ as its proper variables by Remark 2.13 and Definition 3.1.

Definition 3.4. For any normal $[\underset{\sim}{V}^+]$-proposition $\underset{\sim}{A}(x_1, \cdots, x_n)$ with respect to variables $x_1, \cdots, x_n$ and a sequence of term expressions $\underset{\sim}{t}_1, \cdots, \underset{\sim}{t}_n$, we understand by $\underset{\sim}{A}(\underset{\sim}{t}_1, \cdots, \underset{\sim}{t}_n)$ the proposition $A'(\underset{\sim}{t}_1, \cdots, \underset{\sim}{t}_n)$, where $\underset{\sim}{A}'(x_1, \cdots, x_n)$ is the normal $[\underset{\sim}{V}^+]$-proposition given by Theorem 3.3 which is equivalent to $\underset{\sim}{A}(x_1, \cdots, x_n)$ and which has $x_1, \cdots, x_n$ as its proper variables.

Theorem 3.5. *Let $\underset{\sim}{A}(t)$ be any normal $[\underset{\sim}{V}^+]$-proposition with respect to t, and $\underset{\sim}{t}$ be any term expression. Then,*

(UT) *$\underset{\sim}{A}(\underset{\sim}{t})$ is deducible from $(x)\underset{\sim}{A}(x)$.*

(E*T) *$(\exists x)\underset{\sim}{A}(x)$ is deducible from $\underset{\sim}{A}(\underset{\sim}{t})$.*

Proof. By Theorem 2.16 and Definition 3.4.

Lemma 3.6. *Any proposition of the form $(x, \underset{\sim}{x})$ for any variable x and any term expression $\underset{\sim}{x}$ is a normal $\underset{\sim}{V}^+$-proposition with respect to any set of variables.*

Proof. This can be proved by complete induction with respect to the rank of the term expression $\underset{\sim}{x}$ by making use of Definitions 1.15 and 3.1.

Theorem 3.7. *For any normal $[\underset{\sim}{V}^+]$-proposition with respect to a certain set* *of variables, there is a* $\underset{\sim}{V}^+$*-proposition which is equivalent to the* $[\underset{\sim}{V}^+]$*-proposition and normal with respect to the same set of variables.*

Proof. Let $\underset{\sim}{A}(x,\cdots,z)$ be any normal $[\underset{\sim}{V}^+]$-proposition with respect to the set of variables $x,\cdots,z$. To confirm that $\underset{\sim}{A}(x,\cdots,z)$ is normal by Definition 3.1, it is necessary to confirm that some sub-proposition of the form $\underset{\sim}{u}=\underset{\sim}{v}$ for term expressions $\underset{\sim}{u}$ and $\underset{\sim}{v}$ must be confirmed as normal according to the rule (N1) of the definition. Replace every sub-proposition of the form $\underset{\sim}{u}=\underset{\sim}{v}$ of $\underset{\sim}{A}(x,\cdots,z)$ by

$$(\exists u)(\exists v)(u=v \wedge (u,\underset{\sim}{u}) \wedge (v,\underset{\sim}{v})),$$

and let us denote by $\underset{\sim}{B}(x,\cdots,z)$ the proposition obtained by this replacement. Then, $\underset{\sim}{B}(x,\cdots,z)$ is easily provable to be equivalent to $\underset{\sim}{A}(x,\cdots,z)$ by making use of Theorems 2.2, 2.5, and 2.15. $\underset{\sim}{B}(x,\cdots,z)$ is also provable to be a normal $[\underset{\sim}{V}^+]$-proposition with respect to $x,\cdots,z$. Moreover, in any sub-proposition of the form $\underset{\sim}{u}=\underset{\sim}{v}$ of $\underset{\sim}{B}(x,\cdots,z)$, the term expressions $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are variables.

Now, I will prove our theorem for $\underset{\sim}{B}(x,\cdots,z)$ instead of $\underset{\sim}{A}(x,\cdots,z)$ by complete induction with respect to the highest rank ρ of term expressions occurring in $\underset{\sim}{B}(x,\cdots,z)$. If $\rho=0$, $\underset{\sim}{B}(x,\cdots,z)$ is already a $\underset{\sim}{V}^+$-proposition. If $\rho>0$, carry out the reduction A1—A3 of Definition 1.16 for every elementary sub-proposition of the forms $R_i(\underset{\sim}{x}_1,\cdots,\underset{\sim}{x}_{n_i})$, $p(\underset{\sim}{x})$, or $x<\underset{\sim}{y},\underset{\sim}{z}>$ containing any term expression of the highest rank ρ. Let $\underset{\sim}{B}'(x,\cdots,z)$ denote the replaced proposition. Then, $\underset{\sim}{B}'(x,\cdots,z)$ is provable to be equivalent to $\underset{\sim}{B}(x,\cdots,z)$, and $\underset{\sim}{B}'(x,\cdots,z)$ is also provable to be a normal $\underset{\sim}{V}^+$-proposition with respect to $x,\cdots,z$ and to have only term expressions of lower rank than ρ.

