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Nagoya Math. J.
Vol. 38 (1970), 181-183

THE TOPOLOGICAL SUPPORT OF GAUSS MEASURE ON HILBERT SPACE

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dedicated to Professor K. Ono for his sixtieth birthday

1. Introduction

Let X be a Hilbert space. The *topological support* of a Radon probability measure P on X is the least closed subset M of X that carries the total measure 1. A closed subset M of X is called a *linear subvariety* if

$x, y \in M$ implies $x + (1-\alpha)y \in M$ for every $\alpha \in R^1$,

or equivalently if $M = a + Y$ for some $a \in X$ and some closed linear subspace Y of X. A Radon probability measure P on X is called a *Gauss measure* if for every $a \in X$, the image measure of P by the map

$$f_a(x) = (a, x) : X \longrightarrow R^1$$

is a Gauss measure on R^1.

The purpose of this note is to prove

THEOREM. *Let P be a Gauss measure on a Hilbert space X. Then the topological support $S(P)$ of P is a linear subvariety of X.*

This fact is obvious in case X is finite dimensional but we need a small trick to discuss the infinite dimensional case as we shall see below.

2. Proof of the theorem.

Since P is a Gauss measure, its characteristic functional

$$C(z) = \int_X e^{i(z,x)} P(dx)$$

is expressed as

$$C(z) = \exp\left\{ i\,(z, m) - \frac{1}{2} \sum_k v_k (z, e_k)^2 \right\}$$

Received July 3, 1969

where $\{e_k\}$ is an orthonormal sequence (finite or countable) and

$$z \in X, \quad v_k > 0 \quad \sum_k v_k < \infty.$$

By the translation $x \longrightarrow x + m$, we can assume that $m = 0$, namely that

$$C(z) = \exp\left\{-\frac{1}{2}\sum_k v_k(z, e_k)^2\right\}.$$

Let Y be the closed linear subspace spanned by $\{e_k\}$. If $z \perp Y$, then

$$E(e^{it(z,x)}) = C(tz) = 1, \quad E(f(x)) = \int_X f(x)P(dx),$$

for every $t \in R^1$. Therefore we get

$$P(L_z) = 1, \quad L_z = \{x : (z, x) = 0\}.$$

Since $Y = \bigcap_z L_z$, we obtain

(1) $P(Y) = 1$,

because L_z is closed and P is Radon.

Now we will prove that $Y = S(P)$. For this purpose it is enough to prove that

$$P\{x \in X : \| x - a \| < r\} > 0$$

for every $a \in Y$ and every $r > 0$. Suppose to the contrary that we have $a \in Y$ and $r > 0$ such that

$$P\{x \in X : \| x - a \| < r\} = 0.$$

Then we have

(2) $E(e^{-\alpha\|x-a\|^2/2}) \leqq e^{-\alpha r^2/2}, \quad \alpha > 0.$

On the other hand we have by (1)

$$E(e^{-\alpha\|x-a\|^2/2}) = E(e^{-\alpha\sum_k (x_k - a_k)^2/2}), \quad x_k = (x, e_k), \quad a_k = (a, e_k).$$

Since

$$E(e^{i\sum_{k=1}^n z_k x_k}) = \exp\{-\textstyle\sum_{k=1}^n v_i z_k^2/2\}, \quad n = 1, 2, \cdots,$$

x_k, $k = 1, 2, \cdots$ are independent and each x_k is $N(0, v_k)$-distributed on the probability space (X, P). Thus we have

(3) $E(e^{-\alpha\|x-a\|^2/2}) = \prod_k E(e^{-\alpha(x_k-a_k)^2/2})$

$$= \prod_k \exp - \frac{\alpha a_k^2}{2(1-v_k\alpha)} (1+v_k\alpha)^{-1/2}.$$

Comparing (2) with (3) we have

(4) $\prod_k \exp \frac{a_k^2\alpha}{1+v_k\alpha} (1+v_k\alpha) \geq e^{\alpha r^2}.$

Writing I_1 and I_2 for the products corresponding to $k \leq N$ and $k > N$ respectively, we have

$$I_2 \leq \prod_{k>N} e^{a_k^2\alpha} e^{v_k\alpha} = e^{\alpha \sum_{k>N} (v_k + a_k^2)}.$$

Since $\sum v_k$ and $\sum a_k^2$ are both finite, we have

(5) $I_2 \leq e^{\alpha r^2/2}$

for some large N which is independent of α. Fix such N. From (4) and (5) we have

$$\prod_{k=1}^{N} \exp \frac{a_k^2\alpha}{1+v_k\alpha} (1+v_k\alpha) \geq e^{\alpha r^2/2}$$

namely

$$\prod_{k=1}^{N} \exp \frac{a_k^2\alpha}{1+v_k\alpha} \cdot \frac{\prod_{k=1}^{N} (1+v_k\alpha)}{e^{\alpha r^2/2}} \geq 1.$$

Letting $\alpha \uparrow \infty$, we have

$$\prod_{k=1}^{N} e^{a_k^2/v_k} \cdot 0 \geq 1,$$

which is a contradiction. This completes the proof.

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