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ON A PROPOSITIONAL CALCULUS WHOSE DECISION PROBLEM IS RECURSIVELY UNSOLVABLE[1]

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Dedicated to Professor Katuzi Ono on his 60th birthday

§0. Introduction

The purpose of this paper is to present a propositional calculus whose decision problem is recursively unsolvable. The paper is based on the following ideas:

(1) Using Löwenheim-Skolem's Theorem and Surányi's Reduction Theorem, we will construct an infinitely many-valued propositional calculus corresponding to the first-order predicate calculus.

(2) It is well known that the decision problem of the first-order predicate calculus is recursively unsolvable.

(3) Thus it will be shown that the decision problem of the infinitely many-valued propositional calculus is recursively unsolvable.

In this paper, we consider semantically the problem. That is, we define a validity of wff in our logical system and we will discuss on the problem to decide whether or not an arbitrary wff in our system is valid.[2]

§1. Logical system L

We consider a logical system L:

(1) Propositional variables: $F_1, F_2, \cdots, G_1, G_2, \cdots, P_1, P_2, \cdots$.

(2) Truth-values: Let N be the set of natural numbers and $\Omega = \{0,1\}$.

We define functions f, g as follows:

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[1] This research was done while the author stayed at Dept. of Information Science, Univ. of North Carolina at Chapel Hill.

[2] In the first-order predicate calculus, the semantical decision problem is equivalent to the syntactical one by the completeness theorem.

$$f:\ N \to \Omega, \qquad f \in 2^N,$$
$$g:\ N\times N\times N \to 2^N,\ \ g \in (2^N)^{N\times N\times N}.$$

A *truth-value* is defined as a member of $(2^N)^{N\times N\times N}$, i.e. it is such a function g. Let us say here (x, y, z) in $N\times N\times N$ as a *coordinate,* and x, y, z as x-coordinate, y-coordinate, z-coordinate, respectively.

(3) Logical operations[1]:
Monadic operations: X, Y, Z, $\exists_x$, $\exists_y$, $\exists_z$, $\Diamond$, $\neg$,
Duadic operation: $\vee$.

(4) Truth-value functions:
Let us denote as follows:

$$f(\lambda) = *_\lambda,\quad \lambda \in N,\ *_\lambda \in \Omega,$$
$$g(x, y, z) = v_{xyz},\ \ x, y, z \in N,\ v_{xyz} \in 2^N.$$

X: If a truth-value of wff $\mathfrak{A}$ has v_{iii} at (i, i, i) in $N\times N\times N$, the truth-value of $X\mathfrak{A}$ has the same v_{iii} at every (i, y, z) where $y, z = 1, 2, 3, \cdots$.

Y, Z: Those are defined by the similar way to X.

$\vee$, $\neg$: Those are defined by the usual way.

$\exists_x$: We consider all elements (x, j, k) in $N\times N\times N$ where j, k are constants. If there exists a such that a $*_\lambda$ at (a, j, k) of truth-value of wff $\mathfrak{A}$ is 1, then the truth-value of $\exists_x\mathfrak{A}$ has 1 at $*_\lambda$ of every (x, j, k).
If a truth-value of $\mathfrak{A}$ has 0 at $*_\lambda$ of (x, j, k) for every x, then the truth-value of $\exists_x\mathfrak{A}$ has 0 at $*_\lambda$ in every (x, j, k).

$\exists_y, \exists_z$: Those are defined by the similar way to $\exists_x$.

$\Diamond$: For every $g(x, y, z)$,

$$\Diamond\, g(x, y, z) = \begin{cases} \text{every } *_\lambda(\lambda = 1, 2, 3, \cdots) \text{ is 1, if } *_\lambda = 1 \text{ for some } \lambda. \\ \text{every } *_\lambda(\lambda = 1, 2, 3, \cdots) \text{ is 0 otherwise.} \end{cases}$$

The logical system L is considered as a kind of infinitely many-valued propositional logic. In this paper, a truth-value whose $*_\lambda$ $(\lambda = 1, 2, 3, \cdots)$ at every (x, y, z) are all 1 is called the *designated value.* And further a wff $\mathfrak{A}$ is called *valid* if and only if the $\mathfrak{A}$ takes always the designated value independently of truth-values of propositional variables $P_1, P_2, \cdots, P_n$ in $\mathfrak{A}$.

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[1] Using those logical operations, we define $P_1 \wedge P_2 \overset{D}{=} \neg(\neg P_1 \vee \neg P_2)$, $P_1 \supset P_2 \overset{D}{=} \neg P_1 \vee P_2$.