LEMMA 3.8. *Let* $\underset{\sim}{x}$ *and* $\underset{\sim}{y}$ *be*

the term $<<\cdots<x_1,x_2>,\cdots>,x_m>$ $\quad(m\geq 1)$

and *the term expression* $<y_1,\cdots,y_n>$ $\quad(n\geq),$

respectively. Then, any proposition of the forms

$$(p)(\exists q)(x_1)\cdots(x_m)(y_1)\cdots(y_n)(q(\underset{\sim}{x},\underset{\sim}{y})\equiv p(x_1,\cdots,x_m,\underset{\sim}{y}))$$

and

$$(p)(\exists q)(x_1)\cdots(x_m)(y_1)\cdots(y_n)(q(x_1,\cdots,x_m,\underset{\sim}{y})\equiv p(\underset{\sim}{x},\underset{\sim}{y}))$$

is provable in $B(\underset{\sim}{V})$*, where neither* p *nor* q *occur in* $\underset{\sim}{x}$ *and* $\underset{\sim}{y}$.

Proof. By repeated application of Axioms GP5 and GP6.

LEMMA 3.9. *Any proposition of the form*

$$(p)(\exists q)(x_1)\cdots(x_m)(u)(v)(y_1)\cdots(y_n)(q(x_1,\cdots,x_m,u,v,y_1,\cdots,y_n)$$
$$\equiv p(x_1,\cdots,x_m,v,u,y_1,\cdots,y_n))$$

is provable in $B(\underline{V})$ *for* $m,n\geq 0$, *where neither* p *nor* q *appear among* $x_1,\cdots,x_m,u,v,y_1,\cdots,y_n$.

Proof. In the case $m=n=0$, this is nothing but Axiom GP3.
In the case, $m,n\geq 1$, let us denote by $\underline{x}$ and $\underline{y}$

$$<<\cdots<x_1,x_2>,\cdots>,x_m> \text{ and } <y_1,\cdots,y_n>,$$

respectively. Then, from any p, we have g satisfying

$$(x_1)\cdots(x_m)(u)(v)(y_1)\cdots(y_n)(g(\underline{x},v,u,\underline{y})\equiv p(x_1,\cdots,x_m,v,u,\underline{y}))$$

according to Lemma 3.8. For this g, we have h satisfying

$$(x_1)\cdots(x_m)(u)(v)(y_1)\cdots(y_n)(h(\underline{x},u,v,\underline{y})\equiv g(\underline{x},v,u,\underline{y}))$$

by Axiom GP7 according to Theorem 2.16. Now, for this h, we have q satisfying

$$(x_1)\cdots(x_m)(u)(v)(y_1)\cdots(y_n)(q(x_1,\cdots,x_m,u,v,\underline{y})\equiv h(\underline{x},u,v,\underline{y})).$$

This q surely satisfies

$$(x_1)\cdots(x_m)(u)(v)(y_1)\cdots(y_n)(q(x_1,\cdots,x_m,u,v,y_1,\cdots y_n)$$
$$\equiv p(x_1,\cdots,x_m,v,u,y_1,\cdots,y_n)).$$

In the case $m=0,n\geq 1$, we can take such g for any p that satisfies

$$(x)(y)(g(x,y)\equiv p(y)),$$

by Axiom GP2. This g satisfies

$$(x)(u)(v)(y_1)\cdots(y_n)(g(x,v,u,\underline{y})\equiv p(u,v,\underline{y})).$$

According to the result in the preceding case, we can take such h that satisfies

$$(x)(u)(v)(y_1)\cdots(y_n)(h(x,u,v,\underline{y})\equiv g(x,v,u,\underline{y})).$$

For this h, we have q satisfying

$$(u)(v)(y_1)\cdots(y_n)(q(u, v, \underset{\sim}{y}) \equiv (x)h(x, u, v, \underset{\sim}{y}))$$

by Axiom GPU. We can easily prove that

$$(u)(v)(y_1)\cdots(y_n)(q(u, v, \underset{\sim}{y}) \equiv p(v, u, \underset{\sim}{y}))$$

holds for this q.

