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ON A Π_1^0 SET OF POSITIVE MEASURE

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Dedicated to Professor Katuzi Ono for his 60th birthday anniversary

Introduction. Some basis results for arithmetic, hyperarithmetic (*HA*) or Π_1^1 sets which have positive measure (or which are not meager, i.e., of the second Baire category) have been obtained by several authors.[1] For example, every non-meager Σ_3^0 set must have a recursive element (Shoenfield-Hinman, Hinman [2]) but there exists a non-meager Π_3^0 set (as well as of measure 1) that contains no recursive element (Shoenfield [7]), and every Σ_n^0 set (i.e., arithmetic set) of positive measure contains an arithmetic element (Sacks [5], and Tanaka [12]).[2] In view of these results, Hinman [2] asked whether a Σ_3^0 set of positive measure must contain a recursive element. The main aim of this note is to give a negative answer for this question; thus, *there is a* Π_1^0 *set of positive measure with no recursive element* (§1). In §2, we shall mention some remarks on hierarchy problems.

§1. Answer for the question.

LEMMA 1. *For each positive integer k, the measure of every Baire's interval of order k is not greater than* $1/k(k+1)$.

Proof. Let $\{a_1, \cdots, a_k, \cdots\}$ be an arbitrary sequence of positive integers. We define $p_k = [a_1, \cdots, a_k]$ as follows:

$$(1)\qquad \begin{cases} p_0 = [\phi] = 1, \quad p_1 = [a_1] = a_1, \\ p_k = [a_1, \cdots, a_k] = [a_1, \cdots, a_{k-1}]a_k + [a_1, \cdots, a_{k-2}] \quad (k \geq 2), \\ \quad = p_{k-1}a_k + p_{k-2}. \end{cases}$$

Further, let $q_0 = 0$ and $q_k = [a_2, \cdots, a_k]$ $(k \geq 1)$. Then, by (1), we have

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[1] In the present paper, sets means subsets of Baire's zero-space N^N. Measure means the Lebesgue measure, and we shall write $\mu(E)$ for the measure of a measurable set E.

[2] An element of Baire's space is regarded as a 1-place number-theoretic function.

(2) $$q_k = q_{k-1}a_k + q_{k-2} \ (k \geq 2).$$

It is well-known by elementary number theory that the following equations hold:

(3) $$p_k q_{k-1} - p_{k-1} q_k = (-1)^k \qquad (k \geq 1),$$

(4) $$\frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots + \frac{1|}{|a_k} = \frac{q_{k-1}a_k + q_{k-2}}{p_{k-1}a_k + p_{k-2}} \text{ if } k \geq 2.$$

Let $\delta = \langle a_1, \cdots, a_k \rangle$ be an arbitrary Baire's interval of order k. Then by (3) and (4), we have

$$\mu(\delta) = \left| \left(\frac{1|}{|a_1} + \cdots + \frac{1|}{|a_{k-1}} + \frac{1|}{|a_k} \right) - \left(\frac{1|}{|a_1} + \cdots + \frac{1|}{|a_{k-1}} + \frac{1|}{|a_k + 1} \right) \right|$$
$$= 1/(p_{k-1}a_k + p_{k-2})(p_{k-1}a_k + p_{k-1} + p_{k-2}).$$

Since $p_k \geq k$ for all $k \geq 1$, $\mu(\delta) \leqq \frac{1}{(2k-3)(3k-4)}$ if $k \geq 2$. Hence we have

(5) $$\mu(\delta) \leq 1/k(k+1),$$

if $k \geq 3$. Obviously (5) holds for $k = 1$ or 2, too. (Q.E.D.)

In the following, a method by which one can evaluate the outer-measure of a countable set is available.

