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A DETERMINATE LOGIC

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To Professor Katuzi Ono on his 60 th birthday

Let L be a fixed language and K be a set of structures related to L. A sentence ψ in L is said to be K-valid if and only if for every structure $\mathfrak{A}$ in K, $\mathfrak{A} \models \psi$. If there exists a logical system S such that $S \vdash \psi$ is equivalent to "Ψ is K-valid", then S is said to be a K-logic.

The sequence $x_0 x_1 \cdots x_\beta \cdots (\beta < \alpha)$ is expressed by $x_0 \cdots \hat{x}_\alpha$. If f is a map from α into $\{\forall, \exists\}$, $Q^f x_0 \cdots \hat{x}_\alpha$ expresses a quantifier of the length α. If all the values of f are constantly $\forall$ or $\exists$, then $Q^f x_0 \cdots \hat{x}_\alpha$ is also written by $\forall x_0 \cdots \hat{x}_\alpha$ or $\exists x_0 \cdots \hat{x}_\alpha$ respectively and is said to be a homogeneous quantifier. If a quantifier is not homogeneous, it is said to be hetrogeneous. If $\alpha = \omega$ and $f(n) = \forall$ for each even number n and $f(n) = \exists$ for each odd number n, then $Q^f x_0 \cdots \hat{x}_\omega$ is written by $\forall x_0 \exists x_1 \forall x_2 \exists x_3 \cdots$. The function $\bar{f}$ defined by the following conditions is said to be the dual of f.

1. The lengths of f and $\bar{f}$ are the same.
2. $\bar{f}(\beta) = \forall$ if and only if $f(\beta) = \exists$.
3. $\bar{f}(\beta) = \exists$ if and only if $f(\beta) = \forall$.

If f and g are dual, then Q^f and Q^g are also said to be dual. The dual quantifier of $\forall x_0 \exists x_1 \forall x_2 \cdots$ is written by $\exists x_0 \forall x_1 \exists x_2 \cdots$. By a language, we mean a set of logical symbols, individual constants, predicate constants, and variables. We shall consider only the following particular kind of languages.

1. Every quantifier in L is of the form Q^f.

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2. If Q^f is in L, then $Q^{\bar{f}}$ is in L.

3. L has sufficiently many bound variables i.e. the cardinality of the set of all bound variables in L is regular and greater than the length of any quantifier and any predicate constant in L.

4. L has sufficiently many free variables i.e. the cardinality of the set of all formulas and the cardinality of the set of all free variables are the same.

Let $\mathfrak{A}$ be a structure related to L. $\mathfrak{A}$ is said to be determinate if and only if the following holds for every formula ϕ in L.

$Q^f x_0 \cdots \hat{x}_\alpha \phi(x_0, \cdots, \hat{x}_\alpha, a_0, \cdots, \hat{a}_\beta)$ or
$Q^{\bar{f}} x_0 \cdots \hat{x}_\alpha \neg \phi(x_0, \cdots, \hat{x}_\alpha, a_0, \cdots, \hat{a}_\beta)$ is satisfied in $\mathfrak{A}$
for every sequence $a_0, \cdots, \hat{a}_\beta$ of members of the universe of $\mathfrak{A}$.

(This schema is called the axiom of determinateness for the quantifier Q^f.) If K is the class of all the determinate structures, then a K-logic is also said to be a determinate logic. The word "determinate" comes from the axiom of determinateness in [5] and [6]. Roughly speaking, a structure $\mathfrak{A}$ is determinate if there exists a winning strategy for every definable game in $\mathfrak{A}$.

This paper is a continuation of [3]. Only homogeneous quantifiers are considered in [3]. For the system of [3], we proved the completeness theorem, the cut-elimination theorem and the interpolation theorem. However Malitz [4] found a counterexample of our interpolation theorem and later we found an error in case 2 of the proof of our Theorem 5 [3]. In this paper, we shall generalize the system in [3] by introducing hetrogeneous quantifier and prove that the system thus generalized is a determinate logic. Then we shall prove that if a formula ϕ is provable in our determinate logic and the inference for hetrogeneous quantifier is used only once at the end of the proof of ϕ, then ϕ is valid. (See Theorem 3 in § 3 for the precise form.) By using this property, the same method as in [3] proves the following interpolation theorem. Let A and B be formulas without hetrogeneous quantifiers. If $A \longrightarrow B$ is valid, then there exists a formula C with hetrogeneous quantifiers such that $A \longrightarrow C$ and $C \longrightarrow B$ are valid and every predicate constant or individual constant except $=$ in C is contained both in A and B.

§ 1. A logical system.

We use sequents in our logical system. Sequents are of the form $\Gamma \longrightarrow \Delta$, where Γ, Δ, and Greek capital letters in general denote sequences of formulas of the length $< \lambda^+$ and λ is the cardinal number of the set of all formulas in L.

The postulates of our system is the following.

1. Beginning sequents.

$$D \longrightarrow D$$

$$\longrightarrow a = a$$

2. Rules of inference.

2. 1. Structural rule.

$$\frac{\Gamma \longrightarrow \Delta}{\Pi \longrightarrow \Lambda},$$

where every formula occurring in Γ or in Δ is contained in Δ or in Λ respectively.

2. 2. Introduction of $\neg$ in succedent.

$$\frac{\{A_\lambda\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \{\neg A_\lambda\}_{\lambda<\gamma}},$$

where $\{A_\lambda\}_{\lambda<\gamma}$ stands for the sequence $A_0, A_1, \cdots, \hat{A}_\gamma$.

2. 3. Introduction of $\neg$ in antecedent.

$$\frac{\Gamma \longrightarrow \Delta, \{A_\lambda\}_{\lambda<\gamma}}{\{\neg A_\lambda\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}$$

2. 4. Introduction of $\vee$ in succedent.

$$\frac{\Gamma \longrightarrow \Delta, \{A_{\lambda,\mu}\}_{\mu<\beta_\lambda, \lambda<\gamma}}{\Gamma \longrightarrow \Delta, \{\bigvee_{\mu<\beta_\lambda} A_{\lambda,\mu}\}_{\lambda<\gamma}}$$

2. 5. Introduction of $\vee$ in antecedent.

$$\frac{\{A_{\lambda,\mu_\lambda}\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta \text{ for all } \{\mu_\lambda\}_{\lambda<\gamma} \text{ such that } \mu_\lambda < \beta_\lambda (\lambda < \gamma)}{\{\bigvee_{\mu<\beta_\lambda} A_{\lambda,\mu}\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}$$

2. 6. Introduction of $\wedge$ in succedent.

$$\frac{\Gamma \longrightarrow \Delta, \{A_{\lambda,\mu_\lambda}\}_{\lambda<\gamma} \text{ for all } \{\mu_\lambda\}_{\lambda<\gamma} \text{ such that } \mu_\lambda < \beta_\lambda (\lambda < \gamma)}{\Gamma \longrightarrow \Delta, \{\bigwedge_{\mu<\beta_\lambda} A_{\lambda,\mu}\}_{\lambda<\gamma}}$$

2. 7. Introduction of $\wedge$ in antecedent.

$$\frac{\{A_{\lambda,\mu}\}_{\mu<\beta_\lambda, \lambda<\gamma}, \Gamma \longrightarrow \Delta}{\{\bigwedge_{\mu<\beta_\lambda} A_{\lambda,\mu}\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}$$

2. 8. Introduction of Q in succedent.

$$\frac{\Gamma \longrightarrow \Delta, \{A_\lambda(a_\lambda)\}_{\lambda<\gamma}}{\Gamma \longrightarrow \Delta, \{Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}_\lambda)\}_{\lambda<\gamma}}$$

Here $\vec{a}_\lambda$ means a sequence $a_{\lambda,0}, a_{\lambda,1}, \cdots, a_{\lambda,\mu}, \cdots$. If $f_\lambda(\mu) = \forall$, then $a_{\lambda,\mu}$ is said to be an eigenvariable of this inference.

2. 9. Introduction of Q in antecedent.

$$\frac{\{A_\lambda(\vec{a}_\lambda)\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}{Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}_\lambda)\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}$$

If $f_\lambda(\mu) = \exists$, then $a_{\lambda,\mu}$ is said to be an eigenvariable of this inference.

In both 2. 8 and 2. 9 we use the following terminology. When an eigenvariable occurs in $\vec{a}_\lambda, Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}_\lambda)$ in the schema is called a principal formula of this eigenvariable and also a principal formula of the inference. $A_\lambda(\vec{a}_\lambda)$ is called the anxiliary formula of $Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}_\lambda)$. μ is called the order of the variable $a_{\lambda,\mu}$ with respect to $Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}_\lambda)$. If the orders of two variables a and b w.r.t. some one principal formula are μ and ν and $\mu < \nu$, then a is said to be before b w.r.t. that principal formula.

3. Cut.

$$\frac{\Gamma \longrightarrow \Delta, A_\lambda \text{ for all } \lambda < \gamma \quad \{A_\lambda\}_{\lambda<\gamma}, \Pi \longrightarrow \Lambda}{\Gamma, \Pi \longrightarrow \Delta, \Lambda}$$

4. Rules for equality.

4. 1. First rules for equality.

$$\frac{\Gamma^{(\vec{a})} \longrightarrow \Delta^{(\vec{a})}}{\vec{a} = \vec{b}, \Gamma^{(\vec{b})} \longrightarrow \Delta^{(\vec{b})}}; \quad \frac{\Gamma^{(\vec{a})} \longrightarrow \Delta^{(\vec{a})}}{\vec{b} = \vec{a}, \Gamma^{(\vec{b})} \longrightarrow \Delta^{(\vec{b})}},$$

where $\vec{a} = \vec{b}$ means the sequence $\{a_\lambda = b_\lambda\}_{\lambda<\gamma}$ and $\Gamma^{(\vec{b})} \longrightarrow \Delta^{(b)}$

means the result obtained from $\Gamma^{(\vec{a})} \longrightarrow \Delta^{(\vec{a})}$ by replacing some occurrence of a_λ with b_λ for all $\lambda < \gamma$.