§2. Relations between the first-order predicate calculus K and the system L

We shall give some relations between the first-order predicate calculus K and the logical system L.

According to the Surányi Reduction Theorem, we have the following one:

THEOREM. *For every* wff $\mathfrak{A}$ *in* K, *we can construct a* wff $\mathfrak{B}$ *of the following form*:

(I) $(\exists x)(\exists y)(\exists z)M_1 \vee (\exists x)(\exists y)(z)M_2$

where M_1 *and* M_2 *are quantifier-free and contain non but monadic and duadic predicates. And, in this case,* $\mathfrak{A}$ *is equivalent to* $\mathfrak{B}$ *in regard to the universal validity.*

From now on, we shall denote Surányi Reduction Form (I) of an arbitrary wff $\mathfrak{A}$ in K as $\mathfrak{A}^*$.

Now, for wff $\mathfrak{A}^*$ and each subformula $\mathfrak{S}$ of $\mathfrak{A}^*$ in K, let $h(\mathfrak{S})$ be a wff in L obtained by using inductively the following (i)-(iii).

(i) If $\mathfrak{S}$ is a monadic predicate $F(x)$, then

$$h(F(x)) \rightarrow \Diamond XF.$$[1]

where $\rightarrow$ means "correspondence".

(ii) If $\mathfrak{S}$ is a duadic predicate $G(x, y)$, then

$$h(G(x, y)) \rightarrow \Diamond(XG^1 \wedge YG^2)$$[1].

Here, it needs not to consider such a case as $h(H(x, y, z))$ because the form (I) contains only monadic and duadic predicates as shown above.

(iii) If $\mathfrak{S}$ contains logical operations or quantifier, then

$$h(\neg\mathfrak{S}_1) \rightarrow \neg h(\mathfrak{S}_1),$$

$$h(\mathfrak{S}_1 \vee \mathfrak{S}_2) \rightarrow h(\mathfrak{S}_1) \vee h(\mathfrak{S}_2),$$

$$h((\exists x)\mathfrak{S}_1) \rightarrow \exists_x h(\mathfrak{S}_1),$$

$$h((\exists y)\mathfrak{S}_1) \rightarrow \exists_y h(\mathfrak{S}_1),$$

$$h((\exists z)\mathfrak{S}_1) \rightarrow \exists_z h(\mathfrak{S}_1).$$

For example:

$$h((\exists x)(\exists y)(F(x) \,\&\, G(x, y)) \rightarrow \exists_x\exists_y(\Diamond XF \wedge \Diamond(XG^1 \wedge YG^2)).$$

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1) Of course, $h(F(y)) \rightarrow \Diamond YF$, $h(F(z)) \rightarrow \Diamond ZF$, $h(G(y, z)) \rightarrow \Diamond(YG^1 \wedge ZG^2), \cdots$.

We shall write $h(\mathfrak{A}^*)$ as $\tilde{\mathfrak{A}}^*$.

Then, we shall prove, in §3, the following theorems:

THEOREM 1. *If $\tilde{\mathfrak{A}}^*$ is valid in L, then $\mathfrak{A}^*$ is universally valid in K.*

THEOREM 2. *If $\mathfrak{A}^*$ is universally valid in K, then $\tilde{\mathfrak{A}}^*$ is valid in L.*

Now, assume that the decision problem of validity in L is recursively solvable. Then, we have an effective procedure to decide whether or not an arbitrary wff $\tilde{\mathfrak{A}}^*$ in L is valid. Thus, from Theorem 1 and 2 we have also an effective procedure to decide whether or not $\mathfrak{A}^*$ in K is universally valid. But, $\mathfrak{A}^*$ is Surányi's reduction form of $\mathfrak{A}$. Therefore it follows that the decision problem of predicate calculus is recursively solvable. This is contradict with (2) in §0. Thus, we know that the decision problem in L is recursively unsolvable.

§3. Proofs of Theorem 1 and 2

Now, we shall give proofs of Theorem 1 and 2.

Theorem 1:

We prove the following Theorem 1′ which is equivalent to Theorem 1.

THEOREM 1′. *If $\mathfrak{A}^*$ is not universally valid in K, than $\tilde{\mathfrak{A}}^*$ is not valid in L.*

Proof. To prove this theorem we use Löwenheim-Skolem's Theorem which is expressed as follows: a wff $\mathfrak{F}$ in K is universally valid if $\mathfrak{F}$ is valid in an enumerable infinite domain ω.

