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PROLONGATIONS OF G-STRUCTURES TO TANGENT BUNDLES OF HIGHER ORDER

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To Professor Katuzi Ono on the occasion of his 60th birthday

§ Introduction and Notations.

In the previous paper [4] we have studied the prolongations of G-structures to tangent bundles. The purpose of the present paper is to generalize the previous prolongations and to look at them from a wide view as a special case by considering the tangent bundles of higher order. In fact, in some places, the arguments and calculations in [4] are more or less simplified. Since the usual tangent bundle $T(M)$ of a manifold M considers only the first derivatives or first contact elements of M, the previous paper contains, in most parts, only the calculation of derivatives of first order.

Now, since the tangent bundle $\overset{r}{T}M$ to a manifold M of order r concerns with the derivatives of higher order (up to order r), the situations should be much complicated. Nevertheless, the (covariant) functor $\overset{r}{T}: M \to \overset{r}{T}M$ from the category of differentiable manifolds and differentiable maps to the same category, fortunately, has many properties similar to the functor $T: M \to TM$. For instance, (i) $\overset{r}{T}G$ is a Lie group if G is a Lie group, (ii) $\overset{r}{T}R^n$ has a natural vector space structure and (iii) $\overset{r}{T}GL(n)$ can be considered as a Lie subgroup of $GL(n(r+1))$. Therefore, we can follow the procedure in [4] by replacing the functor T with the functor $\overset{r}{T}$.

We mention here that Yano and Ishihara [7] study the prolongations of tensor fields to the tangent bundles of order 2 from the viewpoint of tensor analysis.

In §1, we explain the notion of tangent bundles $\overset{r}{T}M$ of order r to a manifold M, tangent bundles of order 1 coinciding with the usual tangent bundle.

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In §2, 3, we consider the tangent bundles to a Lie group of order r and prove that if a Lie group G operates on a manifold M effectively then the Lie group $\overset{r}{T}G$ operates canonically on $\overset{r}{T}M$ also effectively.

In §4, 5, we consider the vector space $\overset{r}{T}R^n$ and prove that $\overset{r}{T}GL(n)$ operates on $\overset{r}{T}R^n$ as a linear transformation group.

In §6, we consider the tangent bundle of higher order to (principal) fibre bundles.

In §7, we construct a canonical imbedding of $\overset{r}{T}FM$ into $F\overset{r}{T}M$, where FM denotes the frame bundle of M. Using the results in §6, 7 we can define in §8 the prolongation $P^{(r)}$ of order r of a G-structure P to the tangent bundle $\overset{r}{T}M$ for any r.

In §9, we prove that a diffeomorphism Φ: $M \to M'$ is an isomorphism of G-structures P with P' if and only if $\overset{r}{T}\Phi$ is an isomorphism of $P^{(r)}$ with $P'^{(r)}$.

In §10, we prove that a G-structure P is integrable if and only if the prolongation $P^{(r)}$ is integrable.

In §11, we consider some classical G-structures and prove, among others, that if a manifold M has an (resp. an integrable) almost complex sturcture, symplectic structure, pseudo-Riemannian structure or a (completely integrable) differential system, then $\overset{r}{T}M$ has canonically the same kind of structures. Moreover, if M has an almost contact structure, then $\overset{r}{T}M$ has a canonical almost complex structure for r odd and has an almost contact structure for r even.

In this paper, all manifolds and mappings (functions) are assumed to be differentiable of class C^∞, unless otherwise stated. If φ: $M \to N$ is a map of a set M into a set N and if A is a subset of M, we often denote by φ itself the restriction $\varphi|A$ of φ to A, if there is no confusion. If φ_i: $M_i \to N_i$ is a map for $i = 1, 2$, then the map $\varphi_1 \times \varphi_2$: $M_1 \times M_2 \to N_1 \times N_2$ is defined by $(\varphi_1 \times \varphi_2)(x_1, x_1) = (\varphi_1(x_1), \varphi_2(x_2))$ for $x_i \in M_i$, $i = 1, 2$. If $M_1 = M_2 = M$, the map (φ_1, φ_2): $M \to N_1 \times N_2$ is defined by $(\varphi_1, \varphi_2)(x) = (\varphi_1(x), \varphi_2(x))$ for $x \in M$.

In the following, R^n denotes always the n-dimensional real number space. The group of all linear automorphisms of R^n will be denoted by $GL(n, R)$ or simply by $GL(n)$. If $a^i_j \in R$ for $i, j = 1, 2, \cdots, n$, we denote by (a^i_j) the matrix of degree n whose (i, j)-entry is a^i_j.

§1. Tangent bundles of order *r*.

Let $\mathfrak{F}$ be the set of all real valued differentiable functions defined on some neighborhood of R containing zero. Take two functions f and g in $\mathfrak{F}$. For a positive integer r we say f is r-equivalent to g iff $d^\nu f/dt^\nu = d^\nu g/dt^\nu$ at $t=0$ for $\nu = 0, 1, \cdots, r$, and we will denote it by $f \underset{r}{\sim} g$. The relation $\underset{r}{\sim}$ is clearly an equivalence relation in $\mathfrak{F}$. Let M be an n-dimensional manifold, and let $C^\infty(M)$ be the ring of all differentiable functions defined on M. We denote by $\tilde{S}(M)$ (resp. $S(M)$) the set of all maps φ of some open interval $(-\varepsilon, \varepsilon)$ (resp. R) into M, $\infty \geqq \varepsilon > 0$ depending on φ. Let φ and ψ be two maps of $\tilde{S}(M)$. We say that φ is r-equivalent to ψ iff $f\circ\varphi \underset{r}{\sim} f\circ\psi$ for every $f \in C^\infty(M)$ and denote it by $\varphi \underset{r}{\sim} \psi$. The relation $\underset{r}{\sim}$ is also an equivalence relation in $\tilde{S}(M)$. For $\varphi \in \tilde{S}(M)$ we denote by $[\varphi]_r$ the equivalence class in $\tilde{S}(M)$ containing φ.

DEFINITION 1. 1. We call $[\varphi]_r$ the *r-tangent to M at $p \in M$* (or *r-jet*) defined by φ iff $\varphi(0) = p$.

For any r-tangent $[\varphi]_r$ to M there exists $\varphi' \in S(M)$ such that $[\varphi']_r = [\varphi]_r$ by virtue of the following

LEMMA 1. 2. *Let $\varphi \in \tilde{S}(M)$. Then there exist some $\varepsilon_1 > 0$ and $\varphi' \in S(M)$ such that φ is defined on $(-\varepsilon_1, \varepsilon_1)$ and $\varphi(t) = \varphi'(t)$ for $|t| < \varepsilon_1$.*

Proof. Since $\varphi \in \tilde{S}(M)$, there is some $\varepsilon > 0$ such that φ is defined on $(-\varepsilon, \varepsilon)$. We can find a function $g \in C^\infty(R)$ such that $g(t) = t$ for $|t| \leqq \varepsilon/2$ and $g(t) = 0$ for $|t| \geqq 2\varepsilon/3$ and that $|g(t)| \leqq 2\varepsilon/3$ for all $t \in R$. Put $\varepsilon_1 = \varepsilon/2$ and $\varphi' = \varphi \circ g$. It is now clear that φ' and ε_1 satisfy the required conditions.

Q.E.D.

DEEFINTION 1. 3. Let $\overset{r}{T}(M)$ (or $\overset{r}{T}M$) be the set of all r-tangents to M, and for $p \in M$ let $\overset{r}{T}_p(M)$ be the set of all r-tangents to M at p. We define $\overset{r}{\pi}\colon \overset{r}{T}(M) \to M$ by $\overset{r}{\pi}([\varphi]_r) = \varphi(0)$ for $[\varphi]_r \in \overset{r}{T}(M)$.

The notion of 1-tangents to M at p coincides with the notion of usual tangent vectors to M at p. In order to define the manifold structure in $\overset{r}{T}M$ we shall prove the following

LEMMA 1. 4. *Let $\{x_1, x_2, \cdots, x_n\}$ be a local coordinate system on some neighborhood U of $p \in M$. Take* two elements φ and ψ in $S(M)$ such that

$\varphi(0)=\psi(0)=p$. Then $\varphi \underset{r}{\sim} \psi$ if and only if $x_i \circ \varphi \underset{r}{\sim} x_i \circ \psi$ for $i=1,2,\cdots,n$.

Proof. Suppose $\varphi \underset{r}{\sim} \psi$. There exist a neighborhood V of p contained in U and a function $f_i \in C^\infty(M)$ $(i=1,2,\cdots,n)$ such that $f_i|V = x_i|V$. Since $f_i \circ \varphi \underset{r}{\sim} f_i \circ \psi$ and since $x_i \circ \varphi(t) = f_i \circ \varphi(t)$, $x_i \circ \psi(t) = f_i \circ \psi(t)$ for $|t| < \varepsilon$ with some $\varepsilon > 0$, we have $x_i|\varphi \underset{r}{\sim} x_i \circ \psi$ for $i=1,2,\cdots,n$.

Conversely, suppose $x_i \circ \varphi \underset{r}{\sim} x_i \circ \psi$ for $i=1,2,\cdots,n$. Take $f \in C^\infty(M)$. We have to prove $f \circ \varphi \sim f \circ \psi$, i.e. $d^\nu(f\circ\varphi)/dt^\nu = d^\nu(f \circ \Psi)/dt^\nu$ at $t=0$ for $\nu=0,1,2,\cdots,r$. This holds for $\nu=0$, since $\varphi(0)=\psi(0)$. Define $\Psi: U \to R^n$ by $\Psi(q)=(x_1(q), x_2(q),\cdots,x_n(q))$ for $q \in U$. Then the function $F = f \circ \Psi^{-1}$ is an element of $C^\infty(\Psi(U))$ and we have $f(q) = F(x_1(q),\cdots,x_n(q))$ for $q \in U$. Since $f(\varphi(t)) = F(x_1(\varphi(t)),\cdots,x_n(\varphi(t))$, we have the following

$$\frac{d(f\circ\varphi)}{dt} = \sum_{i=1}^n \left[\frac{\partial F}{\partial x_i}\right]_{x=\Psi\varphi(t))} \cdot \frac{d(x_i \circ \varphi)}{dt}, \tag{1. 1}$$

and hence we get

$$\left[\frac{d(f\circ\varphi)}{dt}\right]_{t=0} = \sum_{i=1}^n \left[\frac{\partial F}{\partial x_i}\right]_{x=\Psi(p)} \cdot \left[\frac{d(x_i\circ\varphi)}{dt}\right]_{t=0}.$$

Similarly, we have

$$\left[\frac{d(f\circ\psi)}{dt}\right]_{t=0} = \sum_{i=1}^n \left[\frac{\partial F}{\partial x_i}\right]_{x=\Psi(p)} \cdot \left[\frac{d(x_i\circ\psi)}{dt}\right]_{t=0}.$$

Hence we obtain $[d(f\circ\varphi)/dt]_0 = [d(f\circ\psi)/dt]_0$. Differentiate (1. 1) and evaluate at $t=0$, then we get $[d^2(f\circ\varphi)/dt^2]_0 = [d^2(f\circ\psi)/dt^2]_0$ and so on. Thus we see $f\circ\varphi \underset{r}{\sim} f \circ \psi$. Q.E.D.

We define the local coordinate system $\{\overset{(\nu)}{x_i} | i=1,2,\cdots,n;\ \nu=0,1,\cdots,r\}$ on $(\overset{r}{\pi})^{-1}(U)$ by $\overset{(\nu)}{x_i}([\varphi]_r) = (1/\nu!)[d^\nu(x_i(\varphi(t)))/dt^\nu]_{t=0}$ for $[\varphi]_r \in (\overset{r}{\pi})^{-1}(U)$. It is straightforward to see that $\overset{r}{T}(M)$ has a differentiable manifold structure by these coordinate systems and to see that $\overset{r}{\pi}$ is a differentiable surjective map of maximal rank. It is also clear that $\overset{r}{T}_p(M)$ is diffeomorphic to R^{rn} for any $p \in M$.