For each numbers p and e we shall define a set $M_{p,e}$ as follows:

$$\alpha \in M_{p,e} \longleftrightarrow (\forall x)_{x<p+e+1} (\exists y) [T_1(e, x, y) \ \& \ \alpha(x) = U(y)],$$

and let

$$M_p = \bigcup_{e=0}^{\infty} M_{p,e}.$$

For each p and e, $M_{p,e}$ is either the empty set or a Baire's interval of order $p + e + 1$, and M_p is a Σ_1^0 set which contains *all* recursive elements. By Lemma 1, we have

$$\mu(M_p) \leq \sum_{e=0}^{\infty} \mu(M_{p,e}) \leq \sum_{e=0}^{\infty} \frac{1}{(p+e+1)(p+e+2)} = \frac{1}{p+1}.$$

Thus we obtain the

THEOREM 2. *There exists a Σ_1^0 set M ($\subset N \times N^N$) such that each $M_p = \{\alpha : \langle p, \alpha \rangle \in M\}$ contains all recursive elements and satisfies the following condition:*

$$\mu(M_p) \leq \frac{1}{p+1}\ .^{3)}$$

COROLLARY 3. *There exists a* Π_1^0 *set of positive measure that contains no recursive element.*[3]

This gives a negative answer for Hinman's problem. By a theorem obtained by Sacks [5] and the author [12] (see Introduction), any set obtained in Corollary 3 must contain an *arithmetic* element.

COROLLARY 4. *There exists a* Σ_2^0 *set of measure* 1 *which has no recursive element.*

It follows from Shoenfield-Hinman's Theorem [2; p. 1] (see Introduction) that such a set as in Corollary 4 is *an example of arithmetic, meager* (first Baire category) *sets having measure* 1.[4),5)]

§2. **Some remarks.** 1°) Evidently, there is a Σ_1^0 set E of measure 1 such that $E \not\supset \mathfrak{R}$, where $\mathfrak{R}$ is the set of all 1-place recursive functions.

2°) Contrasting with Corollary 4, *if* E *is a* Π_2^0 *set of measure* 1 *then* E *contains a recursive element.* For, since every Σ_1^0 set of measure 1 is an open dense set, E is co-meager (the complement of a meager set) and hence E is not meager. By the Shoenfield-Hinman Theorem, E contains a recursive element.

3°) There is a Π_1^0 set consisting of a single element that is not arithmetical. (Spector [10; Corollary 2])

4°) It is known as Kripke-Feferman-Harrison's Theorem (e.g. Mathias [4; T 3200]) that every countable Σ_1^1 set contains only HA elements. This can be proved, for example, by the fact that a non-empty Σ_1^1 set with no HA element is dense-in-itself. The elements of a countable Σ_1^1 set are not necessarily enumerated by a HA function, as is obvious; but, by the following proposition, *the elements of a countable* Δ_1^1 *set can be enumerated by a* HA *function:*

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3) N. Tsukada has pointed out that this result can be straightforwardly extended in the case of sets of level $|a|$ for $a \in O$.

4) The referee called my attention to this fact.

5) Theorem 2, Corollaries 3 and 4 hold true for the case of the space 2^N (instead of N^N).

PROPOSITION 5. A countable Σ_1^1 set E can not contain HA elements of arbitrarily high degrees; that is, there is a HA function φ such that

$$(\forall\alpha)[\alpha\in E \to \alpha \leq_T \varphi].$$

Proof. By the Kripke-Feferman-Harrison Theorem, we have

$$(\forall\beta)(\exists a)[\beta\in E \to a\in O \ \& \ \beta\leq_T H_a],$$

where $\beta\leq_T A$ means that β is Turing reducible to A, namely β is recursive in A. Since E is Σ_1^1, the predicate described in the brackets is Π_1^1. Hence, by Kreisel's Lemma [3; Lemma 1], there exists a HA functional $\Psi \in N^{N^N}$ such that

$$(\forall\beta)[\beta \in E \to \Psi\langle\beta\rangle \in O \ \& \ \beta\leq_T H_{\Psi\langle\beta\rangle}].$$

The set $\{\Psi\langle\beta\rangle: \beta\in E\}$ is a Σ_1^1 subset of O. Therefore, by a fact known as a direct consequence of Spector [9; Theorem 1], there exists a number $b \in O$ such that

$$|\Psi\langle\beta\rangle| \leq |b| \qquad \text{for all } \beta\in E.$$

Thus we obtain the following implication:

$$\beta\in E \to \beta\leq_T H_b.$$

This completes the proof.