Using this theorem and our assumption of Theorem 1′, we are able to let a truth-value of $\mathfrak{A}^*$ be $\boldsymbol{F}$(falsity) in ω by some suitable truth-value assignment. Here let us denote elements in ω as $e_1, e_2, e_3, \cdots$, and assume that the following predicates occur in $\mathfrak{A}^*$.

$$\text{(II)}\quad \begin{aligned} &F_1(x),\ F_2(x),\ \cdots,\ F_\alpha(x);\ \cdots;\ F_1(z),\ F_2(z),\ \cdots,\ F_\alpha(z),\\ &G_1(x,x),\ \cdots,\ G_\beta(x,x);\ G_1(x,y),\ \cdots;\ G_1(z,z),\ \cdots,\ G_\beta(z,z). \end{aligned}$$

Here, it is possible to assume that those predicates actually occur in $\mathfrak{A}^*$. For if $F_1(x)$ occurs neither in $(\exists x)(\exists y)(\exists z)M_1$ nor in $(\exists x)(\exists y)(z)M_2$, then it is sufficient to consider a formula $(\exists x)(\exists y)(\exists z)(M_1 \& F_1(x) \vee \neg F_1(x))$ which is equivalent to $(\exists x)(\exists y)(\exists z)M_1$.

Now, according to our assumption a truth-value of $\mathfrak{A}^*$ is $\boldsymbol{F}$ in ω by a truth-value assignment for predicates (II). Say that the truth-value assignment is as follows:

$$\text{(III)}\quad \begin{array}{llcl} F_1(e_1): \boldsymbol{T}, & F_2(e_1): \boldsymbol{F}, & \cdots\cdots\cdots & , F_\alpha(e_1): \boldsymbol{T} \\ F_1(e_2): \boldsymbol{T}, & F_2(e_2): \boldsymbol{T}, & & , F_\alpha(e_2): \boldsymbol{F} \\ \vdots & \vdots & & \vdots \\ G_1(e_1, e_1): \boldsymbol{T}, & G_1(e_2, e_1): \boldsymbol{F}, & \cdots\cdots & \\ G_1(e_1, e_2): \boldsymbol{T}, & G_1(e_2, e_2): \boldsymbol{T}, & & \\ \vdots & \vdots & & \\ G_\beta(e_1, e_1): \boldsymbol{T}, & G_\beta(e_2, e_1): \boldsymbol{F}, & \cdots\cdots & \\ G_\beta(e_1, e_2); \boldsymbol{F}, & G_\beta(e_2, e_2); \boldsymbol{T}, & & \\ \vdots & \vdots & & \end{array}$$

From this truth-value assignment, we construct a truth-value assignment in L as follows:
First of all, we make a correspondence of $\boldsymbol{T}$ and $\boldsymbol{F}$ in K to $(1, 1, 1, 1, 1, \cdots)$ and $(0, 0, 0, 0, 0, \cdots)$ in L respectively. Here, the above-mentioned $(1, 1, 1, 1, 1, \cdots)$ $((0, 0, 0, 0, 0, \cdots))$ stands for $f(\lambda) = 1$ $(f(\lambda) = 0)$ for all λ. Next, we make a correspondence of $e_1, e_2, e_3, \cdots$ to $(1, 1, 1)$, $(2, 2, 2)$, $(3, 3, 3)$ $\cdots$ in $N\times N\times N$ in our definitions.

Now, we consider the following truth-value assignment of $F_1, F_2, \cdots, F_\alpha$ in $\tilde{\mathfrak{A}}^*$. If $F_i(e_j)$ is $\boldsymbol{T}$ (or $\boldsymbol{F}$) in (III), then we give $(1,1,1,1,1, \cdots)$ (or $(0,0,0,0,0, \cdots)$) to F_i at (j, j, j) in $N\times N\times N$.
For example:
If $F_2(e_2)$ is $\boldsymbol{T}$ in (III), then we give $(1,1,1,1,1, \cdots)$ to F_2 at $(2, 2, 2)$ in $N\times N\times N$. If $F_2(e_1)$ is $\boldsymbol{F}$ in (III), then we give $(0,0,0,0,0, \cdots)$ to F_2 at $(1,1,1)$ in $N\times N\times N$. In this case, v_{xyz} of $F_1, F_2, \cdots, F_\alpha$ are arbitrary except $v_{111}, v_{222}, v_{333}, \cdots$. This is always possible.