DEFINITION 1. 5. The manifold TM with the projection π is called *the tangent bundle to M of order r*. If U is an open subset of M, then $(\overset{r}{\pi})^{-1}(U)$ is an open submanifold of $\overset{r}{T}(M)$ which can be identified with $\overset{r}{T}(U)$.

However, it must be noticed that $\overset{r}{T}(M)(M, \overset{r}{\pi})$ is not a vector bundle over M.

We define π^r_s: $\overset{r}{T}(M) \to \overset{s}{T}(M)$ for $r > s$ by $\pi^r_s([\varphi]_r) = [\varphi]_s$ for $\varphi \in S(M)$.

On the other hand, M can be imbedded in $\overset{r}{T}(M)$ by $x \to [\gamma_x]_r$ for $x \in M$, where $\gamma_x \in S(M)$ is defined by $\gamma_x(t) = x$ for $t \in R$.

Let N be another manifold of dimension m. For any map Φ: $M \to N$, we define the induced map $\overset{r}{T}\Phi$: $\overset{r}{T}M \to \overset{r}{T}N$ by $(\overset{r}{T}\Phi)([\varphi]_r) = [\Phi \circ \varphi]_r$ for $\varphi \in S(M)$. It is easy to see that $\overset{r}{T}\Phi$ is well-defined and that $\overset{r}{T}\Phi$ is a differentiable map of $\overset{r}{T}M$ into $\overset{r}{T}N$. We shall call $\overset{r}{T}\Phi$ the tangent to Φ of order r (or simply r-tangent to Φ).

Let π_1 (resp. π_2) be the projection of $M \times N$ onto M (resp. N). We can readily see that $\overset{r}{T}(M \times N)$ can be identified with $\overset{r}{T}M \times \overset{r}{T}N$ by $[\varphi]_r \to ([\pi_1 \circ \varphi]_r, [\pi_2 \circ \varphi]_r)$ for $\varphi \in S(M \times N)$.

We can prove the following Propositions 1. 6 and 1. 7 whose proof will be straightforward.

PROPOSITION 1. 6. *Let M_0, M_1, M_2, M_3 be manifolds. and let Φ: $M_0 \to M_1$, Φ_1: $M_1 \to M_2$, Φ': $M_0 \to M_2$ and Ψ: $M_2 \to M_3$ be maps. Then, we have the following equalities:*

(i) $\overset{r}{T}(\Phi_1 \circ \Phi) = (\overset{r}{T}\Phi_1) \circ (\overset{r}{T}\Phi)$,

(ii) $\overset{r}{T}(\Phi, \Phi') = (\overset{r}{T}\Phi, \overset{r}{T}\Phi')$,

(iii) $\overset{r}{T}(\Phi \times \Psi) = \overset{r}{T}\Phi \times \overset{r}{T}\Psi$,

(iv) $\overset{r}{T}(1_M) = 1_{\overset{r}{T}M}$,

where 1_M stands for the identity map of M.

PROPOSITION 1. 7. *Let π_1(resp. π_2) be the projection of $M_1 \times M_2$ onto M_1 (resp. M_2), and let $\tilde{\pi}_1$ (resp. $\tilde{\pi}_2$) be the projection of $\overset{r}{T}M_1 \times \overset{r}{T}M_2$ onto $\overset{r}{T}M_1$ (resp. $\overset{r}{T}M_2$). Then, we have $\overset{r}{T}\pi_i = \tilde{\pi}_i$ for $i = 1, 2$.*

PROPOSITION 1. 8. *Let M, N be manifolds and let Φ be a map of M into N of maximal rank. Then, $\overset{r}{T}\Phi$ is a map of $\overset{r}{T}M$ into $\overset{r}{T}N$ of maximal rank.*

Proof. We shall prove only for the case $r = 2$, since the proof for $r \geqq 3$ is similar. Let $p_0 \in M$ and put $q_0 = \Phi(p_0)$. We take a coordinate neighborhood U (resp. V) of p_0 (resp. q_0) with coordinate system $\{x_1, \cdots, x_n\}$ (resp. $\{y_1, \cdots, y_m\}$) such that $\Phi(U) \subset V$. Then, $\overset{2}{T}U$ (resp. $\overset{2}{T}V$) has the induced coordinate system $\{x_i, \dot{x}_i, \ddot{x}_i \mid i = 1, 2, \cdots, n\}$ (resp. $\{y_j, \dot{y}_j, \ddot{y}_j \mid j = 1, 2, \cdots, m\}$). Put $F_i(x_1, \cdots, x_n) = y_i(\Phi(x))$ for $x \in U$. Take an element $[\varphi]_2 \in \overset{2}{T}(U)$ with coordinates $\{x_i, \dot{x}_i, \ddot{x}_i\}$, then $x_i(\varphi(t)) = x_i + \dot{x}_i t + \ddot{x}_i t^2 + \varepsilon_i(t)$, where $[d^2\varepsilon_i/dt^2]_0 = 0$. Hence, we have $y_i\Phi(x_1(\varphi(t)), \cdots, x_n(\varphi(t))) = F_i(x_1, \cdots, x_n)$ $+ \sum_j \frac{\partial F_i}{\partial x_j} \dot{x}_j t + \frac{1}{2}\left(\sum_{j,k} \frac{\partial^2 F_i}{\partial x_j \partial x_k} \dot{x}_k \dot{x}_j + 2\sum_j \frac{\partial F_i}{\partial x_j} \ddot{x}_j\right) t^2 + \eta_i(t)$, where $[d^2\eta_i/dt^2]_0 = 0$. Therefore, $(\overset{2}{T}\Phi)([\varphi]_2) = [\Phi \circ \varphi]_2$ has the following coordinates:

$$
(1.\ 2) \qquad \begin{cases} y_i = F_i(x), \quad \dot{y}_i \sum_j \dfrac{\partial F_i}{\partial x_j} \dot{x}_j , \\[2ex] \ddot{y}_i = \dfrac{1}{2} \sum_{j,k} \dfrac{\partial^2 F_i}{\partial x_j \partial x_k} \dot{x}_j \dot{x}_k + \sum_j \dfrac{\partial F_i}{\partial x_j} \ddot{x}_j. \end{cases}
$$

Hence, the map $\overset{2}{T}\Phi$ has the Jacobian matrix J with respect tlo the coordinate systems $\{\overset{(\nu)}{x_i} \mid i = 1, \cdots, n;\ \nu = 0, 1, 2\}$ and $\{\overset{(\nu)}{y_k} \mid k = 1, \cdots, m;\ \nu = 0, 1, 2\}$ as follows:

$$
(1.\ 3) \qquad J = \begin{pmatrix} \left(\dfrac{\partial F_i}{\partial x_k}\right) & 0 & 0 \\[2ex] (J_k^i) & \left(\dfrac{\partial F_i}{\partial x_k}\right) & 0 \\[2ex] (\ddot{J}_k^i) & (\dot{J}_k^i) & \left(\dfrac{\partial F_i}{\partial x_k}\right) \end{pmatrix}
$$

where $\dot{J}_k^i = \sum_j \frac{\partial^2 F_i}{\partial x_j \partial x_k} \dot{x}_j$ and $\ddot{J}_k^i = \frac{1}{2} \sum_{j,l} \frac{\partial^3 F_i}{\partial x_j \partial x_k \partial x_l} \dot{x}_j \cdot \dot{x}_l + \sum_j \frac{\partial^2 F_i}{\partial x_j \partial x_k} \ddot{x}_j$. Since the Jacobian matrix of Φ is $\left(\frac{\partial F_i}{\partial x_k}\right)$, which has the maximal rank, J has also the maximal rank.

COROLLARY 1. 9. *Let Φ be a regular map of M into N, namely the differential $T\Phi$ is an injective map of $T_p(M)$ into $T_{\Phi(p)}(N)$ for every point $p \in M$. Then, $\overset{r}{T}\Phi$ is also a regular map of $\overset{r}{T}M$ into TN.*

Remark 1. 10. We see that if Φ is a regular injective map, then $\overset{r}{T}\Phi$ is also a regular injective map.

§2. Tangent groups of order *r*.

Let G be a Lie group with group multiplication $\mu\colon G\times G\to G$ and with the unit element e.

THEOREM 2. 1. *$\overset{r}{T}G$ is a Lie group with group multiplication $\overset{r}{T}\mu$. The group G is a closed subgroup of $\overset{r}{T}G$ and $\overset{r}{T}_e(G)$ is a closed normal subgroup of $\overset{r}{T}G$ such that*

$$\overset{r}{T}G = G\cdot\overset{r}{T}_e(G),$$

with $G\cap\overset{r}{T}_e(G)=\tilde{e}$, where $\tilde{e}$ is the unit element of $\overset{r}{T}G$. Moreover the projection $\overset{r}{\pi}\colon \overset{r}{T}G\to G$ is a homomorphism. (*cf.* [3] *for* $r=1$)

Proof. For any two elements $\varphi,\ \psi\in S(G)$ (cf. §1), we define $\varphi\cdot\psi\in S(G)$ by $(\varphi\cdot\psi)(t)=\varphi(t)\cdot\psi(t)$ for $t\in R$. Then, we have $(\overset{r}{T}\mu)([\varphi]_r, [\psi]_r)=(\overset{r}{T}\mu)$ $([(\varphi,\psi)]_r)=[\mu\circ(\varphi,\psi)]_r=[\varphi\cdot\psi]_r$ and hence we get

$$(2.\ 1)\qquad (\overset{r}{T}\mu)([\varphi]_r,\ [\psi]_r)=[\varphi\cdot\psi]_r.$$

Since $(\varphi\cdot\psi)\cdot\eta=\varphi\cdot(\psi\cdot\eta)$ for any $\varphi,\psi,\eta\in S(G)$, we see that the multiplication $\overset{r}{T}\mu$ is associative. Define $\gamma_e\in S(G)$ by $\gamma_e(t)=e$ for $t\in R$ and put $\tilde{e}=[\gamma_e]_r$. Clearly $\tilde{e}$ is the unit element with respect to $\overset{r}{T}\mu$. For $\varphi\in S(G)$, we define $\varphi^{-1}\in S(G)$ by $\varphi^{-1}(t)=(\varphi(t))^{-1}$ for $t\in R$. Then $\overset{r}{T}\mu([\varphi]_r, [\varphi^{-1}]_r)=[\varphi\cdot\varphi^{-1}]_r=[\gamma_e]_r=\tilde{e}$ and hence $[\varphi^{-1}]_r$ is the inverse element of $[\varphi]_r$. Now, $[\varphi^{-1}]_r=(\overset{r}{T}\iota)[\varphi]_r$, where $\iota\colon G\to G$ is the map $x\to x^{-1}$ for $x\in G$. Since $\overset{r}{T}\iota$ is a differentiable map of $\overset{r}{T}G$ into itself, we have proved that $\overset{r}{T}G$ is a Lie group with group multiplication $\overset{r}{T}\mu$. Next, since $G=\{[\gamma_a]_r\,|\,a\in G\}$, where $\gamma_a(t)=a$ for $t\in R$, it follows that G is a closed subgroup of $\overset{r}{T}G$. Similarly we see that $\overset{r}{T}_eG$ is a closed normal subgroup of $\overset{r}{T}G$. Next, any $[\varphi]_r\in\overset{r}{T}G$ can be written as $[\varphi]_r=[\gamma_a]_r\cdot[\gamma_{a^{-1}}\cdot\varphi]_r$, where $a=\varphi(0)$ and so $[\gamma_{a^{-1}}\cdot\varphi]_r\in\overset{r}{T}_eG$. The equality $G\cap\overset{r}{T}_eG=\tilde{e}$ is also clear. Finally the projection $\overset{r}{\pi}$ is a homomorphism since (2. 1) holds. Q.E.D.

DEFINITION 2. 2. The Lie group TG with group multiplication $\overset{r}{T}\mu$ will be called *the tangent group to G of order r.*

PROPOSITION 2. 3. *Let Φ be a homomorphism of a Lie group G into a Lie group G'. Then $\overset{r}{T}\Phi$ is also a homomorphism of the tangent group $\overset{r}{T}G$ of order r into $\overset{r}{T}G'$.*

Proof. Let μ' be the group multiplication of G. Since Φ is a homomorphism, we have $\Phi\circ\mu=\mu'\circ(\Phi\times\Phi)$. By Proposition 1. 6 we have $\overset{r}{T}\Phi\circ\overset{r}{T}\mu = \overset{r}{T}\mu'\circ(\overset{r}{T}\Phi\times\overset{r}{T}\Phi)$, which means that $\overset{r}{T}\Phi$ is a homomorphism of $\overset{r}{T}G$ into $\overset{r}{T}G'$.