4. 2. Second rule for equality.
Let Σ be an arbitrary set of variables and $\widetilde{\Sigma}$ be a set consisting of all prime formulas of the form $a = b$ such that a and b belong to Σ. $(\Phi|\Psi)$ is said to be a decomposition of Σ if and only if $\Phi \cup \Psi = \widetilde{\Sigma}$ and $\Phi \cap \Psi = 0$.

$$\frac{\Phi, \Gamma \longrightarrow \Delta, \Psi \text{ for all decompositions } (\Phi|\Psi) \text{ of } \widetilde{\Sigma}}{\Gamma \longrightarrow \Delta}$$

Every formal proof must satisfy the following eigenvariable conditions.

5. 1. If a free variable occurs in two or more places as eigenvariables, the principal formulas of these occurences are the same formula and the orders of this eigenvariable w.r.t. each principal are the same. If a occurs in two different auxiliary formulas $A(\vec{a}_1)$ and $A(\vec{a}_2)$ as an eigenvariable of a principal formula $Q^f\vec{x}A(\vec{x})$ and $a_{1,\mu}$ and $a_{2,\mu}$ are a, then $a_{1,\nu}$ and $a_{2,\nu}$ are the same for all $\nu < \mu$.

5. 2. To each free variable a, an ordinal number named the height $h(a)$ of a must be assigned and satisfy the conditions.

5. 2. 1. The height $h(a)$ of an eigenvariable a is greater than the height $h(b)$ of every free variable b in the principal formula of the eigenvariable a.

5. 2. 2. The height of an eigenvariable a is greater than the height of b if b is before a w.r.t. a principal formula of an eigenvariable a.

5. 3. No variable occurring in an inference as an eigenvariable may occur in the end sequent.

Remark. The following weaker modification of eigenvariable conditions is enough to get a determinate logic. Replace the last half of 5. 1 by the following. If $A(\vec{a})$ is a auxiliary formula of a principal formula $Q^f\vec{x}A(x)$ and a_ν and a_μ are eigenvariables of $Q^f\vec{x}A(x)$ and $\nu \neq \mu$, then a_ν and a_μ are different. If a occurs in two different auxiliary formulas $A(\vec{a}_1)$ and $A(\vec{a}_2)$ as an eigenvariable of a principal formula $Q^f\vec{x}A(\vec{x})$ and $a_{1,\mu}$ and $a_{2,\mu}$ are a, then $a_{1,\nu}$ and $a_{2,\nu}$ are the same for any noneigenvariable $a_{1,\nu}$ of $Q^f\vec{x}A(\vec{x})$, for each $\nu < \mu$.

5. 2. 2. can be replaced by the following 5. 2. 2′.

5. 2. 2′. The height of an eigenvariable a is greater than the height of b if a is an eigenvariable of a principal formula and b is before a w.r.t. this principal formula but b is not an eigenvariable of this principal formula.

§ 2. Examples of cut-free formal proof.

1) If $\Gamma \longrightarrow \Delta$ has no hetrogeneous quantifiers and is valid, then there exists a cut-free proof of $\Gamma \longrightarrow \Delta$ because a completeness theorem and a cut-elimination theorem can be proved for such a sequent. As one of the simplest cases of this kind, we shall show a cut-free proof of the axiom of depending choice.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a_n, a_{n+1}) \longrightarrow F(a_n, a_{n+1})}{\{F(a_m, a_{m+1})\}_{m<\omega} \longrightarrow F(a_n, a_{n+1}) \text{ for every } n<\omega}}{\{F(a_m, a_{m+1})\}_{m<\omega} \longrightarrow \bigwedge_{n<\omega} F(a_n, a_{n+1})}}{\{F(a_m, a_{m+1})\}_{m<\omega} \longrightarrow \forall x_0 \exists x_1 x_2 \cdots \bigwedge_{n<\omega} F(x_n, x_{n+1})}}{\{\forall x \exists y F(x, y)\}_{m<\omega} \longrightarrow \forall x_0 \exists x_1 x_2 \cdots \bigwedge_{n<\omega} F(x_n, x_{n+1})}}{\forall x \exists y F(x, y) \longrightarrow \forall x_0 \exists x_1 x_2 \cdots \bigwedge_{n<\omega} F(x_n, x_{n+1})}$$

In this proof, $h(a_m)$ is defined to be m for each $m<\omega$.

2) The proof of axiom of determinateness.
Let $\vec{a}$ be $\{a_\lambda\}_{\lambda<\alpha}$ and $\vec{b}$ be $\{b_\mu\}_{\mu<\beta}$.

$$\dfrac{\dfrac{A(\vec{a}, \vec{b}) \longrightarrow A(\vec{a}, \vec{b})}{\longrightarrow A(\vec{a}, \vec{b}), \neg A(\vec{a}, \vec{b})}}{\longrightarrow Q^f \vec{x} A(\vec{x}, \vec{b}), Q^f \vec{x} \neg A(\vec{x}, \vec{b})}$$

In this proof, $h(a_\lambda) = 1 + \lambda$ and $h(b_\mu) = 0$.

3) Malitz's example.
Malitz found a counterexample for the interpolation theorem for the homogeneous infinitary language. His example is the following. Let A and B be two well-ordered sets with the same order type. If F and G are two order preserving one-to-one map from A onto B, then F and G are the same. Let $Ln(=,<)$ be a formula which expresses that $<$ together with $=$ is a linear ordering relation. Let Γ be a sequence of the following formulas.

$$Ln(\overset{1}{=}, \overset{1}{<}), Ln(\overset{2}{=}, \overset{2}{<}),$$
$$\forall x \forall y \forall u \forall v(x \overset{1}{=} y \wedge u \overset{2}{=} v \longrightarrow (F(x,u) \longleftrightarrow F(y,v))),$$
$$\forall x \forall y \forall u \forall v(x \overset{1}{=} y \wedge u \overset{2}{=} v \longrightarrow (G(x,u) \longleftrightarrow G(y,v))),$$
$$\forall x \forall y \forall u \forall v(F(x,u) \wedge F(y,v) \longrightarrow (x \overset{1}{<} y \longleftrightarrow u \overset{2}{<} v) \wedge (x \overset{1}{=} y \longleftrightarrow u \overset{2}{=} v))$$
$$\forall x \forall y \forall u \forall v(G(x,u) \wedge G(y,v) \longrightarrow (x \overset{1}{<} y \longleftrightarrow u \overset{2}{<} v) \wedge (x \overset{1}{=} y \longleftrightarrow u \overset{2}{=} v))$$

It should be remarked that all the quantifiers in Γ are universal at the front of a formula. The following sequent is easily proved to be valid.

$$\Gamma, \forall x \exists y F(x,y), \forall x \exists y G(x,y),$$
$$\forall x \exists y F(y,x), \forall x \exists y G(y,x), F(a,b) \longrightarrow G(a,b), \exists x_0 x_1 \cdots \bigwedge_n (x_{n+1} \overset{1}{<} x_n)$$

We are going to get a cut-free proof of this sequent. Let T be the set of all finite sequences of 1 and 2. It is understood that the empty sequence is a member of T. We use τ as a variable expressing a member of T. The set D of free variables is defined as follows.

1) $a \in D$. (a is a a^τ, where τ is an empty-sequence.)
2) $a^\tau \in D$, then $b^{\tau 1}$ and $b^{\tau 2}$ are members of D.
3) $b^\tau \in D$, then $a^{\tau 1}$ and $a^{\tau 2}$ are members of D.
4) All members of D are obtained by 1), 2) and 3).

The members of D are $a, b^1, b^2, a^{11}, a^{12}, a^{21}, a^{22}, b^{111}, b^{112}, \cdots$. Γ' is a sequence of all the formulas which are obtained from a formula in Γ by deleting all the universal quantifiers and replacing bound variables by the members of D. (From one formula, infinitely many formulas will be obtained. Of course, in one instance of substitution, the same member of D should be substituted for the same bound variable in a formula.) Δ' is a sequence of all the formulas of the form

$$F(a^\tau, b^{\tau 1}), \quad F(a^{\tau 1}, b^\tau), \quad G(a^\tau, b^{\tau 2}), \quad G(a^{\tau 2}, b^\tau) \qquad (\tau \in T).$$

In the following lemmas, we state several sequents which are provable in the ordinary first order predicate calculus and so cut-free provable in Gentzen's LK.

LEMMA 1. *The following are LK-provable.*

1) $\Gamma', \Delta' \longrightarrow b^{\tau 11} = b^{\tau}$,
where $b^{\tau 1} = b^{\tau 2}$ *is an abbreviation of* $b^{\tau 1} \overset{2}{=} b^{\tau 2}$. *In the same way,* $a^{\tau 1} = a^{\tau 2}$ *is an abbreviation of* $a^{\tau 1} \overset{1}{=} a^{\tau 2}$.

2) $\Gamma', \Delta' \longrightarrow b^{\tau 22} = b^{\tau}$.

3) $\Gamma', \Delta' \longrightarrow a^{\tau 11} = a^{\tau}$.