In the case $m \geqq 1$, $n = 0$, we can take such a for any p that satisfies

$$(x_1)\cdots(x_m)(u)(v)(a(<\underset{\sim}{x}, v>, u) \equiv p(x_1, \cdots, x_m, v, u))$$

by Axiom GP5. For this a, we can take such b that satisfies

$$(x_1)\cdots(x_m)(u)(v)(z)(b(z, <<\underset{\sim}{x}, v>, u>) \equiv a(<\underset{\sim}{x}, v>, u))$$

by Axiom GP2. For this b, we can take such c that satisfies

$$(x_1)\cdots(x_m)(u)(v)(z)(c(<<\underset{\sim}{x}, v>, u>, z) \equiv b(z, <<\underset{\sim}{x}, v>, u>))$$

by Axiom GP3. For this c, we can take such d that satisfies

$$(x_1)\cdots(x_m)(u)(v)(z)(d(\underset{\sim}{x}, v, u, z) \equiv c(<<\underset{\sim}{x}, v>, u>, z))$$

by Lemma 3.8. For this d, we can take such e that satisfies

$$(x_1)\cdots(x_m)(u)(v)(z)(e(\underset{\sim}{x}, u, v, z) \equiv d(\underset{\sim}{x}, v, u, z))$$

by already proved case $m, n \geqq 1$ of this lemma. For this e, we can take such f that satisfies

$$(x_1)\cdots(x_m)(u)(v)(z)(f(<<\underset{\sim}{x}, u>, v>, z) \equiv e(\underset{\sim}{x}, u, v, z))$$

by Lemma 3.8. For this f, we can take such g that satisfies

$$(x_1)\cdots(x_m)(u)(v)(z)(g(z, <<\underset{\sim}{x}, u>, v>) \equiv f(<<\underset{\sim}{x}, u>, v>, z))$$

by Axiom GP3. For this g, we can take such h that satisfies

$$(x_1)\cdots(x_m)(u)(v)(h(<\underset{\sim}{x}, u>, v) \equiv (z)g(z, <<\underset{\sim}{x}, u>, v>))$$

by Axiom GPU. For this h, we can take such q that satisfies

$$(x_1)\cdots(x_m)(u)(v)(q(x_1, \cdots, x_m, u, v) \equiv h(<\underset{\sim}{x}, u>, v))$$

by Axiom GP6. For this q, we can prove that

$$(x_1)\cdots(x_m)(u)(v)(q(x_1, \cdots, x_m, u, v) \equiv p(x_1, \cdots, x_m, v, u)).$$

LEMMA 3.10. *Let* $x_1, \cdots, x_n$ *and* $y_1, \cdots, y_n$ *be two permutations of the same set of* n *variables. Then, the proposition*

$$(p)(\exists q)(x_1)\cdots(x_n)(q(x_1,\cdots,x_n)\equiv p(y_1,\cdots,y_n))$$

is provable in $B(\underline{V})$, *where neither* p *nor* q *appear among* $x_1,\cdots,x_n$.

Proof. By repeated application of Lemma 3.9.

LEMMA 3.11. *Let* $x,\cdots,z$ *be a sequence of distinct variables,* $s,\cdots,t$ *be a sequence of variables of* $x,\cdots,z$, *maybe containing some repetitions, and* $u,\cdots,w$ *be a sequence of variables other than* $x,\cdots,z$ *but maybe containing some repetitions. Then, the proposition*

$$(p)(\exists q)(x)\cdots(z)(q(s,\cdots,t)\equiv p(u,\cdots,w,s,\cdots,t))$$

is provable in $B(\underline{V})$, *where neither* p *nor* q *appear among* $x,\cdots,z$.

Proof. I will prove this theorem by complete induction with respect to the length λ of the sequence $u,\cdots,w$. $(\lambda\geq 0)$

In the case $\lambda=0$, we have nothing to prove.