PROPOSITION 2. 4. *The projection π^r_s: $\overset{r}{T}G\to\overset{r}{T}G$ for $r>s$ is s a homomorphism of tangent groups.*

Proof. Clear from the equality (2. 1).

PROPOSITION 2. 5. *If G is a Lie subgroup of G', then $\overset{r}{T}(G)$ is also a Lie subgroup of $\overset{r}{T}(G')$.*

Proof. Let Φ: $G\to G'$ be the injection map. Then Φ is a regular map. By Remark 1. 10 and Proposition 2. 3, $\overset{r}{T}\Phi$ is a regular homomorphism of $\overset{r}{T}G$ into $\overset{r}{T}G'$. Let $[\varphi]_r$ be an element of $\overset{r}{T}G$ such that $(\overset{r}{T}\Phi)([\varphi]_r)=\tilde{e}'$ is the unit element of $\overset{r}{T}G'$. Then $[\Phi\circ\varphi]_r=[\gamma'_e]_r$, where γ'_e: $R\to G'$ is defined by $\gamma'_e(t)=e$ for $t\in R$, e being the unit element of G. We see that $\varphi(0)=e$ and that $[\varphi]_r=[\gamma_e]=\tilde{e}$. Hence $\overset{r}{T}\Phi$ is a regular injective homomorphism, which means that $\overset{r}{T}G$ is a Lie subgroup of $\overset{r}{T}G'$. Q.E.D.

§3. Tangent operations of order *r*.

Let G be a Lie group operating on a manifold M differentiably. We denote by ρ: $G\times M\to M$ the operation map of G on M.

PROPOSITION 3. 1. *The tangent group $\overset{r}{T}G$ to G of order r operates on the tangent bundle $\overset{r}{T}M$ of order r by the operation map $\overset{r}{T}\rho$ (for the tangent group TG, see* [3]).

Proof. Since ρ is the operation map of G on M, we have $\rho\circ(\mu\times 1_M) = \rho\circ(1_G\times\rho)$. By Proposition 1. 6 we have $(\overset{r}{T}\rho)\circ(\overset{r}{T}\mu\times 1_{\overset{r}{T}M})=\overset{r}{T}\rho\circ(1_{\overset{r}{T}G}\times\overset{r}{T}\rho)$, which means that $\tilde{a}\cdot(\tilde{b}\cdot\tilde{x})=(\tilde{a}\cdot\tilde{b})\cdot\tilde{x}$ for $\tilde{a},\tilde{b}\in\overset{r}{T}(G)$ and $\tilde{x}\in\overset{r}{T}M$, where we have put $\tilde{a}\cdot\tilde{x} = (\overset{r}{T}\rho)(\tilde{a},\tilde{x})$. Let γ_e: $R\to G$ be the constant map: $\gamma_e(t)=e$ for $t\in R$. Then, for any $[\varphi]_r\in \overset{r}{T}M$ we have $\overset{r}{T}\rho([\gamma_e],[\varphi]_r) = \overset{r}{T}\rho([\gamma_e,\varphi)]_r) = [\rho\circ(\gamma_e,\varphi)]_r = [\gamma_e,\varphi)]_r = [\varphi]_r$, which means that the unit element $\tilde{e}=[\gamma_e]_r$ of $\overset{r}{T}G$ operates on $\overset{r}{T}M$ as the identity map. Hence we have proved that $\overset{r}{T}G$ operates on $\overset{r}{T}M$ by $\overset{r}{T}\rho$. Q.E.D.

DEFINITION 3. 2. The operation map $\overset{r}{T}\rho$ in Proposition 3. 1 will be called *the tangent operation to* ρ *of order* r.

PROPOSITION 3. 3. *If a Lie group* G *operates on* M *effectively* (i.e. $a\cdot x=x$ *for all* $x\in M$ *implies* $a=e$), *then* $\overset{2}{T}G$ *operaties on* $\overset{2}{T}M$ *effectively by the tangent operation of order* 2.

Proof. For $\varphi\in S(G)$ and $\psi\in S(M)$ we define $\varphi\cdot\psi\in S(M)$ by $(\varphi\cdot\psi)(t) = \varphi(t)\cdot\psi(t)$ for $t\in R$. Suppose $\varphi\cdot\psi \underset{2}{\sim} \psi$ for every $\psi\in S(M)$. We have to show that $\varphi\underset{2}{\sim}\gamma_e$, where $\gamma_e\in S(G)$ is defined by $\gamma_e(t)=e$. First, since $\varphi(0)\cdot\psi(0)=\psi(0)$ for any $\psi\in S(M)$, we see that $\varphi(0)\cdot x = x$ for any $x\in M$, whence $\varphi(0)=e$ since G operates effectively on M. Next take a point $p_0\in M$ and fix it. We take a coordinate neighborhhod U (resp. V) of p_0 (resp. of e) in M (resp. in G) with coordinate system $\{x_1,\cdots,x_n\}$ (resp. $\{z_1,\cdots,z_N\}$) such that $x_i(p_0)=0$ for $i=1,2,\cdots,n$ (resp. $z_l(e)=0$ for $l=1,2,\cdots,N$). Define the functions $F_i (i=1,\cdots,n)$ by

$$F_i(z_1,\cdots,z_N;\ x_1,\cdots,x_n) = x_i(\rho(z,x)).$$

Let $\{\overset{(\nu)}{x_i}\,|\,i=1,\cdots,n;\ \nu=0,1,2\}$ (resp. $\{\overset{(\nu)}{z_l}\,|\,l=1,\cdots,N;\ \nu=0,1,2\}$) be the induced coordinate system on $\overset{r}{T}(U)$ (resp. $\overset{r}{T}(V)$). If $\overset{(0)}{x_i}([\psi]_2)=x_i$, $\overset{(1)}{x_i}([\psi]_2)=\dot{x}_i$, $\overset{(2)}{x_i}([\psi]_2)=\ddot{x}_i$, we see that

$$\psi(t) = (\cdots,\ x_i+\dot{x}_i+\ddot{x}_i t^2+\varepsilon_i(t),\cdots)\in U$$

for small $|t|$, where $[d^2\varepsilon_i/dt^2]_0=0$ for $i=1,\cdots,n$. Similarly we see that

$$\varphi(t) = (\cdots,\ \dot{z}_l t+\ddot{z}_l t+\eta_l(t),\cdots)\in V$$

for small $|t|$, where $[d^2\eta_l/dt^2]_0=0$ for $l=1,\cdots,N$. We have the relations $x_i\circ(\varphi\cdot\psi)\underset{r}{\sim} x_i\circ\psi$ $(i=1,2,\cdots,n)$ for every $\psi\in S(M)$. To simplify the notations we define the functions $f_i(t)$ for $i=1,\cdots,n$ by

$$f_i(t) = F_i(\cdots,\varphi_l(t),\cdots;\ \cdots,\psi_i(t),\cdots)$$

and we define the variables $\overset{(\nu)}{y}_\kappa$ for $\kappa = 1, 2, \cdots N+n$; $\nu = 0, 1, \cdots, r$ by $\overset{(\nu)}{y}_\kappa = \nu ! \cdot \overset{(\nu)}{z}_\kappa$ for $\kappa = 1, 2, \cdots, N$ and $\overset{(\nu)}{y}_k = \nu ! \cdot \overset{(\nu)}{x}_{\kappa - N}$ for $\kappa = N+1, \cdots, N+n$. By means of these notations we have the following equalities

$$\frac{df_i}{dt} = \sum_{\kappa=1}^{N+n} \frac{\partial F_i}{\partial y_\kappa} (\dot{y}_\kappa + \ddot{y}_\kappa t + \varepsilon_1^\kappa(t)) \tag{3. 1}$$

$$\frac{d^2 f_i}{dt^2} = \sum_\kappa \frac{\partial F_i}{\partial y_\kappa} (\ddot{y}_\kappa + \overset{(3)}{y}_\kappa t + \varepsilon_2^\kappa(t)) + \sum_{\kappa, \lambda} \frac{\partial^2 F_i}{\partial y_\kappa \partial y_\lambda} (\dot{y}_\kappa + \ddot{y}_\kappa t + \varepsilon_1^\kappa(t)) \cdot (\dot{y}_\lambda + \ddot{y}_\lambda t + \varepsilon_1^\lambda(t)), \tag{3. 2}$$

where $[d\varepsilon_k^\kappa / dt]_0 = 0$ for $k = 1, 2$. Since $f_i(t) = (x_i \circ (\varphi \cdot \psi))(t)$ and since $x_i \circ (\varphi \cdot \psi) \sim x_i \circ \psi$ we obtain the following relations:

$$\sum_{l=1}^{N} \left[\frac{\partial F_i}{\partial z}\right]_{(0,x)} \dot{z}_l + \sum_{j=1}^{n} \left[\frac{\partial F_i}{\partial x_j}\right]_{(0,x)} \dot{x}_j = \dot{x}_i, \tag{3. 3}$$

$$\begin{aligned} &2\sum_l \left[\frac{\partial F_i}{\partial z_l}\right]_{(0,x)} \ddot{z}_l + 2\sum_j \left[\frac{\partial F_i}{\partial x_j}\right]_{(0,x)} \ddot{x}_j \\ &+ \sum_{l,m=1}^{N} \left[\frac{\partial^2 F_i}{\partial z_l \partial z_m}\right]_{(0,x)} \dot{z}_l \cdot \dot{z}_m + 2\sum_{l=1}^{N} \sum_{j=1}^{n} \left[\frac{\partial^2 F_i}{\partial z_l \partial x_j}\right]_{(0,x)} \dot{z}_l \dot{x}_j \\ &+ \sum_{j,k=1}^{n} \left[\frac{\partial F_i}{\partial x_j \partial x_k}\right]_{(0,x)} \dot{x}_j \dot{x}_k = \ddot{x}_i \end{aligned} \tag{3. 4}$$

for $i = 1, 2, \cdots, n$ and for every $(x_i, \dot{x}_i, \ddot{x}_i) \in U$. Now, since $e \cdot x = x$ for any $x \in M$, we have

$$F_i(0, \cdots, 0; x_1, \cdots, x_n) = x_i$$

for $i = 1, 2, \cdots, n$. Therefore, we get $\left[\frac{\partial F_i}{\partial x_j}\right]_{(0,x)} = \delta_j^i$ for $i, j = 1, 2, \cdots, n$. Finally, we obtain. from (3. 3), (3. 4) the following relations:

$$\sum_{l=1}^{N} \left[\frac{\partial F_i}{\partial z}\right]_{(0,x)} \dot{z}_l = 0. \tag{3. 5}$$

$$\begin{aligned} &2\sum \left[\frac{\partial F_i}{\partial z_l}\right]_{(0,x)} \ddot{z}_l + \sum_{l,m} \left[\frac{\partial^2 F_i}{\partial z_l \partial z_m}\right]_{(0,x)} \dot{z}_l \dot{z}_m \\ &+ 2\sum_{l,j} \left[\frac{\partial^2 F_i}{\partial z_l \partial x_j}\right]_{(0,x)} \dot{z}_l \cdot \dot{x}_j = 0 \end{aligned} \tag{3. 6}$$

for every $(x_1, \cdots, x_n) \in U$ and $i = 1, \cdots, n$.

Now, we shall prove the following

LEMMA 3. 4. *Let* $a_1, \cdots, a_N \in R$. *Suppose* $\sum_{l=1}^{N} a_l \left[\frac{\partial F_i}{\partial z_l}\right]_{(0,x)} = 0$ *holds for every* $(x_1, \cdots, x_n) \in U$ *and for* $i = 1, 2, \cdots, n$, *where* U *is an arbitrary coordinate neighborhood in* M. *Then* $a_l = 0$ *for* $l = 1, 2, \cdots, N$.