Next, we consider the following assignment of $G_1^1, G_2^1, \cdots, G_\beta^1; G_1^2, G_2^2, \cdots, G_\beta^2$ in $\tilde{\mathfrak{A}}^*$ corresponding to $G_1, G_2, \cdots, G_\beta$ in $\mathfrak{A}^*$.
If $G_i(e_1, e_1)$ is $\boldsymbol{T}$ in (III), we give $(\tau_1, \tau_2, \tau_3, \cdots)$ to G_i^1 at $(1, 1, 1)$ and $(\tau_1', \tau_2', \tau_3', \cdots)$ to G_i^2 at $(1,1,1)$ by whose value $\Diamond(G_i^1 \wedge G_i^2)$ takes $(1, 1, 1, 1, 1, \cdots)$, where τ_i, τ_i' is in Ω.
If $G_i(e_1, e_2)$ is $\boldsymbol{T}$ in (III), we give the above $(\tau_1, \tau_2, \tau_3, \cdots)$ to G_i^1 at $(1,1,1)$ and $(\tau_1'', \tau_2'', \tau_3'', \cdots)$ to G_i^2 at $(2,2,2)$ by whose value $\Diamond(G_i^1 \wedge G_i^2)$ takes $(1,1,1,1,1, \cdots)$

$\vdots$

In the above explanation, $(1,1,1)$, $(2,2,2)$, $\cdots$ correspond to $e_1, e_2, \cdots$ and G_i^1, G_i^2 to the first argument, the second argument of G_i.
Those $(\tau_1^{(\sigma)}, \tau_2^{(\sigma)}, \cdots)$ at (k, k, k) and $(\tau_1^{(\rho)}, \tau_2^{(\rho)}, \cdots)$ at (l, l, l) must be given

such that $\Diamond(G_i^1 \wedge G_i^2)$ obtained from $G_i(e_k, e_l)$ whose value is $\boldsymbol{F}$ in (III) does not take the value $(1,1,1,1,1, \cdots)$.

By repeated applications of this process, we give values to $G_1^1, G_2^1, \cdots, G_\beta^1; G_1^2, G_2^2, \cdots, G_\beta^2$ at $(1,1,1)$, $(2,2,2)$, $(3,3,3)$, $\cdots$ in $N \times N \times N$ and in this case values at (ν_1, ν_2, ν_3) where at least two of ν_1, ν_2 and ν_3 are different are arbitrary.

This process is always possible too. Because since our $(*_1, *_2, *_3, \cdots)$ is an infinite sequence of 0, 1, it is possible by the definition of truth-value function of $\Diamond$.

That is, for example: let us assume that

$$
1)\quad
\begin{array}{llllll}
① & G_i(e_1, e_1): \boldsymbol{T}, & ② & G_i(e_2, e_1): \boldsymbol{F}, & ④ & G_i(e_3, e_1): \boldsymbol{F}, \cdots \\
③ & G_i(e_1, e_2): \boldsymbol{T}, & ⑤ & G_i(e_2, e_2): \boldsymbol{T}, & ⑧ & G_i(e_3, e_2): \boldsymbol{T}, \cdots \\
⑥ & G_i(e_1, e_3): \boldsymbol{F}, & ⑨ & G_i(e_2, e_3): \boldsymbol{T}, & & G_i(e_3, e_3): \boldsymbol{F}, \cdots
\end{array}
$$

Then, first we enumerate those predicates as the above-mentioned ①, ②, ③, $\cdots$ and we give an assignment as follows:

$$
2)\quad
\begin{array}{ll}
G_i^1(e_1): (1, 0, 1, *_4, *_5, \cdots) & G_i^2(e_1): (1, 0, 0, \cdots) \\
G_i^1(e_2): (0, 0, 0, 0, 1, \cdots) & G_i^2(e_2): (0, 0, 1, 0, 1, 0, 0, 1, \cdots) \\
G_i^1(e_3): (0, 0, 0, 0, 0, 0, 0, 1, \cdots) & G_i^2(e_3): (0, 0, 0, 0, 0, 0, 0, 0, 1, \cdots) \\
\vdots & \vdots
\end{array}
$$

where $G_i^j(e_1)$ $(j = 1, 2)$ means a value of G_i^j at $(1,1,1)$ in $N \times N \times N$ and $G_i^j(e_2)$ means a value of G_i^j at $(2,2,2)$ in $N \times N \times N$, etc..

2) is constructed such that the first 1, the third 1 from the left in $(1, 0, 1, *_4, *_5, \cdots)$ of $G_i^1(e_1)$ correspond to $\boldsymbol{T}$ of the enumeration ①, ③ in 1).