In the case $0<\lambda=l$ let us assume that our proposition holds in the cases $0\leq\lambda<l$. Take any arbitrary p. By Axiom GP1, we can take such h that satisfies

$$(x)(g(x)\equiv p(u,x)).$$

If we denote $u,\cdots,w$ by $u,v,\cdots,w$, this g satisfies

$$(x)\cdots(z)(g(v,\cdots,w,s,\cdots,t)=p(u,v,\cdots,w,s,\cdots,t)).$$

Moreover, by our induction assumption, we have such q that satisfies

$$(x)\cdots(z)(q(s,\cdots,t)\equiv g(v,\cdots,w,s,\cdots,t)).$$

This q surely satisfies

$$(x)\cdots(z)(q(s,\cdots,t)\equiv p(u,v,\cdots,w,s,\cdots,t)).$$

LEMMA 3.12. *Let* $x,\cdots,z$ *be a sequence of mutually different variables, and let* $s,\cdots,t$ *be a sequence of variables of the set* $\{x,\cdots,z\}$, *possibly having some repetition. Let further* $u,\cdots,w$ *be a sequence of all the remaining variables of the set* $\{x,\cdots,z\}$ *which do not occur in* $s,\cdots,t$. *Then, the proposition*

$$(p)(\exists q)(x)\cdots(z)(q(u,\cdots,w,s,\cdots,t)\equiv p(s,\cdots,t))$$

is provable in $B(\underline{V})$, *where neither* p *nor* q *appear among* $x,\cdots,z$.

Proof. I will prove this theorem by complete induction with respect to the length λ of the sequence $u, \cdots, w$,

In the case $\lambda = 0$, we have nothing to prove.

Next let us assume that our theorem holds in any case $0 \leq \lambda < l$, and I will prove the theorem in the case $0 < \lambda = l$. Let us denote any sequence $u, \cdots, w$ of the length λ by $u, v, \cdots, w$, and let us take any arbitrary p. By our induction assumption, we can take such g that satisfies

$$(x) \cdots (z)(g(v, \cdots, w, s, \cdots, t) \equiv p(s, \cdots, t)).$$

For this g, we can take by Axiom GP2 such q that satisfies

$$(u)(y)(q(u, y) \equiv g(y)).$$

Because this q satisfies

$$(x) \cdots (z)(q(u, v, \cdots, w, s, \cdots, t) \equiv g(v, \cdots, w, s, \cdots, t)),$$

it satisfies also

$$(x) \cdots (z)(q(u, v, \cdots, w, s, \cdots, t) \equiv p(s, \cdots, t)).$$

LEMMA 3.13. *Let* $\{x, \cdots, z\}$ *be a set of* n *variables, and* $u, \cdots, w$ *be a sequence of variables admitting repetition, each of which be a variable in the set* $\{x, \cdots, z\}$, *and every variable in the set* $\{x, \cdots, z\}$ *occur at least once in the sequence* $u, \cdots, w$. *Then, the proposition*

$$(p)(\exists q)(x) \cdots (z)(q(x, \cdots, z) \equiv p(u, \cdots, w))$$

is provable in $B(\underline{V})$, *where neither* p *nor* q *appear among* $x, \cdots, z$.

Proof. I will prove this theorem by complete induction with respect to the length λ of the sequence $u, \cdots, w$. Certainly, λ is a number no less than n.

In the case $\lambda = n$, the sequence $u, \cdots, w$ is a permutation of $\{x, \cdots, z\}$. Hence, we have the theorem by virtue of Lemma 3.10.

Now, let us assume that our theory holds for every case $n \leq \lambda < l$, and we will prove our theorem for the case $\lambda = l$. Namely, let $u, \cdots, w$ be a sequence of the length λ, and $v, v, \cdots$ be a sequence of the same set of variables, each having the same repetition as $u, \cdots, w$. Then, for any p we have such g that satisfies

$$(x)\cdots(z)(g(v, v, \cdots) \equiv p(u, \cdots, w)).$$

according to Lemma 3.10. For this g, we have such h that satisfies

$$(v)(t)(h(v, t) \equiv g(v, v, t)),$$

according to Axiom GP4. This h surely satisfies

$$(x)\cdots(z)(h(v, \cdots) \equiv g(v, v, \cdots)).$$

The sequence $v, \cdots$ is a sequence of the length $l-1$ of the variable set $\{x, \cdots, z\}$, and each variable of the set surely appears at least once in the sequence $v, \cdots$, so according to our induction assumption, we can find out such q that satisfies $(x)\cdots(z)(q(x, \cdots, z) \equiv h(v, \cdots))$ which implies

$$(x)\cdots(z)(q(x, \cdots, z) \equiv p(u, \cdots, w)).$$

LEMMA 3.14. *Any proposition of the form*

$$(p)(\exists q)(x)\cdots(z)(q(x, \cdots, z) \equiv p(u, \cdots, w)),$$

where $x, \cdots, z$ *is any sequence of mutually distinct variables without containing* p *and* q, *and* $u, \cdots, w$ *is any sequence of variables* $x, \cdots, z$ *admitting repetitions.*

Proof. This lemma is a consequence of Lemmata 3. 9—3. 13.