By virtue of this lemma, we see from (3. 5) that $\dot{z}_l = 0$ for $l = 1, 2, \cdots, N$ and then from (3. 6) it follows that $\ddot{z}_l = 0$ for $l = 1, 2, \cdots, N$, which proves that $\varphi \underset{2}{\sim} \gamma_e$ and thus the proposition will be proved.

Proof of Lemma 3. 4. Suppose $a_l \neq 0$ for some l. Let $\mathfrak{g}$ be the Lie algebra of G. By taking a linear transformation of the coordinates $\{z_1, \cdots, z_N\}$, if necessary, we can suppose that $[\partial F_i / \partial z_1]_{(0,x)} = 0$ for any $x \in U$ and that $z_i (\exp \sum_{j=1}^{N} t_j X_j) = t_i$ for $i = 1, 2, \cdots, N$, where $\{X_1, \cdots, X_N\}$ is a base of $\mathfrak{g}$. Now let $\tilde{X}_1$ be the vector field on M induced by the one-parameter group $\exp tX_1$. For any point $x \in U$, we have $(\tilde{X}_1)_x = 0$, since $(\tilde{X}_1)_x \cdot x_i = [dx_i((\exp tX_1) \cdot x)/dt]_0 = [dF_i(t, 0, \cdots, 0;\ x)/dt]_0 = [\partial F_i / \partial z_1]_{(0,x)} = 0$ for $i = 1, 2, \cdots, n$. Since U and x are arbitrary, we see that $\tilde{X}_1 = 0$ on M and that $\exp tX_1$ operates trivially on M. It follows that $\exp tX_1 = e$ for any $t \in R$ and hence $X_1 = 0$, which is a contradiction. Thus Lemma 3. 4 is proved and hence the proof of Proposition 3. 3 is complete. Q.E.D.

More generally, we can prove the following

TH: OREM 3. 5. *If a Lie group* G *operates on* M *effectively, then* $\overset{r}{T}G$ *operates on* $\overset{r}{T}M$ *effectively by tangent operation of order* r *for any poistive integer* r.

Proof. Using the notations of the proof of Proposition 3. 3, especially the notations of (3. 1), we define $\varphi_\alpha(t)$ by $\varphi_\alpha(t) = \dot{y}_\alpha + \ddot{y}_\alpha t + \varepsilon_1^\alpha(t)$ for $\alpha = 1, 2, \cdots, N + n$. Then the equality (3. 2) can be written as follows:

$$\frac{d^2 f_i}{dt^2} = \sum \frac{\partial F_i}{\partial y_\alpha} \cdot \varphi_\alpha + \sum \frac{\partial^2 F_i}{\partial y_\alpha \partial y_\beta} \varphi_\alpha \varphi_\beta. \tag{3. 7}$$

By differentiating (3. 7), we obtain the following

$$\frac{d^3 f_i}{dt^3} = \sum \frac{\partial^3 F_i}{\partial y_\alpha \partial y_\beta \partial y_\gamma} \varphi_\alpha \varphi_\beta \varphi_\gamma + 3 \sum \frac{\partial^2 F_i}{\partial y_\alpha \partial y_\beta} \varphi'_\alpha \varphi_\beta + \sum \frac{\partial F_i}{\partial y_\alpha} \varphi''_\alpha. \tag{3. 8}$$

In general, by induction on $\nu = 1, 2, \cdots$, we obtain the following equality

$$(3.9)\qquad \frac{d^\nu f_i}{dt^\nu} = \sum \frac{\partial^\nu F_i}{\partial y_{\alpha_1}\cdots\partial y_{\alpha_\nu}} \varphi_{\alpha_1}\cdots\varphi_{\alpha_\nu}$$
$$+ c_1^{(\nu)}\sum \frac{\partial^{\nu-1} F_i}{\partial y_{\alpha_1}\cdots\partial y_{\alpha_{\nu-1}}} \varphi'_{\alpha_1}\varphi_{\alpha_2}\cdots\varphi_{\alpha_{\nu-1}}$$
$$+ c_{1,1}^{(\nu)}\sum \frac{\partial^{\nu-2} F_i}{\partial y_{\alpha_1}\cdots\partial y_{\alpha_{\nu-2}}} \varphi'_{\alpha_1}\varphi'_{\alpha_2}\varphi_{\alpha_3}\cdots\varphi_{\alpha_{\nu-2}}$$
$$+ c_2^{(\nu)}\sum \frac{\partial^{\nu-2} F_i}{\partial y_{\alpha_1}\cdots\partial y_{\alpha_{\nu-2}}} \varphi''_{\alpha_1}\varphi_{\alpha_2}\cdots\varphi_{\alpha_{\nu-2}}$$
$$+ \cdots + c_{\nu-2}^{(\nu)}\sum \frac{\partial^2 F_i}{\partial y_{\alpha_1}\partial y_{\alpha_2}} \varphi_1^{(\nu-2)}\varphi_{\alpha_2} + \sum \frac{\partial F_i}{\partial y_\alpha}\varphi_\alpha^{(\nu-1)},$$

where $c^{(\nu)}_{\mu_1\cdots\mu_s}$ are some positi e integer for $\sum_{i=1}^{s}\mu_i = 1, 2, \cdots, \nu-2$ and for any $\nu = 1, 2, \cdots$.

Suppose $(\varphi\cdot\psi)\underset{r}{\sim}\psi$ for every $\psi\in S(M)$ as in the proof of Proposition 3. 3. By using (3. 9) and Lemma 3. 4 repeatedly we can show, by induction on ν, that $z_l = 0$ for any $l = 1, 2, \cdots, N$ and $\nu = 0, 1, \cdots, r$, which proves that $\varphi\underset{r}{\sim}\gamma_e$. Q.E.D.

§4. Tangent bundle to R^n of order r.

Let R^n be the real euclidean space of dimension n. For any two r-tangents $[\varphi]_r$, $[\psi]_r$ to R^n, we define their sum by: $[\varphi]_r + [\psi]_r = [\varphi+\psi]_r$, where $(\varphi+\psi)(t) = \varphi(t) + \psi(t)$ for $t\in R$. For any $c\in R$ we define the scalar multiplication of $[\varphi]_r$ by c as follows: $c\cdot[\varphi]_r = [c\cdot\varphi]_r$, where $(c\cdot\varphi)(t) = c\cdot\varphi(t)$ for $t\in R$. Clearly $[\varphi]_r + [\psi]_r$ and $c\cdot[\varphi]_r$ are well-defined.

THEOREM 4. 1. *By the above sum and scalar multiplication the tangent bundle* $\overset{r}{T}R_n$ *to* R^n *of order* r *is a real vector space of dimension* $n(r+1)$.

Proof. Straightforward verification. Q.E.D.

PROPOSITION 4. 2. *Let* $V\oplus W$ *be a direct sum of vector subspaces* V *and* W, *then* $\overset{r}{T}V$ *and* $\overset{r}{T}W$ *are identified with vector subspaces of* $T(V\oplus W)$ *and we have*

$$\overset{r}{T}(V\oplus W) = \overset{r}{T}V\oplus\overset{r}{T}W \quad (\textit{direct sum}).$$

Remark 4. 3. Let $\{x_1, \cdots, x\}$ be the natural coordinate system on R^n and let $\{x_i | i = 1, \cdots, n; \nu\}$ be the induced coo dinate system on $\overset{r}{T}R^n$.

Then the sum and scalar multiplication in $\overset{r}{T}R^n$ in Theorem 4. 1 are as follows:

$$\begin{cases} (\overset{(\nu)}{x_i}) + (\overset{(\nu)}{x'_i}) = (\overset{(\nu)}{x_i} + \overset{(\nu)}{x'_i}), \\ c \cdot (\overset{(\nu)}{x_i}) = (c \cdot \overset{(\nu)}{x_i}). \end{cases}$$

§5. Imbedding of $\overset{r}{T}GL(n)$ into $GL(n(r+1))$.

Let ρ: $GL(n) \times R^n \to R^n$ be the usual operation of the general linear group $GL(n)$ on R^n. By Proposition 3. 1, the tangent group $\overset{r}{T}GL(n)$ to $GL(n)$ of order r operates on $\overset{r}{T}R^n$ by the tangent operation $\overset{r}{T}\rho$ to ρ of order r. Now, by Theorem 4. 1, $\overset{r}{T}R^n$ is a vector space of dimension $n(r+1)$. We shall prove the following

THEOREM 5. 1. *The tangent group $\overset{r}{T}GL(n)$ to $GL(n)$ of order r operates on $\overset{r}{T}R^n$ effectively as a linear group.*

Proof. Since ρ is effective, we see that $\overset{r}{T}\rho$ is effective by Theorem 3. 5. For any $\eta \in S(GL(n))$ and $\varphi \in S(R^n)$, we define $\eta \cdot \varphi \in S(R^n)$ by the equality $(\eta \cdot \varphi)(t) = \eta(t) \cdot (t) = \rho(\eta(t), \varphi(t))$ for $t \in R$. We put $[\eta]_r \cdot [\varphi]_r = \overset{r}{T}\rho([\eta]_r, [\varphi]_r)$. Then we have $[\eta]_r \cdot [\varphi]_r = [\eta \cdot \varphi]_r$. Take an element $[\psi]_r$ of $\overset{r}{T}(R^n)$ and $c \in R$. Then we calculate as follows: $[\eta]_r([\varphi]_r + [\psi]_r) = [\eta]_r \cdot [\varphi + \psi]_r = [\eta \cdot (\varphi + \psi)]_r$ $= [\eta \cdot \varphi + \eta \cdot \psi]_r = [\eta \cdot \varphi]_r + [\eta \cdot \psi]_r = [\eta]_r \cdot [\varphi]_r + [\eta]_r \cdot [\psi]_r$. Similarly, we have $[\eta]_r(c \cdot [\varphi]_r) = [\eta]_r[c \cdot \varphi]_r = [\eta \cdot (c\varphi)]_r = [c \cdot (\eta \cdot \varphi)]_r = c[\eta \cdot \varphi]_r = c([\eta]_r \cdot [\varphi]_r)$. Thus we have proved that $[\eta]_r$ operates on $\overset{r}{T}R^n$ as a linear transformation.

Q.E.D.

DEFINITION 5. 2. Let $\{x_1, \cdots, x_n\}$ be the natural coordinate system on R^n and let $\{\overset{(\nu)}{x_i} | i = 1, \cdots, n; \nu = 0, 1, \cdots, r\}$ be the induced coordinate system on $\overset{r}{T}R^n$. Using these coordinates, Theorem 5. 1 shows that there is a canonical injective homomorphism $j_n^{(r)}$ of $TGL(n)$ into $GL(n(r+1))$.

Let $(y_j^i) \in GL(n)$. Then $\overset{r}{T}GL(n)$ has the induced coordin te system $\{\overset{(\nu)}{y_j^i} | i, j = 1, \cdots, n; \nu = 0, 1, \cdots, r\}$. We denote by Y_ν the $n \times n$-matrix $(\overset{(\nu)}{y_j^i})$ for $\nu = 0, 1, \cdots \xi, r$.