4) $\Gamma', \Delta' \longrightarrow a^{\tau 22} = a^{\tau}$.

Proof. Obviously, $\Gamma', F(a^{\tau 1}, b^{\tau 11}),\ F(a^{\tau 1}, b^{\tau}) \longrightarrow b^{\tau 11} = b^{\tau}$ whence follows 1) trivially. The proof of 2), 3) and 4) are similar.

LEMMA 2. *The following are LK-provable.*

1) $\Gamma', \Delta', b^{\tau} = b^{\tau 12} \longrightarrow a^{\tau 1} = a^{\tau 2}$

2) $\Gamma', \Delta', a^{\tau} = a^{\tau 12} \longrightarrow b^{\tau 1} = b^{\tau 2}$

Proof. Under the hypotheses of Γ' and $\Delta', b^{\tau} = b^{\tau 12}$ implies $a^{\tau 2} = a^{\tau 122}$. Using the previous lemma, we have $a^{\tau 2} = a^{\tau 1}$. The proof of 2) is similar.

LEMMA 3. *The following are provable in LK.*

1) $\Gamma', \Delta', b^{\tau i 1} = b^{\tau i 2} \longrightarrow a^{\tau 1} = a^{\tau 2} \qquad (i = 1, 2)$.

2) $\Gamma', \Delta', a^{\tau i 1} = a^{\tau i 2} \longrightarrow b^{\tau 1} = b^{\tau 2} \qquad (i = 1, 2)$.

Proof. Under the hypotheses of Γ' and Δ', $b^{\tau 11} = b^{\tau 12} \longrightarrow b^{\tau} = b^{\tau 12} \longrightarrow a^{\tau 1} = a^{\tau 2}$ (Lemmas 1 and 2). The other cases are similarly proved.

LEMMA 4. *The following is provable in LK.*

$$\Gamma', \Delta', b^{\tau 1} = b^{\tau 2} \longrightarrow b^{1} = b^{2}.$$

Proof. This is easily proved by the induction on the length of τ, using lemma 3.

LEMMA 5. *The following are provable in LK.*

1) $\Gamma', \Delta', b^{1} = b^{2} \longrightarrow G(a, b^{1})$

2) $\Gamma', \Delta', b^{1} < b^{2} \longrightarrow a^{12} < a$

where $b^{\tau 1} < b^{\tau 2}$ *and* $a^{\tau 1} < a^{\tau 2}$ *are abbreviations of* $b^{\tau 1} \overset{2}{<} b^{\tau 2}$ *and* $a^{\tau 1} \overset{1}{<} a^{\tau 2}$ *respectively.*

3) $\Gamma', \Delta', b^2 < b^1 \longrightarrow a^{21} < a.$

Proof.

1) $\Gamma', G(a, b^2), b^1 = b^2 \longrightarrow G(a, b^1)$

2) $\Gamma', F(a, b^1), b^1 < b^2, \ G(a, b^2), G(a^{12}, b^1) \longrightarrow a^{12} < a$

3) $\Gamma', F(a, b^1), b^2 < b^1, F(a^{21}, b^2) \longrightarrow a^{21} < a.$

LEMMA 6. *The following are provable in LK*

1) $\Gamma', \Delta', b^{\tau 1} = b^{\tau 2} \longrightarrow G(a, b^1)$

2) $\Gamma', \Delta', b^{\tau 1} < b^{\tau 2} \longrightarrow a^{\tau 12} < a^{\tau}$

3) $\Gamma', \Delta', b^{\tau 2} < b^{\tau 1} \longrightarrow a^{\tau 12} < a^{\tau}.$

Proof. The proofs of 2) and 3) are similar to the proof of Lemma 5. 1) follows from Lemma 4 and 1) of Lemma 5.

DEFINITION. $R^i(\tau)$ means $b^{\tau 1} = b^{\tau 2}$ if $i = 0$; $b^{\tau 1} < b^{\tau 2}$, if $i = 1$; and $b^{\tau 2} < b^{\tau 1}$, if $i = 2$. T_0 is a set of all members τ in T such that the length of τ is odd.

The following immediately follows from Lemma 6.

LEMMA 7. *The following is cut-free provable for each sequence of* $i_\tau (= 0, 1, 2)$ $(\tau \in T_0)$.

$$\{R^{i_\tau}(\tau)\}_{\tau \in T_0}, \Gamma', \Delta' \longrightarrow \bigwedge_n t_{n+1} \overset{1}{<} t_n, G(a, b^1),$$

where t_n *is a member of* D *whose length is* $2n$.

LEMMA 8. *The following is cut-free provable.*

$$\Gamma, \Delta', \forall x_0 x_1 \cdots \neg \bigwedge_n (x_{n+1} \overset{1}{<} x_n) \longrightarrow G(a, b^1)$$

Proof. This follows from Lemma 7, since $\forall x \forall y (x \overset{2}{<} y \vee x \overset{2}{=} y \vee y \overset{2}{<} x)$ is contained in Γ.

THEOREM. *The following is cut-free provable.*

$$\Gamma, \Delta, \forall x_0 x_1 \cdots \neg \bigwedge_n (x_{n+1} \overset{1}{<} x_n), F(a, b) \longrightarrow G(a, b)$$

Proof. Take b to be b^1. Then define $h(a^\tau)$ and $h(b^{\tau'})$ to be the length of τ and the length of τ' respectively. Then the theorem follows from Lemma 8.

§ 3. Validity of provable formulas.

First of all, we shall prove the following theorem.

THEOREM 1. *Let* $\mathfrak{A}$ *be a determinate structure and* $\Gamma \longrightarrow \Delta$ *be provable in the system in* § 1. *Then* $\Gamma \longrightarrow \Delta$ *is satisfied in* $\mathfrak{A}$.

Proof. Take an arbitrary formula with a quantifier at the beginning, say

$$Q^f \vec{x} A(\vec{x}, \vec{a}),$$

where $\vec{a}$ is the sequence of all free variables in this formula and the length of $\vec{x}$ is α. For each $\gamma < \alpha$, we introduce a Skolem function

$$g_A^{f,\gamma}(x_{\xi_0}, \cdots, x_{\xi_\mu}, \cdots, \vec{a}) \text{ or } \bar{g}_A^{f,\gamma}(x_{\eta_0}, \cdots, x_{\eta_\mu}, \cdots, \vec{a})$$

according as $f(\gamma) = \exists$ or $f(\gamma) = \forall$, where $\xi_0, \cdots, \xi_\mu, \cdots$ are all ordinals $\xi < \gamma$ satisfying $f(\xi) = \forall$ and $\eta_0, \cdots, \eta_\mu, \cdots$ are all ordinals $\eta < \gamma$ satisfying $f(\eta) = \exists$. We define the following interpretation of $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$ w.r.t. $\mathfrak{A}$.

If $Q^f \vec{x} A(\vec{x}, \vec{a})$ is satisfied in $\mathfrak{A}$, then $g_A^{f,\gamma}$'s are functions satisfying

1. 1. $\forall x_{\xi_0} x_{\xi_1} \cdots A(\tilde{x}_0, \cdots, \vec{a})$,

where $\tilde{x}_\gamma$ is x_γ if $f(\gamma) = \forall$ and $\tilde{x}_\gamma$ is $g_A^{f,\gamma}(x_{\xi_0}, \cdots, \vec{a})$ if $f(\gamma) = \exists$. Let D be the universe of $\mathfrak{A}$ and 0 be a member of D. $\vec{a}$ is understood to be a sequence of members of D. If $Q^f \vec{x} A(\vec{x}, \vec{a})$ is not satisfied in $\mathfrak{A}$, then $g_A^{f,\gamma}$'s are interpreted to be a constant function 0 in $\mathfrak{A}$.

If $Q^f \vec{x} \neg A(\vec{x}, \vec{a})$ is satisfied in $\mathfrak{A}$, then $\bar{g}_A^{f,\gamma}$'s are functions satisfying

1. 2. $\forall x_{\eta_0} x_{\eta_1} \cdots \cdots A(\tilde{x}_0, \cdots, \vec{a})$,

where $\tilde{x}_\gamma$ is x_γ if $f(\gamma) = \exists$ and $\tilde{x}_\gamma$ is $\bar{g}_A^{f,\gamma}(x_{\xi_0}, \cdots, \vec{a})$ if $f(\gamma) = \forall$. If $Q^f \vec{x} \neg A(\vec{x}, \vec{a})$ is not satisfied in $\mathfrak{A}$, then $\bar{g}_A^{f,\gamma}$'s are interpreted to be a constant function 0 in $\mathfrak{A}$.

Now let P be a proof-figure in our logical system. Well-order all the eigenvariables in P in such a way that $h(a_\beta) \leq h(a_\gamma)$ if $\beta < \gamma$ and let the well-ordered sequence be $a_0, a_1, \cdots, a_\beta, \cdots$. We shall define terms $t_0, t_1, \cdots, t_\beta, \cdots$ by transfinite induction on β. Assume that $t_0, \cdots, \hat{t}_\beta$ have been defined; we shall define t_β. Let $Q^f \vec{x} A(\vec{x}, \vec{b})$ and $A(\vec{d}, \vec{b})$ be the principal formula and an auxiliary formula of a_β and let the order of a_β w.r.t. this principal formula be γ i.e., let a_β be d_γ. For each d_ν, let μ_ν be either the already-defined t_γ for which d_ν is a_γ, in case d_ν is an eigenvariable; or else d_ν itself, if d_ν is a free variable not used as an eigenvariable. Likewise for each b_ν let s_ν be obtained in the same way. Therefore $u_0, \cdots, \hat{u}_\gamma$ and $s_0, \cdots, s_\delta$ have been defined for $d_0, \cdots, \hat{d}_\gamma$ and $b_0, \cdots, \hat{b}_\delta$, where δ is the length of b. Then t_β is defined to be $g_A^{f,\gamma}(u_{\xi_0}, \cdots ; s_0, \cdots, \hat{s}_\delta)$ or $\bar{g}_A^{f,\gamma}(u_{\eta_0}, \cdots ; s_0, \cdots, \hat{s}_\delta)$ according as $f(\gamma)$ is $\exists$ or $\forall$. This definition does not depend on the choice of $A(\vec{d}, \vec{b})$ because of 5. 1 of § 1.