Now, I can give a proof of the following theorem, which is the main purpose of this chapter:

THEOREM 3.15. *For any normal* $[\underline{V}^+]$-*proposition* $\underline{A}(x, \cdots, z)$ *with respect to the variable set* $\{x, \cdots, z\}$, *the proposition*

$$(\exists p)(x)\cdots(z)(p(x, \cdots, z) \equiv \underline{A}(x, \cdots, z))$$

is provable in $B(\underline{V})$, *where* p *does not occur in* $\underline{A}(x, \cdots, z)$.

Proof. By virtue of Theorem 3.7, it is enough to prove this theorem only in the case where $\underline{A}(x, \cdots, z)$ is a $\underline{V}^+$-proposition. Any normal $\underline{V}^+$-proposition is confirmed to be normal by making use of the rules (N1)—(N4) of Definition 3.1. I will prove this theorem by complete induction with respect to the number ν of times of applications of these rules to confirm that $\underline{A}(x, \cdots, z)$ is normal with respect to $x, \cdots, z$. ($\nu \geqq 1$.)

In the case $\nu = 1$, $\underline{A}(x, \cdots, z)$ is either of the forms

$$R_i(t_1, \cdots, t_{n_i}), \qquad u<t, s>, \qquad t = s, \qquad \text{or} \qquad f(t),$$

where f does not belong to the variable set $x, \cdots, z$. By Axioms PNR$_i$, OPR, and ER, we can take such $g, h,$ and k that satisfy

$$(t_1)\cdots(t_{n_i})(g(t_1, \cdots, t_{n_i}) \equiv R_i(t_1, \cdots, t_{n_i})),$$
$$(u)(t)(s)(h(u, t, s) \equiv u<t, s>),$$
and
$$(t)(s)(k(t, s) \equiv . \ t = s).$$

Needless to say, we can take these variables $g, h,$ and k so that any one of them does not belong to the variable set $x, \cdots, z$. Hence, according to Lemma 3.14, we can take such p that satisfies.

$$(x)\cdots(z)(p(x, \cdots, z) \equiv g(t_1, \cdots, t_{n_i})),$$
$$(x)\cdots(z)(p(x, \cdots, z) \equiv h(u, t, s)),$$
$$(x)\cdots(z)(p(x, \cdots, z) \equiv k(t, s)),$$
$$(x)\cdots(z)(p(x, \cdots, z) \equiv f(t)),$$

in respective cases. This p surely satisfies

$$(x)\cdots(z)(p(x, \cdots, z) \equiv R_i(t_1, \cdots, t_{n_i})),$$
$$(x)\cdots(z)(p(x, \cdots, z) \equiv u<t, s>),$$
$$(x)\cdots(z)(p(x, \cdots, z) =. \ t = s),$$
$$(x)\cdots(z)(p(x, \cdots, z) \equiv f(t)),$$

in respective cases. Namely,

$$(\exists p)(x)\cdots(z)(p(x, \cdots, z) \equiv \underline{A}(x, \cdots, z))$$

is provable in $B(\underline{V})$ in the case $\nu = 1$.

Next, let us assume our theorem in all the cases $1 \leq \nu < n$, and we will prove our theorem in the case $1 < \nu = n$. Namely, let $\underline{A}(x, \cdots, z)$ be a $\underline{V}^+$-proposition which can be confirmed to be normal with respect to the variable set $x, \cdots, z$ after n time applications of the rules (N1)—(N4). Then, $\underline{A}(x, \cdots, z)$ must be a proposition of the forms:

$\underline{B}(x, \cdots, z) \longrightarrow \underline{C}(x, \cdots, z)$, $\underline{B}(x, \cdots, z) \wedge \underline{C}(x, \cdots, z)$, or $\underline{B}(x, \cdots, z) \vee C(x, \cdots, z)$

for a pair of normal $\underline{V}^+$-propositions $\underline{B}(x, \cdots, z)$ and $\underline{C}(x, \cdots, z)$ with respect to the variable set $x, \cdots, z$ (First sub-case), or, $(u)\underline{B}(u, x, \cdots, z)$ or $(\exists u)\underline{B}(u, x, \cdots, z)$ for a normal $\underline{V}^{+}$-proposition $\underline{B}(u, x, \cdots, z)$ with respect to the variable set $u, x, \cdots, z$ (Second sub-case).