Proposition 5. 3. *The homomorphism $j_n^{(r)}$ is given by the following equality:*

$$j_n^{(r)}(\cdots,\overset{(\nu)}{y_j^i},\cdots,)=\begin{pmatrix} Y_0 & 0 & \cdots & \cdots & 0 \\ Y_1 & Y_0 & \ddots & & \vdots \\ \vdots & Y_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ Y_r & \cdots & \cdots & Y_1 & Y_0 \end{pmatrix}$$

Proof. We shall prove the proposition only for the case $r=2$, since the proof for the case $r\geqq 3$ is similar. Let $[\varphi]_2\in \overset{2}{T}GL(n)$ be such that $[\varphi]_2=(y_j^i,\ \dot{y}_j^i,\ \ddot{y}_j^i)$. Let $[\xi]_2\in\overset{2}{T}R^n$ be such that $[\xi]_2=(x_i,\dot{x}_i,\ddot{x}_i)$. Then we can assume that

$$\text{(5. 1)}\qquad \begin{cases} \varphi(t)=(y_j^i+\dot{y}_j^i t+\ddot{y}_j^i t^2), \\ \xi(t)=(x_i+\dot{x}_i t+\ddot{x}_i t^2) \end{cases}$$

for $t\in R$. From (5. 1) it follows that $(\varphi\cdot\xi)(t)=\varphi(t)\cdot\xi(t)=(\sum_i(y_j^i+\dot{y}_j^i t+\ddot{y}_j^i t^2)(x_i+\dot{x}_i t+\ddot{x}_i t^2))=(\sum_i y_j^i x_i+\sum_i(\dot{y}_j^i x_i+y_j^i\dot{x}_i)t+\sum_i(y_j^i\ddot{x}_i+\dot{y}_j^i\dot{x}_i+\ddot{y}_j^i x_i)t^2+\sum_i(\dot{y}_j^i\ddot{x}_i+\ddot{y}_j^i\dot{x}_i)t^3+\sum_i\ddot{y}_j^i\ddot{x}_i t^4)$. Therefore, we get $[\varphi]_2[\xi]_2=[\varphi\cdot\xi]_2=(\sum_i y_j^i x_i,\ \sum_i(\dot{y}_j^i x_i+y_j^i\dot{x}_i),\ \sum_i(y_j^i\ddot{x}_i+\dot{y}_j^i+\dot{x}_i+\ddot{y}_j^i x_i))$,

and hence we obtain

$$j_n^{(2)}([\varphi]_2)=\begin{pmatrix} y_j^i & 0 & 0 \\ \dot{y}_j^i & y_j^i & 0 \\ \ddot{y}_j^i & \dot{y}_j^i & y_j^i \end{pmatrix}$$

which proves the proposition. Q.E.D.

§6. Tangential fibre bundle of order r.

Let $E(M,\pi,F,G)$ be a fibre bundle with bundle space E, base M, projection π, fibre F and structure group G. We shall prove the following

Proposition 6. 1. *$\overset{r}{T}E(\overset{r}{T}M,\ \overset{r}{T}\pi,\ \overset{r}{T}F,\ \overset{r}{T}G)$ is a fibre bundle with bundle space $\overset{r}{T}E$, base $\overset{r}{T}M$, projection $\overset{r}{T}\pi$, fibre $\overset{r}{T}F$ and structure gro·p $\overset{r}{T}G$.*

Proof. First, since G operates on F effectively, $\overset{r}{T}G$ operates on $\overset{r}{T}F$ effectively by virtue of Theorem 3. 5. Let $\{U_\alpha\}$ be an open covering of M such that E is trivial over U_α with trivialization Ψ_α: $\pi^{-1}(U_\alpha) \to U_\alpha \times F$ and with transition functions $g_{\alpha\beta}$, i.e. $\Psi_\alpha \circ \Psi_\beta^{-1}(x, y) = (x, g_{\alpha\beta}(x) \cdot y)$ for $x \in U_\alpha \cap U_\beta$ and $y \in F$. Clearly $\{\overset{r}{T}U_\alpha\}$ is an open covering of $\overset{r}{T}M$ and $\overset{r}{T}\Psi_\alpha$ is a diffeomorphism of $(\overset{r}{T}\pi)^{-1}(\overset{r}{T}U_\alpha)$ onto $\overset{r}{T}U_\alpha \times \overset{r}{T}F$. We shall verify the following

$$(\overset{r}{T}\Psi_\alpha) \circ (\overset{r}{T}\Psi_\beta)^{-1}([\varphi]_r, [\psi]_r) = ([\varphi]_r, ((\overset{r}{T}g_{\alpha\beta})[\varphi]_r) \cdot [\psi]_r) \tag{6. 1}$$

for $[\varphi]_r \in \overset{r}{T}(U_\alpha \cap U_\beta)$ and $[\psi]_r \in \overset{r}{T}F$. We denote by ρ: $G \times F \to F$ the operation of G on F and by π_1: $U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta$ (resp. π_2: $U_\alpha \cap U_\beta \times F \to F$) the projection. Similarly we define $\tilde{\pi}_1$: $\overset{r}{T}(U_\alpha \cap U_\beta) \times TF \to \overset{r}{T}(U_\alpha \cap U_\beta)$ and $\tilde{\pi}_2$. Then, we have the following equalities

$$\pi_1 \circ \Psi_\alpha \circ \Psi_\beta^{-1} = \pi_1, \quad \pi_2 \circ \Psi_\alpha \circ \Psi_\beta^{-1} = \rho \circ (g_{\alpha\beta} \times 1_F). \tag{6. 2}$$

Taking the tangent to (6. 2) of order r, we get, by Propositions 1. 6 and 1. 7, the following

$$\begin{cases} \tilde{\pi}_1 \circ T\Psi_\alpha \circ T\Psi_\beta^{-1} = \tilde{\pi}_1, \\ \tilde{\pi}_2 \circ \overset{r}{T}\Psi_\alpha \circ \overset{r}{T}\Psi_\beta^{-1} = \overset{r}{T}\rho \circ (\overset{r}{T}g_{\alpha\beta} \times 1_{\overset{r}{T}F}), \end{cases} \tag{6. 3}$$

which proves (6. 1). Therefore, we have proved that $\overset{r}{T}E$ is a fibre bundle with transition functions $\{\overset{r}{T}g_{\alpha\beta}\}$. Q.E.D.

DEFINITION 6. 2. We shall call the fibre bundle $\overset{r}{T}E(\overset{r}{T}M, \overset{r}{T}\pi, \overset{r}{T}F, \overset{r}{T}G)$ *the tangential fibre bundle to* E *of order* r.

Let $P(M, \pi, G)$ be a principal fibre bundle with bundle space P, base M, projection π and structure group G, and let $\{U_\alpha\}$ be an open covering of M such that P is trivial over U_α and let $\{g_{\alpha\beta}\}$ be the transition function with respect to this covering $\{U_\alpha\}$. We denote such a principal fibre bundle by $P(M, \pi, G) = \{U_\alpha, g_{\alpha\beta}\}$. (For the general theory of fibre bundles, see [5]). Then, by the proof of Proposition 6. 1 we obtain the following

COROLLARY 6. 3. *From a principal fibre bundle* $P(M, \pi, G) = \{U_\alpha, g_{\alpha\beta}\}$ *we get a principal fibre bundle* $\overset{r}{T}P(\overset{r}{T}M, \overset{r}{T}\pi, \overset{r}{T}G) = \{\overset{r}{T}U_\alpha, \overset{r}{T}g_{\alpha\beta}\}$ *for any positive integer* r.

§7. Imbedding of $\overset{r}{T}FM$ into $F\overset{r}{T}M$.

Let $F(M)\,(M, \pi, GL(n))$ be the frame bundle of an n-dimensional manifold M as in [4]. We shall prove the following

THEOREM 7. 1. *For any manifold M, there is a canonical injection $j_M^{(r)}: \overset{r}{T}FM \to F\overset{r}{T}M$ of the tangential fibre bundle $\overset{r}{T}FM$ to FM of order r into the frame bundle of $\overset{r}{T}M$ such that $j_M^{(r)}(x\cdot g) = j_M^{(r)}(x)\cdot j_n^{(r)}(g)$ for $x \in \overset{r}{T}FM$, $g \in \overset{r}{T}GL(n)$ and that the following diagram is commutative:*

$$\begin{array}{ccc} \overset{r}{T}FM & \xrightarrow{\quad j_M^{(r)} \quad} & F\overset{r}{T}M \\ \Big\downarrow \overset{r}{T}\pi & & \Big\downarrow \tilde{\pi} \\ \overset{r}{T}M & \xrightarrow{\quad 1_{\overset{r}{T}M} \quad} & \overset{r}{T}M\,, \end{array}$$

where π: $FM \to M$ (resp. $\tilde{\pi}$: $F\overset{r}{T}M \to \overset{r}{T}M$) is the projection.

Proof. We shall use the same notations as in the proof of Theorem 2. 4 [4]. We denote by $J_{\alpha\beta}^{(r)}$ the Jacobian matrix with respect to the coordinate systems $\{\overset{(\nu)}{x}_{\alpha,i} \mid i = 1, \cdots, n;\ \nu = 0, 1, \cdots, r\}$ and $\{\overset{(\nu)}{x}_{\beta,i} \mid i = 1, \cdots, n;\ \nu = 0, 1, \cdots, r\}$. Using the same arguments as the proof of Theorem 2. 4 [4], in order to prove the Theorem 7. 1, it is sufficient to verify the following relation:

$$J_{\alpha\beta}^{(r)} = j_n^{(r)} \circ \overset{r}{T}J_{\alpha\beta} \text{ on } \overset{r}{T}(U_\alpha) \cap \overset{r}{T}(U_\beta). \tag{7. 1}$$

We shall prove (7. 1) only for $r = 2$, since the proof for the case $r \geqq 3$ is similar. Put $\overset{(\nu)}{x}_i = \overset{(\nu)}{x}_{\alpha,i}$ and $\overset{(\nu)}{y}_i = \overset{(\nu)}{x}_{\beta,i}$ for $i = 1, 2, \cdots, n$; $\nu = 0, 1, \cdots, r$. By expressing y_i as a function $f_i(x_1, \cdots, x_n)$ of $x_1, \cdots, x_n$, we get from (1. 3) the following relation:

$$J_{\alpha\beta}^{(2)} = \begin{pmatrix} J_{\alpha\beta} & 0 & 0 \\ \dot{J}_{\alpha\beta} & J_{\alpha\beta} & 0 \\ \ddot{J}_{\alpha\beta} & \dot{J}_{\alpha\beta} & J_{\alpha\beta} \end{pmatrix} \tag{7. 2}$$

where $\dot{J}_{\alpha\beta} = (\dot{J}_k^i)$ with $\dot{J}_k^i = \sum_j \frac{\partial^2 f_i}{\partial x_j \partial x_k} \dot{x}_j$ and $\ddot{J}_{\alpha\beta} = (\ddot{J}_k^i)$ with $\ddot{J}_k^i = \frac{1}{2} \sum_{j,l} \frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l} \dot{x}_j \dot{x}_l + \sum_j \frac{\partial^2 f_i}{\partial x_j \partial x_k} \ddot{x}_j$. Putting $J_k^i = \frac{\partial f_i}{\partial x_k}$ we get the following

(7. 3) $$\dot{J}^i_k = \sum_j \frac{\partial J^i_k}{\partial x_j}\dot{x}_j, \quad \ddot{J}^i_k = \frac{1}{2}\sum_{j,l}\frac{\partial^2 J^i_k}{\partial x_j \partial x_l}\dot{x}_j\dot{x}_l + \sum_j \frac{\partial J^i_k}{\partial x_j}\ddot{x}_j.$$

Now, consider the map $J = J_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(n)$. We can calculate the coordinates $(\overset{(\nu)}{y}{}^i_j \mid i, j = 1, \cdots, n;\ \nu = 0, 1, 2)$ of the image of $(\overset{(\nu)}{x}_i \mid i = 1, \cdots, n;\ \nu = 0, 1, 2)$ by the map $\overset{2}{T}J$ as follows:

(7. 4) $$\begin{cases} \overset{(0)}{y}{}^i_j = J^i_j(x), \quad \overset{(1)}{y}{}^i_j = \sum_k \frac{\partial J^i_j}{\partial x_k}\dot{x}_k, \\ \overset{(2)}{y}{}^i_j = \frac{1}{2}\sum_{k,l}\frac{\partial^2 J^i_j}{\partial x_k \partial x_l}\dot{x}_k\dot{x}_l + \sum_k \frac{\partial J^i_j}{\partial x_k}\ddot{x}_k. \end{cases}$$

By Proposition 5. 3 and (7. 4), (7. 3) we obtain

$$j^{(2)}_n \circ \overset{2}{T}J_{\alpha\beta} = J^{(2)}_{\alpha\beta} \text{ on } \overset{2}{T}(U_\alpha) \cap \overset{2}{T}(U_\beta).$$ Q.E.D.

§8. Prolongations of G-structures to tangent bundles of order r.

DEFINITION 8. 1. Let G be a Lie subgroup of $GL(n)$. We denote by $G^{(r)}$ the image of $\overset{r}{T}G$ by the homomorphism $j^{(r)}_n$, i.e.

(8. 1) $$G^{(r)} = j^{(r)}_n(\overset{r}{T}G).$$

Clearly, $G^{(r)}$ is a Lie subgroup of $GL(n(r+1))$.