Now substitute $t_0, t_1, \cdots, t_\beta, \cdots$ for $a_0, a_1, \cdots, a_\beta, \cdots$ respectively in P. Let P' be the proof-figure thus obtained from P. The end-sequents of P' and P are the same because the end-sequent of P has no eigenvariables. We shall show that every sequent of P' is satisfied in $\mathfrak{A}$. We have only to show that if the upper sequents of an inference in P' are satisfied in $\mathfrak{A}$, then the lower sequent of this inference is also satisfied in $\mathfrak{A}$. Since the other cases are obvious, we only consider the inferences on quantifiers. The introduction of Q in the antecedent in P' is of the following form

2. 1. $$\frac{\cdots\cdots\cdots, A(\vec{u}, \vec{s}), \cdots, \Gamma \longrightarrow \Delta}{\cdots, Q^f \vec{x} A(\vec{x}, \vec{s}), \cdots, \Gamma \longrightarrow \Delta},$$

where u_γ is of the form $g_A^{f,\gamma}(u_{\xi_0}, \cdots, \vec{s})$ if $f(\gamma) = \exists$.

The introduction of Q in the succedent in P' is of the following form

2. 2. $$\frac{\Gamma \longrightarrow \Delta, \cdots, A(\vec{u}', \vec{s}), \cdots}{\Gamma \longrightarrow \Delta, \cdots, Q^f \vec{x} A(\vec{x}, \vec{s}), \cdots,},$$

where u'_γ is of the form $\bar{g}_A^{f,\gamma}(u'_{\eta_0}, \cdots, \vec{s})$, if $f(\gamma) = \forall$. Therefore, what we have to show is

3. 1. $Q^f \vec{x} A(\vec{x}, \vec{s}) \longrightarrow A(\vec{u}, \vec{s})$ for 2. 1 and

3. 2. $A(\vec{u}', \vec{s}) \longrightarrow Q^f \vec{x} A(\vec{x}, \vec{s})$ for 2. 2.

However 3. 1 immediately follows from 1. 1. Now we shall consider 3. 2.

Assume that $\neg Q^f\vec{x}A(\vec{x}, \vec{s})$ holds in $\mathfrak{A}$. Since $\mathfrak{A}$ is determinate, $Q^{\bar{f}}\vec{x} \neg A(\vec{x}, \vec{s})$ holds in $\mathfrak{A}$. Therefore what we have to show is

$$Q^{\bar{f}}\vec{x} \neg A(\vec{x}, \vec{s}) \longrightarrow \neg A(\vec{u}, \vec{s}),$$

which follows from 1. 2.

Since the determinateness of $\mathfrak{A}$ is used only for 3. 2 and the axiom of the determinateness always holds for a homogeneous quantifier, we have the following theorem.

THEOREM 2. *Let a proof-figure P in our determinate logic satisfy the following condition. Every quantifier introduced in the succedent in P is homogeneous. Then the end-sequent of P is valid.*

We can show a little more. First we shall define a logical system *VSS* which is a valid subsystem of our determinate logic.

DEFINITION. A proof-figure P in our determinate logic is said to be a proof-figure in *VSS* if every inference I in P on the introduction of Q in succedent is homogeneous or has the form

4. 1. $$\frac{\Gamma \longrightarrow \Delta, A(\vec{d})}{\Gamma \longrightarrow \Delta, Q^f\vec{x}A(\vec{x})}$$

where no eigenvariable in P used prior to $\Gamma \longrightarrow \Delta, Q^f\vec{x}A(\vec{x})$ occurs in $\Gamma \longrightarrow \Delta, Q^f\vec{x}A(\vec{x})$.

THEOREM 3. *If a sequent S is provable in VSS, then S is valid.*

Proof. Define $\bar{g}_A^{f,\tau}$ only for homogeneous f and $g_A^{f,\tau}$ as in the proof of Theorem 1. Then define substitution also as in the proof of Theorem 1 except that all eigenvariables in the inference of 4. 1 remain unsubstituted. Then P will be transformed to P'. What we have to show is that every sequent S' in P' is satisfied in $\mathfrak{A}$. This is shown by the transfinite induction on the complexity of the proof to S. We can repeat the proof of Theorem 1 except in the following case. S is infered by the inference I

$$\frac{\Gamma \longrightarrow \Delta, A(\vec{d}, \vec{b})}{\Gamma \longrightarrow \Delta, Q^f\vec{x}A(\vec{x}, \vec{b})},$$

where Q^f is not homogeneous. In order to illustrate the proof, we assume that $Q^f\vec{x}$ is $\forall x_0 \exists x_1 \forall x_2 \exists x_3 \cdots$ and $\vec{d}$ is $d_0, d_1, d_2, \cdots$. Since I satisfies 4. 1 and $h(d_0) < h(d_1) < h(d_2) < \cdots$, $(\Gamma \longrightarrow \Delta, A(\vec{d}, \vec{b}))'$ is of the form $\Gamma' \longrightarrow \Delta'$,

$A(d_0, t_1(d_0, \bar{s}), d_2, t_3(d_0, d_2, \bar{s}), \cdots, \bar{s})$. It follows from the inductive hypothesis that $\Gamma' \longrightarrow \Delta', A(d_0, t_1(d_0, \bar{s}), d_2, t_3(d_0, d_2, \bar{s}), \cdots, \bar{s})$ is satisfied in $\mathfrak{A}$ for every sequence $d_0, d_2, d_4, \cdots$ of members of $\mathfrak{A}$. Therefore $\Gamma' \longrightarrow \Delta', Q^f \vec{x} A(\vec{x}, \bar{s})$ is satisfied in $\mathfrak{A}$.

We shall consider another logical system.

DEFINITION. A figure P is said to be a proof-figure in *RHS* (restricted homogeneous system) if P satisfies the following conditions

5. 1. All quantifiers in P are $\exists$.

5. 2. P satisfies all conditions of proof-figure in § 1 except 5. 1–5. 3.

5. 3. Every inference in P of type 2. 9 in § 1 is of the following form

$$\frac{\{A_\lambda(\vec{a}_\lambda)\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}{\{\exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda)\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta}$$

where no $a_{\lambda,\mu}$ occurs in

$$\{\exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda)\}_{\lambda<\gamma}, \Gamma \longrightarrow \Delta.$$

Then we have the following proposition.

PROPOSITION 1. *If $\Gamma \longrightarrow \Delta$ is provable in RHS and the heights h's are defined for all free variables in $\Gamma \longrightarrow \Delta$. Then there exists a proof-figure P' to $\Gamma \longrightarrow \Delta$ in RHS such that the heights in P' of free variables in $\Gamma \longrightarrow \Delta$ are the same with h.*

Proof. We may assume that the same eigenvariable is never used in two different places. (Otherwise, we can reletter some eigenvariables.) Then we define heights of free variables from the bottom so that the proof-figure in *RHS* satisfies the conditions 5. 1–5. 3 in § 1. Since our proof-figure satisfies 5. 3 in the previous definition, this is easily done.

§ 4. A completeness theorem.

First we shall prove the following theorem.

THEOREM 1. *Let $\Gamma \longrightarrow \Delta$ be a sequent. Then there exists a cut-free proof of $\Gamma \longrightarrow \Delta$ in our determinate logic or else there exists a structure $\mathfrak{A}$ (possibly not determinate) such that every formula in Γ is satisfied in $\mathfrak{A}$ and no formula in Δ is satisfied in $\mathfrak{A}$.*

Proof. Every free variable in Γ or Δ is treated as an individual constant. Let D_0 be an arbitrary non-empty set containing all individual constants in Γ and Δ. Let D be the closure of D_0 w.r.t. all the functions $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$ for all formulas A in our language, i.e., D is generated by all $g_A^{f,\gamma}$'s and $\bar{g}_A^{f,\gamma}$'s from D_0. (Actually it is enough that D is closed w.r.t. all the functions $g_A^{f,\gamma}$ and $\bar{g}_A^{f,\gamma}$ for all subformulas A of a formula in Γ or Δ.) In this proof, a member of $D-D_0$ is treated as a free variable and a member of D_0 is treated as an individual constant. Let E be the set of all formulas of the form $s=t$, where s and t are members of D. Let $(\Phi|\Psi)$ be an arbitrary decomposition of E and consider the following sequent 1. 1.

1. 1. $\Phi, \Gamma \longrightarrow \Delta, \Psi$

If all the sequents of the form 1. 1 are provable without cut, then $\Gamma \longrightarrow \Delta$ is also provable without cut. If there exists a counterexample for a sequent of the form 1. 1, then it is also a counterexample for $\Gamma \longrightarrow \Delta$.

Let S be $\Gamma \longrightarrow \Delta$. We shall define the figure $P(S)$ by the following way.

1) The lowest sequent is S.