In the first sub-case, $\underline{B}(x, \cdots, z)$ as well as $\underline{C}(x, \cdots, z)$ can be confirmed to be normal by a number of times of applications of the rules (N1)—(N4) less than n. Hence, according to our induction assumption, we have such free g and h that satisfy

$$(x)\cdots(z)(g(x, \cdots, z) \equiv \underline{B}(x, \cdots, z))$$

and

$$(x)\cdots(z)(h(x, \cdots, z) \equiv \underline{C}(x, \cdots, z)).$$

Now, for these g and h, we have such p that satisfies

$$(t)(p(t) \equiv .\ g(t) \longrightarrow h(t)),$$

$$(t)(p(t) \equiv .\ g(t) \wedge h(t)),$$

or

$$(t)(p(t) \equiv .\ g(t) \vee h(t)),$$

in respective cases, according to Axioms GPI, GPC, or GPD. For this p, we can easily prove

$$(x)\cdots(z)(p(x, \cdots, z) \equiv .\ \underline{B}(x, \cdots, z) \longrightarrow \underline{C}(x, \cdots, z)),$$

$$(x)\cdots(z)(p(x, \cdots, z) \equiv .\ \underline{B}(x, \cdots, z) \wedge \underline{C}(x, \cdots, z)),$$

or

$$(x)\cdots(z)(p(x, \cdots, z) \equiv .\ \underline{B}(x, \cdots, z) \vee \underline{C}(x, \cdots, z)),$$

in each one of respective cases. Thus, we have

$$(\exists p)(x)\cdots(z)(p(x, \cdots, z) \equiv \underline{A}(x, \cdots, z)).$$

In the second sub-case, $\underline{B}(u, x, \cdots, z)$ can be confirmed to be normal with respect to the variable set $u, x, \cdots, z$ after $n-1$ times of applications of the rules (N1)—(N4). Hence, according to our induction assumption, we have such g that satisfies

$$(u)(x)\cdots(z)(g(u, x, \cdots, z) \equiv \underline{B}(u, x, \cdots, z)).$$

For this g, we can take such p that satisfies

$$(t)(p(t) \equiv (u)g(u, t))$$

or

$$(t)(p(t) \equiv (\exists u)g(u, t)),$$

in each one of respective cases, according to Axioms GPU or GPE. For this p, we can prove easily

$$(x)\cdots(z)(p(x,\cdots,z)\equiv \underset{\sim}{A}(x,\cdots,z))$$

in the second sub-case.

THEOREM 3.16. *For any* $[\underline{V}^{+}]$*-proposition* $\underset{\sim}{A}(t)$ *which is normal with respect to* t *but possibly containing* x *and* y*, the proposition*

$$(x)(y)(x=y\longrightarrow(\underset{\sim}{A}(x)\equiv \underset{\sim}{A}(y)))$$

holds in $B(\underline{V})$.

Proof. Let $\underset{\sim}{A}(t)$ be any normal $[\underline{V}^{+}]$-proposition with respect to t, and x and y be any pair of arbitrary variables satisfying $x=y$. According to Theorem 3.15, we can take such p that satisfies

$$(t)(p(t)\equiv \underset{\sim}{A}(t)),$$

especially, $p(x)\equiv \underset{\sim}{A}(x)$ and $p(y)\equiv \underset{\sim}{A}(y)$. According to Definition 1.13, $(q)(q(x)\equiv q(y))$, especially

$$p(x)\equiv p(y).$$

holds. Hence, we have $\underset{\sim}{A}(x)\equiv \underset{\sim}{A}(y)$.

(4) Extensions of the basic theory $B(\underline{V})$.

Let $\underline{V}$ be any primitive vocabulary, and $\underline{V}^{+}$ be the vocabulary containing two more primitive notions S and T as before. Any $[\underline{V}^{+}]$-theory is called an *extension* of $B(\underline{V})$ if and only if the theory can be axiomatically formulated by supplying a finite number of $\underline{V}^{+}$-proposition axioms to the axiom system $\sum(\underline{V})$.

A vast class of formal $\underline{V}$-theories usually developed axiom-schematically on some logics between the classical logic LK and the minimal logic LM are expected to be extensions of $B(\underline{V})$. As any extensions of $B(\underline{V})$ is axiomatic (finitely axiomatizable), these axiom-schematic formal theories turn out to be finitely axiomatizable by formulating them as extensions of $B(\underline{V})$.