Let $P(M, \pi, G)$ be a G-structuure on M (for the general theory of G-structures see, for instance [1], [2], [4] or [6]). We denote by $\pi^{(r)}$ the restriction of the projection $\pi\colon F\overset{r}{T}M \to \overset{r}{T}M$ to the subbundle $P^{(r)} = j^{(r)}_M(\overset{r}{T}P)$. Then we obtain a $G^{(r)}$-structure $P^{(r)}(\overset{r}{T}M, \pi^{(r)}, G^{(r)})$ on the tangent bundle $\overset{r}{T}M$ to M of order r. We shall call $P^{(r)}$ *the prolongation of order* r of the G-structure P to the tangent bundle $\overset{r}{T}M$ to M of order r.

We can easily see the following

PROPOSITION 8. 2. *If M is completely parallelizable, then $\overset{r}{T}M$ is also completely parallelizable.*

PROPOSITION 8. 3. *There is a canonical bundle homomorphism β^r_s of $P^{(r)}$ into $P^{(s)}$ for $r > s$,* i.e. *the following diagram*

$$\begin{array}{ccc} P^{(r)} & \xrightarrow{\beta_s^r} & P^{(s)} \\ \downarrow{\scriptstyle \pi^{(r)}} & & \downarrow{\scriptstyle \pi^{(s)}} \\ \overset{r}{T}M & \xrightarrow{\pi_s^r} & \overset{s}{T}M \end{array}$$

is commutative and there is a canonical homomorphism $h_s^r\colon G^{(r)} \to G^{(s)}$ *such that*

$$\beta_s^r(x \cdot a) = \beta_s^r(x) \cdot h_s^r(a)$$

for $x \in P^{(r)}$ *and* $a \in G^{(r)}$.

§9. Prolongations of isomorphisms of G-structures.

THEOREM 9. 1. *Let M and M' be two manifolds and $f\colon M \to M'$ be a diffeomorphism betw en them. Then, we have the following commutative diagram:*

$$\begin{array}{ccc} \overset{r}{T}FM & \xrightarrow{j_M^{(r)}} & F\overset{r}{T}M \\ \downarrow{\scriptstyle \overset{r}{T}Ff} & & \downarrow{\scriptstyle F\overset{r}{T}f} \\ \overset{r}{T}FM' & \xrightarrow{j_{M'}^{(r)}} & F\overset{r}{T}M' \end{array}.$$

Proof. We use the same notations Φ_α, Φ'_α, f_α as in the proof of Theorem 4. 2 [4]. On the other hand, let

$$\Psi_\alpha\colon \overset{r}{T}U_\alpha \times GL(n(r+1)) \to F\overset{r}{T}U_\alpha$$

$$\Psi'_\alpha\colon \overset{r}{T}V_\alpha \times GL(n(r+1)) \to F\overset{r}{T}V_\alpha$$

be the local trivializations of $F\overset{r}{T}M$ (resp. $F\overset{r}{T}M'$) over $\overset{r}{T}U_\alpha$ (resp. $\overset{r}{T}V_\alpha$) induced by the coordinate system on U_α (resp. V_α). Define $f_\alpha^{(r)}\colon \overset{r}{T}U_\alpha \times GL(n(r+1)) \to \overset{r}{T}V_\alpha \times GL(n(r+\ ))$ by the following

$$f_\alpha^{(r)} = \Psi_\alpha'^{-1} \circ F\overset{r}{T}f \circ \Psi_\alpha .$$

Let $j_\alpha^{(r)} = 1_{\overset{r}{T}U} \times j_n^{(r)}$ and $j_\alpha'^{(r)} = 1_{\overset{r}{T}V_\alpha} \times j_n^{(r)}$. By the same arguments · as the proof of Th. 4. 2 [4], in order to prove the Theorem 9. 1, it is now sufficient to prove the commutativity of the following diagram:

$$(9.\,1)\qquad \begin{array}{ccc} \overset{r}{T}U_\alpha\times\overset{r}{T}GL(n) & \xrightarrow{\;j_\alpha^{(r)}\;} & \overset{r}{T}U_\alpha\times GL(n(r+1)) \\ \Big\downarrow \overset{r}{T}f_\alpha & & \Big\downarrow f_\alpha^{(r)} \\ \overset{r}{T}V_\alpha\times\overset{r}{T}GL(n) & \xrightarrow{\;j_\alpha'^{(r)}\;} & \overset{r}{T}V_\alpha\times GL(n(r+1))\,. \end{array}$$

We shall prove the commutativity of (9. 1) only for the case $r=2$, since the case for $r\geqq 3$ is similar. Using the same notations $y_i, f_i(x), w_i^k, z_i^k$ as in Th. 4. 2 [4] (we use y_i instead of y^i, etc), we introduce the notations $f_\kappa(x)$, x_κ, y_κ for $\kappa=1,2,\cdots,3n$ by the following

$$(9.\,2)\qquad \begin{cases} f_{i+n}=\sum \dfrac{\partial f_i}{\partial x_k}\dot{x}_k, \\ f_{i+2n}=\dfrac{1}{2}\sum \dfrac{\partial^2 f_i}{\partial x_k\partial x_l}\dot{x}_k\dot{x}_l+\sum\dfrac{\partial f_i}{\partial x_k}\ddot{x}_k, \\ x_{i+n}=\dot{x}_i,\ \ x_{i+2n}=\ddot{x}_i,\ \ y_{i+n}=\dot{y}_i,\ \ y_{j+2n}=\ddot{y}_i \end{cases}$$

for $i=1,2,\cdots,n$. Let $\{x_\kappa, \tilde{w}_\lambda^\kappa \mid \kappa,\lambda=1,2,\cdots,3n\}$ (resp. $\{y_\kappa, \tilde{z}_\lambda^\kappa\mid\kappa,\lambda=1,2,\cdots,3n\}$) be the coordinate system on $F\overset{2}{T}U_\alpha$ (resp. $F\overset{2}{T}V_\alpha$) induced by the coordinate system $\{x_\kappa\}$ (resp. $\{y_\kappa\}$). Now since the map f_α: $U_\alpha\times GL(n)\to V_\alpha\times GL(n)$ is expressed as follows:

$$(9.\,3)\qquad f_\alpha:\ y_i=f_i(x),\ \ z_i^j=\sum w_i^k\frac{\partial f_j}{\partial x_k}\ (i,j=1,2,\cdots,n),$$

we obtain the expression of $\overset{2}{T}f_\alpha$ as follows:

$$(9.\,4)\qquad \begin{cases} y_i=f_i(x),\ \ z_i^j=\sum\limits_k w_i^k\dfrac{\partial f_j}{\partial x_k}, \\ \dot{y}_i=\sum\limits_k\dfrac{\partial f_i}{\partial x_k}\dot{x}_k, \\ \dot{z}_i^j=\sum\limits_{k,l}w_i^k\dfrac{\partial^2 f_j}{\partial x_k\partial x_l}\dot{x}_l+\sum\limits_k\dfrac{\partial f_i}{\partial x_k}\dot{w}_i^k, \\ \ddot{y}_i=\dfrac{1}{2}\sum\limits_{k,l}\dfrac{\partial^2 f_i}{\partial x_k\partial x_l}\dot{x}_k\dot{x}_l+\sum\limits_k\dfrac{\partial f_i}{\partial x_k}\ddot{x}_k, \\ \ddot{z}_i^j=\dfrac{1}{2}\Big(\sum\limits_{k,l,m}w_i^k\dfrac{\partial^3 f_j}{\partial x_k\partial x_l\partial x_m}\dot{x}_l\dot{x}_m+\sum\limits_{k,l}\dfrac{\partial^2 f_j}{\partial x_k\partial x_l}\dot{x}_l\dot{w}_i^k\Big) \\ \quad+\sum\limits_{k,l}w_i^k\dfrac{\partial^2 f_j}{\partial x_k\partial x_l}\ddot{x}_l+\dfrac{1}{2}\sum\limits_{k,l}\dfrac{\partial^2 f_j}{\partial x_k\partial x_l}\dot{x}_l\dot{w}_i^k+\sum\limits_k\dfrac{\partial f_j}{\partial x_k}\ddot{w}_i^k\,. \end{cases}$$

By Proposition 5. 3 we get the following

$$(9.\ 5)\qquad (j_\alpha'^{(2)}\circ \overset{2}{T}f_\alpha)(x_\kappa, w_\lambda^\kappa) = \left(y_\kappa, \begin{pmatrix} z_i^j & 0 & 0 \\ \dot{z}_i^j & z_i^j & 0 \\ \ddot{z}_i^j & \dot{z}_i^j & z_i^j \end{pmatrix}\right),$$

where y_κ and z_λ^κ are given by (9. 4).

On the other hand, since $f: U_\alpha \to V_\alpha$ is expressed by $y_i = f_i(x_1, \cdots, x_n)$ $(i = 1, \cdots, n)$, we have the expression of $\overset{2}{T}f$ as follows:

$$\overset{2}{T}f: \begin{cases} y_i = f_i(x), \quad \dot{y}_i = \sum_k \dfrac{\partial f_i}{\partial x_k}\dot{x}_k, \\ \ddot{y}_i = \dfrac{1}{2}\sum_{k,l}\dfrac{\partial^2 f_i}{\partial x_k \partial x_l}\dot{x}_k\dot{x}_l + \sum_k \dfrac{\partial f_i}{\partial x_j}\ddot{x}_k. \end{cases}$$

Therefore, we get the expression of $f_\alpha^{(2)}$ as follows:

$$f_\alpha^{(2)}: \begin{cases} y_i = f_i(x), \quad \dot{y}_i = \sum_k \dfrac{\partial f_i}{\partial x_k}\dot{x}_k, \\ \ddot{y}_i = \dfrac{1}{2}\sum_{k,l}\dfrac{\partial^2 f_i}{\partial x_k \partial x_l}\dot{x}_k\dot{x}_l + \sum_k \dfrac{\partial f_i}{\partial x_k}\ddot{x}_k, \\ \tilde{z}_\lambda^\kappa = \sum_{\mu=1}^{3n} \tilde{w}_\lambda^\mu \dfrac{\partial f_\kappa}{\partial x_\mu} \end{cases}$$

for $\kappa, \lambda = 1, 2, \cdots, 3n$ and $i = 1, 2, \cdots, n$. Now, we calculate $\tilde{z}_\lambda^\kappa$ by (9. 2) as follows:

$$\begin{aligned} \tilde{z}_\kappa^j &= \sum_k \tilde{w}_\kappa^k \frac{\partial f_j}{\partial x_k}, \\ \tilde{z}_\kappa^{n+j} &= \sum_{k,l} \tilde{w}_\kappa^k \frac{\partial^2 f_j}{\partial x_l \partial x_k}\dot{x}_l + \sum_k \tilde{w}_\kappa^{n+k}\frac{\partial f_j}{\partial x_k}, \\ \tilde{z}_\kappa^{2n+j} &= \sum_k \tilde{w}_\kappa^k \frac{\partial f_{2n+j}}{\partial x_k} \sum_k \tilde{w}_\kappa^{n+k}\frac{\partial f_{2n+j}}{\partial \dot{x}_k} \sum_k \tilde{w}_\kappa^{2n+k}\frac{\partial f_{2n+j}}{\partial \ddot{x}_k} \\ &= \sum_k \tilde{w}_\kappa^k \left(\frac{1}{2}\sum_{l,m}\frac{\partial^3 f_j}{\partial x_m \partial x_l \partial x_k}\dot{x}_m\dot{x}_l + \sum_l \frac{\partial^2 f_j}{\partial x_l \partial x_k}\ddot{x}_l\right) \\ &\quad + \sum_{k,l}\tilde{w}_\kappa^{n+k}\frac{\partial^2 f_j}{\partial x_k \partial x_l}\dot{x}_k + \sum_k \tilde{w}_\kappa^{2n+k}\frac{\partial f_j}{\partial x_k}. \end{aligned}$$

for $\kappa = 1, 2, \cdots, 3n$ and $j = 1, 2, \cdots, n$. By Proposition 5. 3 and the above calculations, we have the following equalities:

$$(9.\ 6)\qquad f_\alpha^{(2)}\circ j_\alpha^{(2)}(x_\mu;\ w_i^k,\ \dot{w}_i^k,\ \ddot{w}_i^k)=f_\alpha^{(2)}\left(x_\mu;\begin{pmatrix} w_i^k & 0 & 0\\ \dot{w}_i^k & w_i^k & 0\\ \ddot{w}_i^k & \dot{w}_i^k & w_i^k\end{pmatrix}\right)$$

$$=\left(f_i(x),\ \dot{y}_i,\ \ddot{y}_i;\begin{pmatrix} \tilde{z}_i^j & 0 & 0\\ \tilde{z}_i^{n+j} & \tilde{z}_i^j & 0\\ \tilde{z}_i^{n+j} & \tilde{z}_i^{n+j} & \tilde{z}_i^j\end{pmatrix}\right),$$

where we see that $\tilde{z}_i^j=\sum_k w_i^k\dfrac{\partial f_j}{\partial x_k}$, $\tilde{z}_i^{n+j}=\sum_{k,l} w_i^k\dfrac{\partial^2 f_j}{\partial x_l\partial x_k}\dot{x}_l+\sum_k \dot{w}_i^k\dfrac{\partial f_i}{\partial x_k}$ and $\tilde{z}_j^{2n+j}=\sum_k w_i^k\left(\dfrac{1}{2}\sum_{l,m}\dfrac{\partial^3 f_j}{\partial x_m\partial x_l\partial x_k}\dot{x}_m\dot{x}_l+\sum_l\dfrac{\partial^2 f_j}{\partial x_l\partial x_k}\ddot{x}_l\right)+\sum_{k,l}\dot{w}_i^k\dfrac{\partial^2 f_j}{\partial x_k\partial x_l}\dot{x}_l+\sum_k \ddot{w}_i^k\dfrac{\partial f_j}{\partial x_k}=\ddot{z}_i^j$. Therefore, we obtain, by (9. 5) and (9. 6), the commutativity of (9. 1) for $r=2$. Q.E.D.

By the same arguments as the proof of Th. 4. 3 [4] we can prove the following

THEOREM 9. 2. *Let Φ be a diffeomorphism of a manifold M onto a manifold M'. Let P (resp. P') be a G-structure on M (resp. M'). Then Φ is an isomorphism of P with P' if an only if $\overset{r}{T}\Phi$ is an isomorphism of $P^{(r)}$ with $P'^{(r)}$.*

COROLLARY 9. 3. *Let Φ be a diffeomorphism of M onto itself, and let P be a G-structure on M. Then Φ is an automorphism of P if and only if $\overset{r}{T}\Phi$ is an automorphism of the prolongation $P^{(r)}$ of order r.*

§10. Integrability of prolongations of G-structures.

In this section, we shall prove that the prolongation of an integrable G-structure (see Def. 5. 1 [4]) of order r is also integrable and vice versa.

PROPOSITION 10. 1. *Let $\{x_1,\cdots,x_n\}$ be a local coordinate system on a neighborhood U in M, on which we give a G-structure P. Let ϕ be a cross section of P over U, which is expressed by $\phi(x)=(\cdots,\sum\phi_j^i(x)(\partial/\partial x_i)_x,\cdots)$ for $x\in U$. Define $\phi^{(r)}$ by $\phi^{(r)}=j_M^{(r)}\circ\overset{r}{T}\phi$. Then $\phi^{(r)}$ is a cross section of the prolongation $P^{(r)}$ over $\overset{r}{T}U$ and is expressed with respect to the induced coordinate system $\{\overset{(\nu)}{x}_i\,|\,i=1,\cdots,n;\ \nu=0,1,\cdots,r\}$ as follows:*

$$(10.\ 1)\quad \phi^{(r)}(\cdots,\overset{(\nu)}{x}_i,\cdots)=\left(\cdots,\sum_{i=1}^n\phi_j^i(x)\left(\frac{\partial}{\partial\overset{(\nu)}{x}_i}\right)_X+\sum_{\mu>\nu}\sum_{i=1}^n F_{j,\mu}^{i,\nu}(X)\left(\frac{\partial}{\partial\overset{(\mu)}{x}_i}\right)_X,\cdots\right),$$

where $X=(\cdots, x_i, \cdots)\in \overset{r}{T}U$ *and* $F^{i;\nu}_{j,\mu}(X)$ *is a polynomial of* $\overset{(\lambda)}{x}_k(\lambda\geqq\mu;\ k=1, \cdots, n)$ *without constant term and with coefficients, which are partial derivatives of* $\phi^l_m(l, m=1, \cdots, n)$.

Proof. Let $\pi\colon F(M)\to M$ and $\tilde{\pi}\colon F\overset{r}{T}M\to \overset{r}{T}M$ be the projections. Let Φ_U and Ψ_U be the local trivialization of FM and $F\overset{r}{T}M$ over U and $\overset{r}{T}U$, respectively. We see that

$$j^{(r)}_M|\overset{r}{T}FM=\Psi_U\circ(1_{\overset{r}{T}U}\times j^{(r)}_n)\circ(\overset{r}{T}\Phi_U)^{-1}.$$

Using Proposition 1. 6, we have the following equalities:

$$\tilde{\pi}\circ\phi^{(r)}=\tilde{\pi}\circ j^{(r)}_M\overset{r}{T}\phi=\overset{r}{T}\pi\circ\overset{r}{T}\phi=\overset{r}{T}(\pi\circ\phi)=\overset{r}{T}1_U=1_{\overset{r}{T}U}.$$

Since $\phi^{(r)}(\overset{r}{T}U)=j^{(r)}_M\circ\overset{r}{T}\phi(TU)=j^{(r)}_M\overset{r}{T}(\phi(U))\subset j^{(r)}_M\overset{r}{T}P=P^{(r)}$, we see that $\phi^{(r)}$ is a cross section of $P^{(r)}$ over $\overset{r}{T}U$.

We shall prove (10. 1) only for the case $r=2$, since the case $r\geqq 3$ is similar. Put $f(x)=(\phi^i_j(x))\in GL(n)$ for $x\in U$, then we have $\Phi_U^{-1}\circ\phi=(1_U, f)$. Hence, we have $\phi^{(2)}=\Psi_U\circ(1_{\overset{2}{T}U}\times j^{(2)}_n)\circ(\overset{2}{T}\Phi)^{-1}\circ\overset{2}{T}\phi=\Psi_U\circ(1_{\overset{2}{T}U}\times j^{(2)}_n)\circ\overset{2}{T}(1_U, f)=\Psi_U\circ(1_{\overset{2}{T}U}\times j^{(2)}_n\circ\overset{2}{T}f)$. Therefore, using the expression (1. 2) of $\overset{2}{T}f$ and Proposition 5. 3 we get the expression of $\phi^{(2)}$ as follows:

$$\phi^{(2)}(x,\dot{x},\ddot{x})=\Psi_U\left((x,\dot{x},\ddot{x});\ \begin{pmatrix}\phi^i_j & 0 & 0\\ \dot{\phi} & \phi^i_j & 0\\ \ddot{\phi}^i_j & \dot{\phi}^i_j & \phi^i_j\end{pmatrix}\right)$$

$$=\Big(\cdots, \sum_i\Big(\phi^i_j\Big(\frac{\partial}{\partial x_i}\Big)_X+\dot{\phi}^i_j\Big(\frac{\partial}{\partial \dot{x}_i}\Big)_X+\ddot{\phi}^i_j\Big(\frac{\partial}{\partial \ddot{x}_i}\Big)_X\Big), \cdots,$$

$$\sum_i\Big(\phi^i_j\Big(\frac{\partial}{\partial \dot{x}_i}\Big)_X+\dot{\phi}^i_j\Big(\frac{\partial}{\partial \ddot{x}_i}\Big)_X\Big), \cdots, \sum_i\phi^i_j\Big(\frac{\partial}{\partial \ddot{x}_i}\Big)_X, \cdots\Big),$$

where
$$\dot{\phi}^i_j=\sum_k\frac{\partial\phi^i_j}{\partial x_k}\dot{x}_k,\quad \ddot{\phi}^i_j=\frac{1}{2}\sum_{k,l}\frac{\partial^2\phi^i_j}{\partial x_k\partial x_l}\dot{x}_k\dot{x}_l+\sum_k\frac{\partial\phi^i_j}{\partial x_k}\ddot{x}_k.$$

These functions $\dot{\phi}^i_j$ and $\ddot{\phi}^i_j$ have the properites stated in the proposition. Thus the proposition is proved. Q.E.D.

Remark 10. 2. By the properties of the functions $F^{i;\nu}_{j,\mu}(X)$, we see that $F^{i;\nu}_{j,\mu}$ vanishes if the functions ϕ^l_m are constants for $l, m=1, 2, \cdots, n$. The function $F^{i;\nu}_{j,\mu}(X)$ also vanishes at $X=(\cdots,\overset{(\nu)}{x}_i,\cdots)$ with $\overset{(\lambda)}{x}_k=0$ for all $\lambda\geqq\mu$ and $k=1,\cdots,n$, since $F^{i;\nu}_{j,\mu}$ is a polynomial of $\overset{(\lambda)}{x}_k$ without constant term.

THEOREM 10. 3. *Let P be a G-structure on a manifold M. Then, P is integrable if and only if the prolongation $P^{(r)}$ of P order r is integrable for any r.*

Proof. Suppose P is integrable. Let $x_0\in M$ be any point of M and let $\{x_1,\cdots,x_n\}$ be a local coordinate system on a neighborhood U of x_0 such that

$$\phi(x)=\left(\cdots,\left(\frac{\partial}{\partial x_i}\right)_x,\cdots\right)\in P \text{ for any } x\in U.$$

Then, by Proposition 10. 1 and Remark 10. 2, $\phi^{(r)}$ is a cross section of $P^{(r)}$ and is expressed with respect to the induced coordinate system $\{\overset{(\nu)}{x}_i\,|\,i=1,\cdots,n;\ \nu=0,1,\cdots,r\}$ as follows: $\phi^{(r)}(\cdots,\overset{(\nu)}{x}_i,\cdots)=(\cdots,(\partial/\partial\overset{(\nu)}{x}_i)_X,\cdots)$ for $X=(\cdots,\overset{(\nu)}{x}_i,\cdots)\in\overset{r}{T}U$. Since $\phi^{(r)}(X)\in P^{(r)}$ and since x_0 is arbitrary, we have proved that $P^{(r)}$ is integrable.