2) Immediate ancestor of S are all the sequents of the form 1. 1.

3) When a sequent $\Pi \longrightarrow \Lambda$ is

$$\{\neg C_\lambda\}_{\lambda<\gamma}, \Gamma' \longrightarrow \Delta', \{\neg D_\mu\}_{\mu<\delta},$$

where Γ' and Δ' have no formulas whose outermost logical symbol is $\neg$, and $\Pi \longrightarrow \Lambda$ is constructed by 2) or 8), the immediate ancestor of $\Pi \longrightarrow \Lambda$ is

$$\{D_\mu\}_{\mu<\delta}, \Gamma' \longrightarrow \Delta', \{C_\lambda\}_{\lambda<\gamma}.$$

4) When a sequent $\Pi \longrightarrow \Lambda$ is

$$\{\bigvee_{\lambda<\gamma_\mu} C_{\lambda,\mu}\}_{\mu<\gamma}, \Gamma' \longrightarrow \Delta', \{\bigvee_{\rho<\delta_\sigma} D_{\rho,\sigma}\}_{\sigma<\delta},$$

where Γ' and Δ' have no formulas whose outermost logical symbol is $\vee$, and when $\Pi \longrightarrow \Lambda$ is constructed by 3), then the immediate ancestors of $\Pi \longrightarrow \Lambda$ are

$$\{C_{\lambda_\mu,\mu}\}_{\mu<\gamma}, \Gamma' \longrightarrow \Delta', \{D_{\rho,\sigma}\}_{\rho<\delta_\sigma,\sigma<\delta}$$

for all sequences $\{\lambda_\mu\}_{\mu<\gamma}$ such that $\lambda_\mu < \gamma_\mu$.

5) When a sequent $\Pi \longrightarrow \Lambda$ is

$$\{\bigwedge_{\lambda<\gamma_\mu} C_{\lambda,\mu}\}_{\mu<\gamma}, \Gamma' \longrightarrow \Delta', \{\bigwedge_{\rho<\delta_\sigma} D_{\rho,\sigma}\}_{\sigma<\delta},$$

where Γ' and Δ' have no formulas whose outermost logical symbol in $\wedge$, and when $\Pi \longrightarrow \Lambda$ is constructed by 5), then the immediate ancestors of $\Pi \longrightarrow \Lambda$ are

$$\{C_{\lambda,\mu}\}_{\lambda<\gamma_\mu,\mu<\gamma}, \Gamma' \longrightarrow \Delta', \{D_{\rho_\sigma,\sigma}\}_{\sigma<\delta}$$

for all sequences $\{\lambda_\mu\}_{\mu<\gamma}$ such that $\lambda_\mu < \gamma_\mu$.

6) When a sequent $\Pi \longrightarrow \Lambda$ is

$$\{Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}, \vec{s})\}_{\lambda<\delta}, \Gamma' \longrightarrow \Delta'$$

where Γ' has no formulas whose outermost logical symbol is Q, and when $\Pi \longrightarrow \Lambda$ is constructed by 5), then the immediate ancestors of $\Pi \longrightarrow \Lambda$ are

$$\{A_\lambda(\vec{t}_{\lambda,\mu}, \vec{s}_\lambda)\}_{\mu,\lambda<\delta} \Gamma' \longrightarrow \Delta'$$

for all $\vec{t}_{\lambda,\mu}$ satisfying the following.
$\vec{t}_{\lambda,\mu}$ is $\{t_{\lambda,\mu,0}, \cdots, t_{\lambda,\mu,\nu}, \cdots\}_{\nu<\gamma}$ where γ is the length of $\vec{x}_\lambda$. If $\xi_0, \xi_1, \cdots$ are all ordinals $<\gamma$ such that $f(\xi) = \forall$ and if $\eta_0, \eta_1, \cdots$ are all ordinals $<\gamma$ such that $f(\eta) = \exists$, then $t_{\lambda,\mu,\xi_0}, t_{\lambda,\mu,\xi_1}, \cdots$ is an arbitrary sequence of members of D and $t_{\lambda,\mu,\eta} = g^{f,\eta}_{A_\lambda}(t_{\lambda,\mu,\xi_0}, \cdots, \vec{s}_\lambda)$. $\vec{t}_{\lambda,\mu}$ runs over all such sequences.

7) When a sequent $\Pi \longrightarrow \Lambda$ is

$$\Gamma' \longrightarrow \Delta', \{Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{s}_\lambda)\}_{\lambda<\delta}$$

where Δ' has no formulas whose outermost logical symbol is Q, and when $\Pi \longrightarrow \Lambda$ is constructed by 6), then the immediate ancestors of $\Pi \longrightarrow \Lambda$ are

$$\Gamma' \longrightarrow \Delta', \{A_\lambda(\vec{t}_{\lambda,\mu}, \vec{s}_\lambda)\}_{\mu,\lambda<\delta},$$

for all $\vec{t}_{\lambda,\mu}$ satisfying the following.
$\vec{t}_{\lambda,\mu}$ is $\{t_{\lambda,\mu,0}, \cdots, t_{\lambda,\mu,\nu}, \cdots\}_{\nu<\gamma}$ where γ is the length of $\vec{x}_\lambda$. If $\xi_0, \xi_1, \cdots$ are all ordinals $<\gamma$ such that $f(\xi) = \forall$, and, if $\eta_0, \eta_1, \cdots$ are all ordinals $<\gamma$ such that $f(\eta) = \exists$, then $t_{\lambda,\mu,\eta_0}, t_{\lambda,\mu,\eta_1}, \cdots$ are arbitrary members of D and $t_{\lambda,\mu,\xi} = \bar{g}^{f,\xi}_{A_\lambda}(t_{\lambda,\mu,\eta_0}, \cdots, \vec{s}_\lambda)$. $\vec{t}_{\lambda,\mu}$ runs over all such sequences.

8) When a sequent $\Pi \longrightarrow \Lambda$ is

$$\{s_\lambda = t_\lambda\}_{\lambda<\beta}, \Gamma' \longrightarrow \Delta',$$

where Γ' has no formulas of the form $s = t$ and when $\Pi \longrightarrow \Lambda$ is constructed by 7), then the immediate ancestor of $\Pi \longrightarrow \Lambda$ is the sequent $\Pi^1 \longrightarrow \Lambda^1$, where Π^1 and Λ^1 are sequences of all the formulas obtained from a formula in Π and Λ respectively, by arbitrary interchange of s_μ and $t_\mu (\mu < \beta)$. (So Π^1 and Λ^1 obviously include Π and Λ respectively.)

A branch of $P(S)$ is an infinite sequence $S = S_0, S_1, S_2, \cdots$ such that S_{n+1} is an immediate ancestor of S_n. We have two cases.

Case 1. In every branch of $P(S)$, there exists at least one sequent of the form

$$\Gamma_1, D, \Gamma_2 \longrightarrow \Delta_1, D, \Delta_2 \text{ or } \Gamma \longrightarrow \Delta_1, \quad s = s, \Delta_2$$

Case 2. There exists at least one branch of $P(S)$, in which there are no sequents of the form

$$\Gamma_1, D, \Gamma_2 \longrightarrow \Delta_1, D, \Delta_2 \text{ or } \Gamma \longrightarrow \Delta_1, \quad s = s, \Delta_2$$

In case 1, S is provable without cut. First we define the height of the free variable as follows.

2. 1. If $a \in D_0$, then $h(a) = 0$.

2. 2. If a is $g_A^{f,\tau}(b_0, \cdots, b_\xi, \cdots)$ or $\bar{g}_A^{f,\tau}(b_0, \cdots, b_\xi, \cdots)$,

then $h(a)$ is the supremum of all $h(b_\xi) + 1$'s.

It is easily shown that $P(S)$ satisfies the conditions 5. 1 and 5. 3 in § 1. In this proof, a figure P is said to be a semi-proof if and only if P satisfies all the conditions of a proof-figure except 5. 1–5. 3 in § 1. P is said to be a quasi-proof if and only if P satisfies all the conditions of a proof-figure except 5. 3 in § 1. Now consider the following conditions on P.

3. 1. P is a cut-free semiproof.

3. 2. Every individual constant or free variable in P occurs in $P(S)$ and every inference on Q in P occurs in $P(S)$.

If P satisfies 3. 1 and 3. 2, then P obviously satisfies 5. 1–5. 2 in § 1 and therefore P is a cut-free quasi-proof. Now consider the condition C on a sequent S' that S' has a quasi-proof P satisfying 3. 1 and 3. 2. Let

S' be in $P(S)$. It is easily seen that if every ancestor of S' satisfies C, then S' satisfies C. Suppose that S is not provable without cut. Then S does not satisfy C. Then some ancestor of S, say S_1, does not satisfy C. Continuing this argument, we get a sequence $S, S_1, S_2, \cdots$, where S_{n+1} is an ancestor of S_n and does not satisfy C for each n. This contradicts the hypothesis of Case 1.

In case 2, there exists a structure $\mathfrak{A}$ in which every formula in Γ is true and every formula in Δ is false. In the rest of this proof, we fix one branch, whose existence is assumed in the hypothesis of Case 2, and consider only the formulas and sequents in this branch, i.e. a sequent always means a sequent in this branch. We only have to define an interpretation which makes all the sequents in this branch false with respect to D.