Although there still remain some problems to be discussed later, I will describe here by some examples how I am planning to construct mathematical theories on the basis of the basic object theory. As simple examples of such kind of theories, I will adopt here the object theories which I have developed in my former papers [4] and [5], and thereafter I will give a remark for formal theories standing on intermediate logics.

(4. 1) **Preliminary remarks for the object theories OZ and OF.**

The object theory **OZ** as well as the object theory **OF** is a formal theory standing on the classical logic **LK.** Both theories have the binary relation "$\in$" as their sole primitive notion. (In my original paper [4] for **OZ,** this relation has been denoted by "$\Subset$".) Accordingly, both theories are $\{\in\}$-theories.

In both theories, I use notation for relation products. Namely, let $x\varphi y$ and $x\theta y$ be a pair of binary relations between x and y. Then, the relation product $\varphi\theta$ is defined by

$$x\varphi\theta y \equiv (\exists z)(x\varphi z \wedge z\theta y).$$

It is also convenient to use the comprehension operator notation "{)" which maps binary relations φ into binary relation $\{\varphi)$. It is defined by

$$x\{\varphi)y \equiv (t)(t \in x \equiv t\varphi y).$$

Let us also define $x = y$ as usual by

$$x = y \equiv (t)(t \in x \equiv t \in y).$$

Then $\{\in)$ is nothing but the equality relation. It is very interesting that we can prove the following:

If any binary relation φ is unique in the sense that

$$(y)(\exists z)(x)(x = z \equiv x\varphi y),$$

the relation φ can be expressed in the form $\{\theta)$ for a suitable binary relation θ.

In any extension of $B(\underline{V})$ for any primitive vocabulary $\underline{V}$, any variable p can be regarded as a binary relation by defining it as follows:

$$xpy \equiv p(x, y).$$

Any extension of $B(\underline{V})$ having the axiom

(ALK) $\quad (p)(x)(\sim\sim p(x) \longrightarrow p(x))$

can be regarded as a theory having the classical logic for $\underline{V}$-propositions. This can be proved by making use of Theorem 3.15.

(4. 2) **The object theory OZ.**

The object theory **OZ** has been originally introduced in my paper [4] as a theory standing on the classical logic **LK**, having "$\in$" as its sole primitive notion, and being defined by a single axiom schema. To introduce the axiom schema, we define the relations "$\subseteq$" and "σ" successively as follows:

$$x \subseteq y \equiv (t)(t \in x \longrightarrow t \in y),$$
$$x\sigma y \equiv (p)((s)(s \subseteq\in p \equiv s \in\in p) \wedge y \in\in p. \longrightarrow x \in\in p).$$

Then, the axiom schema of **OZ** is

$$(m)(\exists y)(x)(x \in y \equiv. \ x\sigma m \wedge \underline{A}(x)),$$

where $\underline{A}(x)$ is a proposition having no occurrence of y.

To introduce **OZ** as an extention of $B(\{\in\})$, it would be necessary to supply axioms which indicate that we can deal with $\{\in\}$-propositions in the classical logic and that the just mentioned axiom schema holds for every $\{\in\}$-proposition $\underline{A}(x)$. As we have remarked in (4. 1), the first request can be expressed by the axiom (ALK). The second request can be expressed by the axiom

(AOZ) $\quad (p)(m)(\exists y)(x)(x \in y \equiv. \ x\sigma m \wedge p(x)).$

As we have shown in [4], the Zermelo set theory without the axiom of choice can be constructed on the theory **OZ**, which would be regarded as a so simple extension of $B(\{\in\})$. Naturally, the object theory thus defined can be regarded as finitely axiomatizable in the minimal logic **LM.**

(4. 3) **The object theory OF.**

The object theory **OF** has been introduced in my paper [5] as a theory standing on the classical logic **LK**, having "$\in$" as its sole primitive notion, and being defined by a single axiom schema. To introduce the axiom schema, we define the relations "$\hat{\in}$", "$\hat{\subseteq}$", and "$\hat{\sigma}$" successively as follows:

$$x \mathrel{\hat{\in}} y \equiv x\{\in) \in y,$$
$$x \mathrel{\hat{\subseteq}} y \equiv (t)(t \mathrel{\hat{\in}} x \longrightarrow t \mathrel{\hat{\in}} y),$$
$$x\hat{\sigma}y \equiv (p)((s)(s \in\in p \equiv s \mathrel{\hat{\subseteq}}\in p) \wedge y \in\in p. \longrightarrow x \in\in p).$$

Then, the axiom schema of **OF** is

$$(m)(\exists y)y\{\{R)\hat{\sigma})m$$

for any binary relation R having no occurrence of y.