Now, notice that in an enumerable infinite domain N the operation $(\exists x)$ $((\exists y), (\exists z))$ can be interpreted as an infinite disjunction on x-coordinate (y-coordinate, z-coordinate). For example: $(\exists x)\,\mathfrak{A}(x, y, z)$ is interpreted as

$$\mathfrak{A}(1, y, z) \vee \mathfrak{A}(2, y, z) \vee \cdots.$$

And also we notice that

(1) if $F(e_1, e_2)$ is a truth-value of $F(x, y)$, it is considered by the definition of X, Y, Z as a value at $(1, 2, z)$ in $N \times N \times N$ where $z = 1, 2, 3, \cdots$.

(2) if $F(e_1, e_2)$ is a truth-value of $F(y, x)$, it is considered as a value at $(2, 1, z)$ in $N \times N \times N$ where $z = 1, 2, 3, \cdots$.

(3) if $F(e_2, e_2)$ is a truth-value of $F(x, x)$, it is considered as a value at $(2, y, z)$ in $N\times N\times N$ where $y, z = 1, 2, 3, \cdots,$

(4) and so on.

Thus, from the above-mentioned truth-value assignment, the construction of $\tilde{\mathfrak{A}}^*$ and the interpretation of existential quantifier in the domain N, we are able to let $\tilde{\mathfrak{A}}^*$ be not valid in L. Therefore, we get Theorem 1′.

Throem 2:

We shall prove the following Theorem 2′ which is equivalent to Theorem 2.

THEOREM 2′. *If $\tilde{\mathfrak{A}}^*$ is not valid in L, then $\mathfrak{A}^*$ is not universally valid in K.*

Proof. Let us notice that $\tilde{\mathfrak{A}}^*$ is of a form $\exists_x \exists_y \exists_z \tilde{M}_1^* \vee \exists_x \exists_y \neg \exists_z \neg \tilde{M}_2^*$ where $\tilde{M}_1^*$ and $\tilde{M}_2^*$ correspond to M_1 and M_2 respectively. From this matter and the assumption of this theorem, we can give truth-values for propositions $F_1, F_2, \cdots, F_\alpha$; $G_1^1, G_2^1, \cdots, G_\beta^1$: $G_1^2, G_2^2, \cdots, G_\beta^2$ in $\tilde{\mathfrak{A}}^*$ by which $\tilde{\mathfrak{A}}^*$ takes $(0, 0, 0, 0, 0, \cdots)$ at every (x, y, z) in $N\times N\times N$.

Here we make a correspondence of $(0, 0, 0, 0, 0, \cdots)$, $(1, 1, 1, 1, 1, \cdots)$ to $\boldsymbol{F}$, $\boldsymbol{T}$ as mentioned above. Then, we consider only values at (i, i, i) in $N\times N\times N$ in the assignment where $i = 1, 2, 3, \cdots$.

Now, let us say that the truth-value assignment is as follows:

$$
\begin{array}{ll}
F_1(e_1): (t_{111}, t_{112}, \cdots) & F_2(e_1): (t_{211}, t_{212}, \cdots)\cdots \\
F_1(e_2): (t_{121}, t_{122}, \cdots) & F_2(e_2): (t_{221}, t_{222}, \cdots) \\
\vdots & \\
G_1^1(e_1): (\tau_{111}^1, \tau_{112}^1, \cdots) & G_1^2(e_1): (\tau_{111}^2, \tau_{112}^2, \cdots)\cdots \\
G_1^1(e_2): (\tau_{121}^1, \tau_{122}^1, \cdots) & G_1^2(e_2): (\tau_{121}^2, \tau_{122}^2, \cdots) \\
\vdots & \vdots
\end{array}
$$

where $F_1(e_1), F_1(e_2), \cdots$ mean values of F_1 at $(1, 1, 1)$, $(2, 2, 2)$, $\cdots$ in $N\times N\times N$ as before.

Then, we take $(1, 1, 1)$, $(2, 2, 2)$, $\cdots$ as an infinite domain ω. Further we take a value of $\diamond XF_i$ obtained from $F_i(e_i)$ as a truth-value of predicate $F_i(x)$ for $x = (i, i, i)$ and also a value of $\diamond (XG_i^1 \wedge YG_i^2)$ obtained from $G_i^1(e_k)$,

$G_i^2(e_l)$ as a truth-value of predicate $G_i(x, y)$ for $x = (k, k, k)$, $y = (l, l, l)$ and so on. From the definitions of X, Y, Z and the h: $h(F_i(x)) \to \Diamond XF_i$, $h(G_i(x, y)) \to \Diamond (XG_i^1 \wedge YG_i^2)$, $\cdots$, we know that $\mathfrak{A}^*$ takes $\boldsymbol{F}$ in the enumerable infinite domain ω.

Thus, we get Theorem 2′.

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