LEMMA 1. *If a formula* $\neg A$ *occurs in the antecedent (or succedent) of a sequent, then the formula* A *occurs in the succedent (or antecedent) of a sequent.*

LEMMA 2. *If a formula* $\bigvee_{\lambda<\beta} A_\lambda$ *occurs in the antecedent (or succedent) of a sequent, then a formula* A_λ *for some (or every)* $\lambda<\beta$ *occurs in the antecedent (or succedent) of a sequent.*

LEMMA 3. *If a formula* $\bigwedge_{\lambda<\beta} A_\lambda$ *occurs in the antecedent (or succedent) of a sequent, then a formula* A_λ *for every (or some)* $\lambda<\beta$ *occurs in the antecedent (or succedent) of a sequent.*

LEMMA 4. *If* $Q^f\vec{x}A(\vec{x},\bar{s})$ *occurs in an antecedent of a sequent and* $\xi_0, \xi_1, \cdots$ *are all ordinals such that* $f(\xi)=\forall$ *and* $\eta_0, \eta_1, \cdots$ *are all ordinals such that* $f(\eta)=\exists$, *then for an arbitrary sequence* $t_{\xi_0}, t_{\xi_1}, \cdots$ *of members of* D, *the formula* $A(\bar{t})$ *is in an antecedent of a sequent, where* $t_\eta = g_A^{f,\eta}(t_{\xi_0}, \cdots, \bar{s})$ *for each* $\eta=\eta_0, \eta_1, \cdots$.

LEMMA 5. *If* $Q^f\vec{x}A(\vec{x},\bar{s})$ *occurs in a succedent of a sequent and* $\xi_0, \xi_1, \cdots$ *are all ordinals such that* $f(\xi)=\forall$ *and* $\eta_0, \eta_1, \cdots$ *are all ordinals such that* $f(\eta)=\exists$, *then for an arbitrary sequence* $t_{\eta_0}, t_{\eta_1}, \cdots$ *of members of* D, *the formula* $A(\bar{t})$ *is in a succedent of a sequent, where* $t_\xi = \bar{g}_A^{f,\xi}(t_{\eta_0}, \cdots, \bar{s})$ *for each* $\xi=\xi_0, \xi_1, \cdots$.

These lemmas are obvious.

LEMMA 6. *If a formula occurs in an antecedent of a sequent, then it does not occur in a succedent of any sequent.*

The proof is by transfinite induction on the complexity of a formula using Lemmas 1–5.

LEMMA 7.

1) *For every member t of D, the formula $t = t$ occurs in the antecedent of a sequent.*

2) *Let s and t be members of D, and if $s = t$ occurs in the antecedent of a sequent, then $t = s$ occurs in the antecedent of a sequent.*

3) *Let t_1, t_2 and t_3 be members of D, and if $t_1 = t_2$ and $t_2 = t_3$ occur in the antecedent of a sequent, then the formula $t_1 = t_3$ occurs in an antecedent of a sequent.*

4) *Let $s_\lambda, t_\lambda (\lambda < \beta)$ be members of D. If $A(s_0, \cdots, s_\lambda, \cdots)$ and $\{s_\lambda = t_\lambda\}_{\lambda<\beta}$ occur in the antecedent of a sequent, then $A(u_0, \cdots, u_\lambda, \cdots)$ occurs in the antecedent of a sequent for each sequence $u_0, \cdots, u_\lambda, \cdots$ such that u_λ is s_λ or t_λ.*

Proof. 1) $t = t$ must be contained in Φ and Ψ in 1. 1. Since $t = t$ cannot be contained in Ψ because of the hypothesis of Case 2, $t = t$ must be contained in Φ.

2) Let $s = t$ occur in the antecedent of a sequent and $t = s$ occur in the succedent of a sequent, then there is a sequent which contains $s = t$ in the antecedent and $t = s$ in the succedent. By the construction 8) of $P(S)$, there must be a sequent of the form $\Gamma_1 \longrightarrow \Delta_1, s = s, \Delta_2$ which is a contradiction.

3) and 4) can be proved by the similar way.

According to Lemma 7, D can be decomposed into equivalence-classes by $=$. Let $D/=$ be the set of equivalence-classes so obtained; from now on we write a class of $D/=$ by a representative of it. We define a structure $\mathfrak{A}$ over $D/=$ as follows. Let s be a variable in D. Then the value of s w.r.t. $\mathfrak{A}$ is defined to be the class represented by s. If P be a predicate constant, then $P(t_0, \cdots, t_\lambda, \cdots)$ is defined to be true w.r.t. $\mathfrak{A}$ if $P(t_0, \cdots, t_\lambda, \cdots)$ is in the antecedent of a sequent and is defined to be false w.r.t. $\mathfrak{A}$ otherwise. By the transfinite induction on the complexity of A, we shall prove that A is true w.r.t. $\mathfrak{A}$ if A is in the antecedent of a sequent and A is false w.r.t. $\mathfrak{A}$ if A is in the succedent of a sequent. Since the other cases are easy, we only consider the cases where A is $Q^f \vec{x} A(\vec{x}, \bar{s})$.

4. 1. $Q^f\vec{x}A(\vec{x}, \bar{s})$ in the antecedent of a sequent. In this case, it follows from the inductive hypothesis and 6) of the construction of $P(S)$ that $A(\bar{t}; \bar{s})$ is true w.r.t. $\mathfrak{A}$ for every $\bar{t}$ satisfying the following condition: *If* $\xi_0, \xi_1, \cdots$ *are all ordinals such that* $f(\xi) = \forall$ *and if* $\eta_0, \eta_1, \cdots$ *are all* ordinals such that $f(\eta) = \exists$, then $t_\eta = g_A^{f, \eta}(t_{\xi_0}, \cdots, \bar{s})$ for every η. This implies that $Q^f\vec{x}A(\vec{x}, \bar{s})$ is true w.r.t. $\mathfrak{A}$.

4. 2. $Q^f\vec{x}A(\vec{x}, \bar{s})$ is in the succedent of a sequent. In this case, it follows from the inductive hypothesis and 7) of the construction of $P(S)$ that $A(\bar{t}, \bar{s})$ is false w.r.t. $\mathfrak{A}$ for every $\bar{t}$ satisfying the following condition. If $\xi_0, \xi_1, \cdots$ are all ordinals such that $f(\xi) = \forall$ and, if $\eta_0, \eta_1, \cdots$ are all ordinals such that $f(\eta) = \exists$, then $t_\xi = \bar{g}_A^{f, \xi}(t_{\eta_0}, \cdots, \bar{s})$. This implies that $\neg A(\bar{t}, \bar{s})$ is true w.r.t. $\mathfrak{A}$ for every such $\bar{t}$. Hence follows that $Q^{\bar{f}}\vec{x} \neg A(\vec{x}, \bar{s})$ is true w.r.t. $\mathfrak{A}$. Since $Q^{\bar{f}}\vec{x} \neg A(\vec{x}, \bar{s}) \longrightarrow \neg Q^f\vec{x}A(\vec{x}, \bar{s})$ is satisfied in all the structures, $Q^f\vec{x}A(\vec{x}, \bar{s})$ is false w.r.t. $\mathfrak{A}$.

The following theorem is a completeness theorem for our determinate logic.

THEOREM 2. *Let* $\Gamma \longrightarrow \Delta$ *be a sequent. Then* $\Gamma \longrightarrow \Delta$ *is provable in our determinate logic or there exists a determinate structure* $\mathfrak{A}$ *such that every formula in* Γ *is satisfied in* $\mathfrak{A}$ *and no formula in* Δ *is satisfied in* $\mathfrak{A}$.

Let D and D_0 be the same as in the proof of Theorem 1. Now Γ_0 is defined to be a sequent of all formulas of the form

$$Q^f\vec{x}A(\vec{x}, \bar{s}) \vee Q^{\bar{f}}\vec{x}A(\vec{x}, \bar{s}),$$

where $A(\vec{x}, \bar{s})$ is an arbitrary formula in our language and $\vec{x}$ and $\bar{s}$ are only free variables in $A(\vec{x}, \bar{s})$ and $\bar{s}$ is an arbitrary sequence of members of D. $\tilde{\Gamma}$ is defined to be Γ_0, Γ. Without loss of generality, we may assume that no member of D_0 is ever used as an eigenvariable in any quasi-proof. Then we have the following theorem.

THEOREM 3. $\tilde{\Gamma} \longrightarrow \Delta$ *has a cut-free quasi-proof whose end-sequent is* $\tilde{\Gamma} \longrightarrow \Delta$ *or else there exists a determinate structure* $\mathfrak{A}$ *such that every formula in* $\tilde{\Gamma}$ *is satisfied in* $\mathfrak{A}$ *and every formula in* Δ *is not satisfied in* $\mathfrak{A}$.

Theorem 3 implies Theorem 2 as follows. Since every formula in Γ_0 is provable in our determinate logic as in § 2, 2), $\Gamma \longrightarrow \Delta$ is obtained from $\tilde{\Gamma} \longrightarrow \Delta$ by a cut as follows

$$\frac{\longrightarrow B_0, \cdots, \longrightarrow B_\beta, \cdots, B_0, \cdots, B_\beta, \cdots, \Gamma \longrightarrow \Delta \quad (\vdots P)}{\Gamma \longrightarrow \Delta}$$

where $B_0, \cdots, B_\beta, \cdots$ is Γ_0. It is easily seen that the thus obtained figure satisfies all the properties of a proof-figure including 5. 3 in § 1.

The proof of Theorem 3 is obtained from the proof of Theorem 1 by replacing a proof-figure and Γ by a quasi-proof and $\tilde{\Gamma}$ respectively. Since $\tilde{\Gamma}$ includes Γ_0, it is easily shown that $\mathfrak{A}$ is determinate.