To introduce **OF** as an extension of $B(\{\in\})$, it would be necessary to supply the axiom (ALK) and the axiom

$$(p)(m)(\exists y)y\{\{p)\hat{\sigma})m.$$

As we have shown in [5], the Fraenkel set-theory without the axiom of choice can be constructed on the theory **OF** which would be regarded as a so simple extension of $B(\{\in\})$. The object theory thus defined is surely finitely axiomatizable.

(4. 4) **Axiom of choice.**

The axiom of choice is occasionally formulated as an axiom schema, so it would be worth to examine if the axiom of choice can be formulated nicely in an extension of $B(\{\in\})$. Among many possible approaches to that purpose, I will suggest the way which has been taken in my paper [6]. Namely, we can formulate the generalized axiom of choice by asserting existence of such relation R that satisfies

$$(x)(x \ni\in x \longrightarrow x \ni\{R)x),$$

where $\ni$ is the inverse relation of $\in$. Similarly, we can formulate the axiom of choice by asserting existence of such R represented by a set that

$$(x)(x \in m \wedge x \ni\in x. \longrightarrow x \ni\{R)x).$$

In extensions of $B(\{\in\})$, these propositions can be formulated nicely as

(GCH) $\quad (\exists p)(x)(x \ni\in x \longrightarrow x \ni\{p)x),$

(CH) $\quad (m)(\exists p)(x)(x \in m \wedge x \ni\in x. \longrightarrow x \ni\{p)x).$

(4. 5) **Formal theories standing on intermediate logics.**

There are a great number of intermediate logics which can be characterized by axiom schemata. Usual axiom schemata would be formulated by propositions in extensions of $B(\underline{V})$. For example, the intuitionistic logic can be characterized by the axiom schema

$$\sim \underline{A} \longrightarrow (\underline{A} \longrightarrow \underline{B}).$$

This axiom schema can be formulated in any extension of $B(\underline{V})$ by the axiom

(LJA) $\quad (p)(q)(x)(\sim p(x) \longrightarrow (p(x) \longrightarrow q(x))).$

Also, Peirce's rule can be also axiomatized by

(PA) $\quad (p)(q)(x)(((p(x) \longrightarrow q(x)) \longrightarrow p(x)) \longrightarrow p(x))$

and *tertium non datur* can be axiomatized by

(TND) $\quad (p)(x)(p(x) \vee \sim p(x)).$

In this way, a great number of intermediate logics would turn out to be finitely axiomatizable on the minimal logic by making use of the device used for introducing $B(\underline{V})$. Also, a great number of axiom schematic formal theories standing on these logics would turn out to be finitely axiomatizable in the same way.

(5) **Concluding remarks.**

Any extension of $B(\underline{V})$ for a suitable $\underline{V}$ can be regarded as finitely axiomatizable on the logic LM. The logic LM can be regarded as a sublogic of the intuitionistic logic LJ without using the negation notion of the original logic LJ and defining new "negation: $\sim$" by

$$\sim \underline{A} \equiv \underline{A} \longrightarrow \lambda,$$

taking up a proposition constant λ. Hence, any extension of $B(\underline{V})$ can be regarded as finitely axiomatizable in the logic LJ. According to the result of my former paper [1], any axiomatizable formal theory standing on the logic LJ can be regarded as tabooistic. Hence, any extension of $B(\underline{V})$ for any $\underline{V}$ can be regarded as tabooistic.

For any tabooistic theory, we can interpret logical constants, $\wedge$, $\vee$, and $(\exists\ )$, and elementary propositions so that any proposition is provable in the tabooistic theory if and only if its interpretation is tautogical in the primitive logic LO. In other words, any tabooistic theory are purely logically constructible in the primitive logic LO.

Accordingly, any extension of $B(\underline{V})$ for suitable $\underline{V}$ is purely logically constructible in LO. Because a vast class of formal theories including most of popular mathematical theories can be regarded as extensions of $B(\underline{V})$ for suitable $\underline{V}$'s, we know that a vast class of formal theories including most of popular mathematical theories are purely logically constructible in LO.

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