Conversely, suppose $P^{(r)}$ is integrable for some r. To prove that P is integrable, we use the same arguments as the proof of Prop. 5. 5 [4]. Take a point $p\in M$ and take a coordinate neighborhood U of p with coordinate system $\{x_1,\cdots,x_n\}$ such that there is a local cross section $\phi\colon U\to P$ of P over U. Then, by Proposition 10. 1, $\phi^{(r)}=j^{(r)}_M\circ\overset{r}{T}\phi$ is a cross section of $P^{(r)}$ over $\overset{r}{T}U$. Now, let X_0 be the element of $\overset{r}{T}U$ having coordinates $\{\overset{(\nu)}{x}_i\}$ with $x_i=x_i(p)$ and $\overset{(\nu)}{x}_i=0$ for all $\nu\geqq1$ and $i=1,\cdots,n$. Since $P^{(r)}$ is integrable, there can be found a coordinate neighborhood $\tilde{U}$ of X_0 with coordinate system $\{y_1,y_2,\cdots,y_N\}$ $(N=n(r+1))$ such that $\tilde{U}\subset\overset{r}{T}U$ and that, if we define $\tilde{\phi}_0$ by $\tilde{\phi}_0(X)=((\partial/\partial y_1)_X,\cdots,(\partial/\partial y_N)_X)$, $\tilde{\phi}_0$ is a cross section of $P^{(r)}$ over $\tilde{U}$. Since $\phi^{(r)}|\tilde{U}$ and $\tilde{\phi}_0$ are both cross sections of $P^{(r)}$ over $\tilde{U}$, there exists a map $\tilde{g}\colon\tilde{U}\to G^{(r)}$ such that

$$\phi^{(r)}(X)=\tilde{\phi}_0(X)\cdot\tilde{g}(X) \tag{10. 2}$$

holds for $X\in\tilde{U}$. By Proposition 5. 3, there is a map $g\colon\tilde{U}\to G$ such that $\tilde{g}(X)$ has the following form:

$$\tilde{g}(X)=\begin{pmatrix} g(X) & & 0\\ & g(X) & \\ * & & \ddots\ g(X)\end{pmatrix}. \tag{10. 3}$$

Since $\{y_1, \cdots, y_N\}$ and $\{\overset{(\nu)}{x}_i\}$ are both coordinate systems on $\tilde{U}$ we have differentiable functions f_κ such that $y_\kappa = f_\kappa(\cdots, \overset{(\nu)}{x}_i, \cdots)$ for $(\cdots, \overset{(\nu)}{x}_i, \cdots) \in \tilde{U}$ and $\kappa = 1, 2, \cdots, N$. Now if $\phi(x) = (\cdots, \sum_i \phi_j^i(x)(\partial/\partial x_i)_x, \cdots)$ for $x \in U$, then by Proposition 10. 1, (10. 2) can be written as follows:

$$\sum_i \phi_j^i(x)\left(\frac{\partial}{\partial \overset{(0)}{x}_i}\right)_X + \sum_{\mu, i} F_{j\mu}^{i,0}(X)\left(\frac{\partial}{\partial \overset{(\mu)}{x}_i}\right)_X \tag{10. 4}$$
$$= \sum_i g_j^i(X)\left(\frac{\partial}{\partial y_i}\right)_X + \sum_{\kappa=n+1}^{N} \tilde{g}_j^\kappa(X)\left(\frac{\partial}{\partial y_\kappa}\right)_X$$

for $j = 1, 2, \cdots, n$, where $\tilde{g}(X) = (g_\lambda^\kappa(X))$ for $X \in \tilde{U}$. Since $(\partial/\partial \overset{(\nu)}{x}_i)_X = \sum(\partial f_\kappa/\partial \overset{(\nu)}{x}_i) \cdot (\partial/\partial y_\kappa)_X$, (10. 4) can be written as follows:

$$\sum_{i,\kappa} \phi_j^i \cdot \frac{\partial f_\kappa}{\partial \overset{(0)}{x}_i} \cdot \left(\frac{\partial}{\partial y_\kappa}\right)_X + \sum_{i,\mu,\kappa} F_{j,\mu}^{i,0}(X)\frac{\partial f_\kappa}{\partial \overset{(\mu)}{x}_i}\left(\frac{\partial}{\partial y_\kappa}\right)_X \tag{10. 5}$$
$$= \sum_i g_j^i(X)\left(\frac{\partial}{\partial y_i}\right)_X + \sum_{\kappa=n+1}^{N} \tilde{g}_j^\kappa(X)\left(\frac{\partial}{\partial y_\kappa}\right)_X.$$

Comparing the coefficients of $(\partial/\partial y_k)_X$ for $k \leqq n$ in (10. 5), we have

$$\sum_i \phi_j^i(x)\frac{\partial f_k}{\partial \overset{(0)}{x}_i} + \sum_{i,\mu} F_{i,\mu}^{i,0}(X)\frac{\partial f_k}{\partial \overset{(\mu)}{x}_i} = g_j^k(X) \tag{10. 6}$$

for $j, k = 1, 2, \cdots, n$. Now, define maps $\bar{f}_k\colon U' \to R$ and $\bar{g}\colon U' \to G$ by $\bar{f}_k(x) = f_k(x, 0, \cdots, 0)$ and $(\bar{g}(x)^{-1})_j^i = g_j^i(x, 0, \cdots, 0)$ for $i, j, k = 1, \cdots, n$ and $x \in U' = \pi(\tilde{U})$.

Putting $\overset{(\nu)}{x}_k = 0$ $(k = 1, 2, \cdots, n;\ \nu = 1, 2, \cdots, r)$ in (10. 6) and using Remark 10. 2 we obtain

$$\sum_i \phi_j^i(x)\frac{\partial \bar{f}_k}{\partial x_i} = (\bar{g}(x)^{-1})_j^k \tag{10. 7}$$

Now, by the same arguments as in the proof of Prop. 5. 5 [4, pp. 88-89], we see that there exists a coordinate neighborhood U_0 of p with coordinate system $\{\bar{x}_1, \cdots, \bar{x}_n\}$ such that the map $\bar{\phi}$, defined by $\bar{\phi}(x) = ((\partial/\partial\bar{x}_1)_x, \cdots, (\partial/\partial\bar{x}_n)_x)$ for $x \in U_0$, is a cross section of P over U_0. Thus P is integrable.

Q.E.D.

§11. Prolongations of classical G-structures.

(I) $G = GL(n, C)$.

Let J be a linear automorphism of R^{2n} such that $J^2 = -1_{R^{2n}}$ and let $GL(n, C;\ J)$ be the group of all $a \in GL(2n)$ such that $a \circ J = J \circ a$. It is easy to see that $\overset{r}{T}J$ is a linear automorphism of $R^{2n(r+1)} = \overset{r}{T}(R^{2n})$ such that $(\overset{r}{T}J)^2 = -1$. We shall prove the following

Proposition 11. 1. *If $G = GL(n, C;\ J)$, then $G^{(r)} \subset GL(n(r+1),\ C;\ \overset{r}{T}J)$.*

Proof. Take an element $\tilde{a} \in G^{(r)}$. We have to prove that $(\tilde{a} \circ \overset{r}{T}J)(X) = ((\overset{r}{T}J) \circ \tilde{a})(X)$ for every $X \in \overset{r}{T}(R^{2n})$. Now, we can find maps $\varphi \in S(G)$ and $\psi \in S(R^{2n})$ (cf. Notations in §1) such that $\tilde{a} = [\varphi]_r$ and $X = [\psi]_r$. First, it is readily seen that $\varphi \cdot (J \circ \psi) = J \circ (\varphi \cdot \psi)$ (cf. Notations in Th. 5. 1). Therefore, we have $\tilde{a}(\overset{r}{T}J(X)) = [\varphi]_r([J \circ \psi)]_r = [\varphi \cdot [J \circ \psi)]_r = [J \circ (\varphi \cdot \psi)]_r = \overset{r}{T}J([\varphi \cdot \psi]_r) = \overset{r}{T}J([\varphi]_r \cdot [\psi]_r) = \overset{r}{T}J(\tilde{a}(X))$. Q.E.D.

By the same arguments as the proof of Theorem 6. 3 [4], we obtain the following

Theorem 11. 2. (1) *If a manifold M has an almost complex structure, $\overset{r}{T}M$ has a canonical almost complex structure for every r.*

(2) *If a manifold M has a complex structure, then $\overset{r}{T}M$ has a canonical complex structure for every r.*

(II) $G = S_p(m)$.

Consider a skew-symmetric non-degenerate bilinear form f on R^{2m}. Let $S_p(m, f)$ be the group of all $a \in GL(2m)$ which leaves f invariant. We denote by π_r the projection of $\overset{r}{T}R = R^{r+1}$ onto R defined by $\pi_r([\varphi]_r) = (1/r!)[d^r\varphi/dt^r]_0$ for $\varphi \in S(R) = C^\infty(R)$.

Lemma 11. 3. *If f is a skew-symmetric non-degenerate bilinear form on R^{2m}, then $f^{(r)} = \pi_r \circ (\overset{r}{T}f)$ is also a skew-symmetric non-degenerate bilinear form on $R^{2m(r+1)} = \overset{r}{T}R^{2m}$.*

Proof. We take the skew-symmetric matrix $(a^i_j) \in GL(2m)$ such that $f(x, y) = \sum a^i_j x_i y_j$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ with $n = 2m$. Let $\{\overset{(\nu)}{x}_i\}$ be the induced coordinate system on $R^{n(r+1)}$. Take an element

$[\varphi]_r$ (resp. $[\psi]_r$) of $\overset{r}{T}R^n$ with coordinates $\{\overset{(\nu)}{x_i}\}$ (resp. $\{\overset{(\nu)}{y_i}\}$). We can assume that $\varphi(t) = (\cdots, \sum_{\nu=0}^{r} \overset{(\nu)}{x_i} t^\nu, \cdots)$ and $\psi(t) = (\cdots, \sum_{\nu=0}^{r} \overset{(\nu)}{y} t^\nu, \cdots)$. It is now straightforward to see that the following equality holds:

$$f^{(r)}([\varphi]_r, [\psi_r]) = \sum_{i,j} \sum_{\nu=0}^{r} a_{ij} \overset{(\nu)}{x_i} \overset{(r-\nu)}{y_i}, \tag{11. 1}$$

which shows that $f^{(r)}$ is a skew-symmetric non-degenerate bilinear form on $R^{n(r+1)}$.

PROPOSITION 11. 4. *If $G = S_p(m, f)$, then $G^{(r)} \subset S_p(m(r+1), f^{(r)})$.*

Proof. Similar to the of Proposition 11. 1.

By the same arguments as the proof of Th. 6. 6 [4] we obtain the following

THEOREM 11. 5. *If a manifold M has a (resp. an almost) symplectic struct ure then $\overset{r}{T}M$ has a canonical (almost) symplectic structure.*

(III) $G = GL(V, W)$.

We have the following Proposition whose proof will be omitted.

PROPOSITION 11. 6. *If a manifold M has a k-dimensional (completely integrable) differential system, then $\overset{r}{T}M$ has a canonical $k(r+1)$-dimensional (completely integrable) differential system.*

(IV) $G = O(k, n-k)$.

Let g be a symmetric non-degenerate bilinear form on R^n of signature $(k, n-k)$ and let π_r: $\overset{r}{T}R \to R$ be the same projection as in (II) and let $g^{(r)}$ be the map $g^{(r)} = \pi_r \circ (\overset{r}{T}g)$: $\overset{r}{T}R^n \times \overset{r}{T}R^n \to R$. We denote by $O(k, n-k, g)$ or simply $O(g)$ the group of all $a \in GL(n)$ such that a leaves g invariant.

LAMMA 11. 7. *The notations being as above, $g^{(r)}$ is a symmetric non-degenerate bilinear form on $R^{n(r+1)}$ of signature $(n(r+1)/2, n(r+1)/2)$ if r is odd and of signature $\left(k + \frac{rn}{2}, n - k + \frac{rn}{2}\right)$ if r is even.*

Proof. If the bilinear form g is expressed by a symmetric matrix $A = (a^i_j) \in GL(n)$, then by the same computation as the proof of (11. 1) in Lemma 11. 3, we see that $g^{(r)}$ is expressed by the following matrix

$$A^{(r)} = \begin{pmatrix} 0 & & & A \\ & & A & \\ & \cdot & & \\ A & & & 0 \end{pmatrix} \quad (A\colon (r+1)\text{-times}).$$

Since A is of signature $(k, n-k)$, $A^{(r)}$ is of signature $(n(r+1)/2,\ n(r+1)/2)$ if r is odd and of signature $(k+(rn/2),\ n-k+(rn/2))$ if r is even.

Q.E.D.

LEMMA 11. 8. *If $G = O(g)$, then $G^{(r)} \subset O(g^{(r)})$, the signature of $g^{(r)}$ being given in Lemma* 11. 7.

Proof. Omitted.

By the Lemma 11. 8, we obtain the following

THEOREM 11. 9. *If M has a pseudo-Riemannian metric, then $\overset{r}{T}M$ has a canonical pseudo-Riemannian metric for every r.*

(V) $G = GL(n, C) \times 1 \subset GL(2n+1)$.

LEMMA 11. 10. *Let $G = GL(n, C) \times 1 \subset GL(2n+1)$. Then, $G^{(r)} \subset GL((2n+1)(r+1)/2, C)$ if r is odd and $G^{(r)} \subset GL((2nr+2n+r)/2, C) \times 1$ if r is even.*

Proof. We shall omit the proof, which is similar to the proof of Lemma 6. 14 [4].

By Lemma 11. 10. we obtain the following

THEOREM 11. 11. *If M has an almost contact structure, then* (i) *$\overset{r}{T}M$ has a canonical almost complex structure for any odd r and* (ii) *$\overset{r}{T}M$ has a canonical almost contact structure for even r.*

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