DEFINITION. *HLS* (a homogeneous logic system) is obtained from our determinate logic by adding a restriction that every quantifier be homogeneous.

Then the similar argument to § 3 and Theorem 1 in this section implies the following theorem.

THEOREM 4. *If $\Gamma \longrightarrow \Delta$ is provable in HLS, then $\Gamma \longrightarrow \Delta$ is valid. Conversely, if every quantifier in $\Gamma \longrightarrow \Delta$ is homogeneous, then $\Gamma \longrightarrow \Delta$ is provable without cuts in HLS or there exists a structure $\mathfrak{A}$ such that every formula in Γ is satisfied in $\mathfrak{A}$ and every formula in Δ is not satisfied in $\mathfrak{A}$.*

Remark. We cannot improve Theorem 2 by replacing "provable" by "provable without cuts". In order to see this, let α be the initial ordinal of the cardinality of ω_2. Let $f \in \omega_2$. Then $\psi(f)$ is defined to be $a_0 = i_0 \wedge a_1 = i_1 \wedge \cdots$, where $i_k = 0$ or 1 according as $f(k) = 0$ or 1. The formula $\psi(f)$ expresses the function f. Now let $A \subseteq \omega_2$. Then A is expressed by the formula $\bigvee_{f \in A} \psi(f)$, where $\bigvee_{f \in A}$ is defined in terms of $\bigvee_{\alpha}$. Now the theorem in [1] implies that there exists a set $A \subseteq \omega_2$ such that the axiom of determinateness fails for the game defined by A. If a formula ψ expresses A, then

$$\forall x(x = 0 \vee x = 1) \longrightarrow 0 = 1,$$
$$\neg(\forall x_0 \exists x_1 \forall x_2 \cdots \psi(x_0, x_1, \cdots) \vee \exists x_0 \forall x_1 \exists x_2 \cdots \neg\psi(x_0, x_1, \cdots))$$

is provable in our determinate logic, where ψ is constructed by $0, 1, =, \bigwedge_{\omega}$, and $\bigvee_{\alpha}$. This means that $\forall x(x = 0 \vee x = 1) \longrightarrow 0 = 1$ is provable in our determinate logic if our language has $\bigvee_{\alpha}$. However this is not provable without cuts even if our language has $\bigvee_{\alpha}$.

§ 5. An interpolation theorem.

First, we shall define some proof-theoretic notion.

DEFINITION. Let P be a semi-proof without cut and I be an inference in P. Let A be a formula in an upper sequent of I and B be a formula in the lower sequent of I. B is said to be the immediate successor of A if and only if the following is satisfied.

Case 1). I is 2. 1 in § 1.

If A is a formula in Γ in 2. 1, then B is the first formula in Π, which is equal to A. If A is formula in Δ in 2. 1, then B is the first formula in Λ, which is equal to A.

Case 2). I is one of 2. 2–2. 9 or 4. 2 and A is a formula in Γ or Δ in the upper sequent of I.

If A is the α-th formula in Γ or Δ, then B is the α-th formula in Γ or Δ in the lower sequent respectively.

Case 3). If I is 2. 2 or 2. 3 and A is A_λ, then B is $\neg A_\lambda$. If I is 2. 4 or 2. 5 and A is $A_{\lambda,\mu}$ or A_{λ,μ_λ}, then B is $\bigvee_{\mu<\beta_\lambda} A_{\lambda,\mu}$. If I is 2. 6 or 2. 7 and A is A_{λ,μ_λ} or $A_{\lambda,\mu}$, then B is $\bigwedge_{\mu<\beta_\lambda} A_{\lambda,\mu}$. If I is 2. 8 or 2. 9 and A is $A_\lambda(\vec{a}_\lambda)$, then B is $Q^{f_\lambda}\vec{x}_\lambda A_\lambda(\vec{x}_\lambda)$. If I is 4. 1 and A is the α-th formula in $\Gamma^{(\vec{a})}$ of $\Delta^{(\vec{a})}$, then B is the α-th formula in $\Gamma^{(\vec{b})}$ or $\Delta^{(\vec{b})}$ respectively.

B is said to be a successor of A, if there exists a sequence $A_0, A_1, \cdots, A_n$ such that $A = A_0$ and $B = A_n$ and A_{i+1} is the immediate successor of A_i for each $i < n$.

Now our interpolation theorem is the following form.

THEOREM 1. *If a sequent* $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ *is valid and does not have any hetrogeneous quantifier, then there exists a formula C such that both the sequents*

$$\Gamma_1 \longrightarrow \Delta_1, C \quad \textit{and} \quad C, \Gamma_2 \longrightarrow \Delta_2$$

are valid and every free variable or predicate constant in C, except $=$, *occurs in both* Γ_1, Δ_1 *and* Γ_2, Δ_2. (*Remark that C may have hetrogeneous quantifiers and or also longer logical connective or quantifiers than logical symbols in L*).

Our proof follows the proof of Theorem 5 in [3]. At first, we shall prove the following lemma.

LEMMA 1. *Let P be a cut-free proof-figure to* $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ *in HLS and satisfy the following conditions* 1. 1–1. 3.

1. 1. *Every quantifier in P is* $\exists$.

2. 2. *Every inference in P on the introduction of Q is an inference on the introduction of* $\exists$ *in succedent.*

Then there exist cut-free proof-figures P_1 *and* P_2 *in RHS and C satisfying the following conditions.*

2. 1. *The end-sequent of* P_1 *is* $C, \Gamma_1 \longrightarrow \Delta_1$ *and the end-sequent of* P_2 *is* $\Gamma_2 \longrightarrow \Delta_2, C$.

2. 2. *Every free variable or predicate constant in C, except =, occurs in both* Γ_1, Δ_2 *and* Γ_2, Δ_2 *or =.* (*Remark that* 1. 1 *is not an essential restriction on P because* $\forall$ *can be expressed by* $\neg$ *and* $\exists$.)

Proof. The proof is by transfinite induction on the complexity of *P*.

Case 1: *P* consists of a single beginning sequent. The theorem is obvious.

Case 2: The last inference of *P* is of the form

$$\frac{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1', \{A_\lambda(\vec{a}_\lambda)\}_{\lambda<\beta_1}, \Delta_2', \{B_\mu(b_\mu)\}_{\mu<\beta_2}}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1', \{\exists x_\lambda A_\lambda(\vec{x}_\lambda)\}_{\lambda<\beta_1}, \Delta_2', \{\exists \vec{y}_\mu B_\mu(\vec{y}_\mu)\}_{\mu<\beta_2}},$$

where Δ_1 is $\Delta_1', \{\exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda)\}_{\lambda<\beta_1}$ and Δ_2 is $\Delta_2', \{\exists \vec{y}_\mu B_\mu(\vec{y}_\mu)\}_{\mu<\beta_2}$.

By the inductive hypothesis, there exists $C'(\vec{a}, \vec{b})$ satisfying the following conditions.

1) $C'(\vec{a}, \vec{b}),\ \Gamma_1 \longrightarrow \Delta_1', \{A_\lambda(\vec{a}_\lambda)\}_{\lambda<\beta_1}$ and $\Gamma_2 \longrightarrow \Delta_2', \{B_\mu(\vec{b}_\mu)\}_{\mu<\beta_2}, C'(\vec{a}, \vec{b})$ are provable in *RHS*.

2) Every free variable and predicate constants in $C'(\vec{a}, \vec{b})$ is either = or contained in both $\Gamma_1, \Delta_1', \{A_\lambda(\vec{a}_\lambda)\}_{\lambda<\beta_1}$ and $\Gamma_2, \Delta_2', \{B_\mu(\vec{b}_\mu)\}_{\mu<\beta_2}$. $\vec{a}$ is a sequence of all variables in $C'(\vec{a}, \vec{b})$ which are not in Γ_1, Δ_1. $\vec{b}$ is a sequence of all variables in $C'(\vec{a}, \vec{b})$ which are not in Γ_2, Δ_2. Then a required formula C is $\exists \vec{x} \forall \vec{y} C'(\vec{x}, \vec{y})$, where $\forall$ is considered as an abbreviation of $\neg \exists \neg$.

Case 3: The last inference of *P* is of the form

$$\frac{\Gamma_1'^{(\vec{a})}, \Gamma_2'^{(\vec{a})} \longrightarrow \Delta_1^{(\vec{a})}, \Delta_2^{(\vec{a})}}{\vec{a}_1 = \vec{b}_1, \vec{a}_2 = \vec{b}_2, \Gamma_1'^{(\vec{b})}, \Gamma_2'^{(\vec{b})} \longrightarrow \Delta_1^{(\vec{b})}, \Delta_2^{(\vec{b})}}$$

where Γ_1 is $\vec{a}_1 = \vec{b}_1, \Gamma_1'^{(\vec{b})}$ and Γ_2 is $\vec{a}_2 = \vec{b}_2, \Gamma_2'^{(b)}$. This can be divided in two steps, namely; first, the substitution of $\vec{a}_1$ for $\vec{b}_1$; then the substitution of $\vec{a}_2$ for $\vec{b}_2$. So we may assume that $\vec{a}_1 = \vec{b}_1$ is empty. By the inductive hypothesis, there exists a formula $C'(\vec{a}, \vec{b})$ which satisfies the theorem for $\Gamma_1'^{(\vec{a})}, \Gamma_2'^{(\vec{a})} \longrightarrow \Delta_1^{(\vec{a})}, \Delta_2^{(\vec{a})}$. $\vec{a}$ is a sequence consisting of all variables in $C'(\vec{a}, \vec{b})$ which are not in Γ_1, Δ_1 and $\vec{b}$ is a sequence of all variables in $C'(\vec{a}, \vec{b})$ which are not in Γ_2, Δ_2. Then take C to be $\exists\vec{x}\forall\vec{y}(\bigwedge_{\mu} \breve{a}_{2,\mu} = \breve{b}_{2,\mu} \wedge C'(\vec{x}, \vec{y}))$, where $\breve{a}_{2,\mu}$ or $\breve{b}_{2,\mu}$ means some x or some y if $a_{2,\mu}$ or $b_{2,\mu}$ are in $\vec{a}$ or $\vec{b}$ and $a_{2,\mu}$ or $b_{2,\mu}$ otherwise.