5°) It is a difficult work that one performs any enumeration of a countable CA (i.e., co-analytic) subset of N^N. Now one knows Mansfield-Solovay's Theorem [11; Appendix II], [4; T3206] and [6]: Let E be a Σ_2^1-in-α set (α is a code of E). If E has a non constructible-from-α element, then E contains a perfect subset. By the theorem, we shall try to do this work for a countable PCA set, thus:

For the sake of simplicity, we shall deal with effective case, i.e., with a countable Σ_2^1 set E, instead of a classical PCA set. By the above theorem we have

$$E \subset L\cap N^N. \tag{1}$$

Since E is Σ_2^1, by Shoenfield's Theorem [4; T3101] together with (1) E is a constructible set. Since $\alpha\in L\cap N^N \to \alpha\in F``\aleph_1^L \subset F``\aleph_1$, we have

$$E \in L \ \& \ E \subset F^{\text{``}} \aleph_1 \ \& \ \text{Card}(E) = \aleph_0.$$

(L and F are Gödel's.) Hence by [8; p. 317] we have

$$E \in F^{\text{``}} \aleph_1; \text{ i.e., } Od(E) < \aleph_1.$$

Thus we obtain

PROPOSITION 6.[6] Let E be a countable $\sum_2^1$ set. Then E itself is constructible and $Od(E) < \aleph_1$.

Let $\sigma = Od(E)$. Then $(\forall\beta)[\beta \in E \rightarrow Or(\beta) < \sigma]$.[7] Note that $Or(\beta) \leq Od(\beta)$. Let φ be a code for the countable ordinal σ. We shall inductively define α as follows:

$$\begin{cases} \alpha(0) = (\mu i)_{i \in \omega}(\exists\beta)[\omega \times \omega \cdot F(\varphi_i) = \beta \ \& \ \beta \in E], \\ \alpha(n+1) = (\mu i)_{i \in \omega}(\exists\beta)[\omega \times \omega \cdot F(\varphi_i) = \beta \ \& \ \beta \in E \ \& \ (\forall k)_{k \leq n}(i \neq \alpha(k))]. \end{cases}$$

Then we can see that α is Δ_3^1-in-φ. Let $\beta_n = \omega \times \omega \cdot F(\varphi_{\alpha(n)})$. ($\beta_n$ is a different notation from φ_i.) Then $E = \{\beta_0, \beta_1, \beta_2, \cdots\}$. Now, since

$$\beta_n(x) = y \longleftrightarrow (\exists\varepsilon)(\exists\beta)[M(\varphi, \varepsilon) \ \& \ A(\varphi, \varepsilon, \beta, \alpha(n)) \ \& \ \beta(x) = y],$$

it is $\sum_3^1$-in-φ and hence Δ_3^1-in-φ. Consequently, *E can be enumerated by a Δ_3^1-in-φ function.* We do not know, however, what φ is.

If φ is a constructible function (e.g., if $\aleph_1^L = \aleph_1$ then it is the case), then

$$(\exists\varphi)[\varphi \in L \cap N^N \ \& \ W(\varphi) \ \& \ (\forall\beta)[\beta \in E \rightarrow (\exists i)[Or(\beta) < \varphi_i]]].$$

Hence we can choose a Δ_3^1 function φ satisfying the bracketed condition. After all, *E can be enumerated by a Δ_3^1 function.*

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6) This proposition is due to Y. Sampei.

7) We use freely some results and notations in Addison [1].

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