Case 4). The last inference of P is of the form

$$\frac{\Phi, \Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2, \Psi}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2} \quad \text{for all } (\Phi|\Psi).$$

By the inductive hypothesis, there exist formulas $C_{(\Phi|\Psi)}$ such that $C_{(\Phi|\Psi)}, \Gamma_1 \longrightarrow \Delta_1$ and $\Phi, \Gamma_2 \longrightarrow \Delta_2, \Psi, C_{(\Phi|\Psi)}$ are provable in *RHS*. So $\bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}, \Gamma_1 \longrightarrow \Delta_1$ and $\Gamma_2 \longrightarrow \Delta_2, \bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}$ are provable in *RHS*. Let $\vec{a}$ be a sequence of all free variables in $\bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}$ which do not appear in Γ_2, Δ_2. We rewrite $\bigvee_{(\Phi|\Psi)} C_{(\Phi|\Psi)}$ as $C'(\vec{a})$. Then take C to be $\forall\vec{x}C'(\vec{x})$.

Other cases: The proof is similar to the above.

Now we shall consider the proof of Theorem 1.

Proof of Theorem 1. Since $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ is valid, there exists a cut-free proof-figure P to $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ in *HLS*. From § 4 follows that P may be assumed to satisfy the following condition.

3. 1. If a variable occurs in two different auxiliary formulas as an eigenvariable, then these two formulas are the same.

Moreover we assume the following on P without loss of generality.

3. 2. Every quantifier in P is $\exists$.

3. 3. The height of a free variable in $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ is less than the height of any eigenvariable in P.

3. 4. The heights of two different variables in P are different.

Let $\Gamma_1' \longrightarrow \Delta_1'$ be a sequent in P. $\Phi(\Gamma_1', \Delta_1')$ be the sequence $A_0, A_1, \cdots, A_\mu, \cdots$ of all A_μ's such that A_μ is of the form $\neg\exists\vec{x}A(\vec{x}) \vee A(\vec{a})$ where $\exists\vec{x}A(\vec{x})$ is a principal formula of an introduction of $\exists$ in antecedent above

$\Gamma_1' \longrightarrow \Delta_1'$ and $A(\vec{a})$ is its auxiliary formula. Replacing $\Gamma_1' \longrightarrow \Delta_1'$ by $\Phi(\Gamma_1', \Delta_1'), \Gamma_1' \longrightarrow \Delta_1'$ and inserting some appropriate structural inferences, we get a new figure P' satisfying the following conditions.

4. 1. P' satisfies 1. 1 and 1. 2 in Lemma 1.

4. 2. The end-sequent of P' is of the following form

$$\{\neg \exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{c}_\lambda) \vee A_\lambda(\vec{a}_\lambda, \vec{c}_\lambda)\}, \Gamma_1,$$
$$\{\neg \exists \vec{y}_\mu B_\mu(\vec{y}_\mu, \vec{d}_\mu) \vee B_\mu(\vec{b}_\lambda, \vec{d}_\mu)\}, \quad \Gamma_2 \longrightarrow \Delta_1, \Delta_2.$$

4. 3. The height of any $c_{\lambda,\alpha}$ is less than the height of any $a_{\lambda,\beta}$. The height of any $d_{\mu,\alpha}$ is less than the height of any $b_{\mu,\beta}$.

4. 4. Every free variable or predicate constant except $=$ in $\exists \vec{z}_\lambda \exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{z}_\lambda)$ occurs in Γ_1, Δ_1 and every free variable or predicate constant except $=$ occuring in $\exists \vec{z}_\mu \exists \vec{y}_\mu B_\mu(\vec{y}_\mu, \vec{z}_\mu)$ occurs in Γ_2, Δ_2.

4. 5. Any $a_{\lambda,\alpha}$ and $b_{\mu,\beta}$ are different. (Otherwise we can modify P' so that P' satisfies 4. 5, because P satisfies 3. 1.)

Applying Lemma 1, we have $C(\vec{a})$ such that the following conditions are satisfied.

5. 1. $C(\vec{a}), \{\neg \exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{c}_\lambda) \vee A_\lambda(\vec{a}_\lambda, \vec{c}_\lambda)\}, \ \Gamma_1 \longrightarrow \Delta_1$ and $\{\neg \exists \vec{y}_\mu B_\mu(\vec{y}_\mu, \vec{d}_\mu) \vee B_\mu(\vec{b}_\lambda, \vec{d}_\mu)\}, \ \Gamma_2 \longrightarrow \Delta_2, C(\vec{a})$ are provable in *RHS* and let Q_1 and Q_2 be proof-figures to these sequents in *RHS*.

5. 2. Every free variable or predicate constant except $=$ occuring in $C(\vec{a})$ is in both $\{A_\lambda(\vec{a}_\lambda, \vec{c}_\lambda)\}, \Gamma_1, \Delta_1$ and $\{B_\mu(\vec{b}_\lambda, \vec{d}_\mu)\}, \ \Gamma_2, \Delta_2$.

5. 3. $\vec{a}$ is the sequence of all variables in $C(\vec{a})$ which are not in both Γ_1, Δ_1 and Γ_2, Δ_2 and well-ordered according to heights.

Then consider the following figure

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\, Q_1}{C(\vec{a}), \{\neg \exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{c}_\lambda) \vee A_\lambda(\vec{a}_\lambda, \vec{c}_\lambda)\}, \Gamma_1 \longrightarrow \Delta_1}}{C(\vec{a}), \{\exists \vec{x}_\lambda'(\neg \exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{c}_\lambda) \vee A_\lambda(\vec{x}_\lambda', \vec{c}_\lambda))\}, \Gamma_1 \longrightarrow \Delta_1}}{C(\vec{a}), \{\forall \vec{z}_\lambda \exists \vec{x}_\lambda'(\neg \exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{z}_\lambda) \vee A_\lambda(\vec{x}_\lambda', \vec{z}_\lambda))\}, \Gamma_1 \longrightarrow \Delta_1}}{Q^f \vec{x} C(\vec{x}), \{\forall \vec{z}_\lambda \exists \vec{x}_\lambda'(\neg \exists \vec{x}_\lambda A_\lambda(\vec{x}_\lambda, \vec{z}_\lambda) \vee A_\lambda(\vec{x}_\lambda', \vec{z}_\lambda))\}, \Gamma_1 \longrightarrow \Delta_1}$$

where f is defined as follows.

6. 1. If a_α is not contained in Γ_1, Δ_1 or Γ_2, Δ_2 and a_α is one of $b_{\mu,\gamma}$, then $f(\alpha) = \exists$.

6. 2. If a_α is not contained in Γ_1, Δ_1 or Γ_2, Δ_2 and a_α is one of $a_{\lambda,\gamma}$, then $f(\alpha) = \forall$.

6. 3. If a_α is contained in Γ_1, Δ_1 but not in Γ_2, Δ_2, then $f(\alpha) = \forall$.

6. 4. If a_α is contained in Γ_2, Δ_2, but not in Γ_1, Δ_1, then $f(\alpha) = \exists$.

6. 5. If 6. 1–6. 4 are not the case, then $f(\alpha) = \exists$.

The heights in $\vec{a}_\lambda, \vec{c}_\lambda, C(\vec{a}), \Gamma_1, \Delta_1$ are defined to be the heights in P. The heights of all other variables in Q_1 can be defined adequately according to Proposition 1 in **§ 3** so that the whole proof-figure satisfies 5. 1–5. 3 in **§ 1.** This means $Q^f\vec{x}C(\vec{x}), \Gamma_1 \longrightarrow \Delta_1$ is valid. The validity of $\Gamma_2 \longrightarrow \Delta_2, Q^f\vec{x}C(\vec{x})$ is also easily shown by observing the following proof-figure in VSS.

$$\begin{array}{c} \vdots\, Q_2 \\ \{\neg \exists \vec{y}_\mu B_\mu(\vec{y}_\mu, \vec{d}_\mu) \vee B_\mu(\vec{b}_\mu, \vec{d}_\mu)\}, \Gamma_2 \longrightarrow \Delta_2, C(\vec{a}) \\ \hline \{\exists \vec{y}'_\mu(\neg \exists \vec{y}_\mu B_\mu(\vec{y}_\mu, \vec{d}_\mu) \vee B_\mu(\vec{y}'_\mu, \vec{d}_\mu)\}, \Gamma_2 \longrightarrow \Delta_2, C(\vec{a}) \\ \hline \{\forall \vec{z}_\mu \exists \vec{y}_\mu(\neg \exists \vec{y}_\mu B_\mu(\vec{y}_\mu, \vec{z}_\mu) \vee B_\mu(\vec{y}'_\mu, \vec{z}_\mu)\}, \Gamma_2 \longrightarrow \Delta_2, C(\vec{a}) \\ \hline \{\forall \vec{z}_\mu \exists \vec{y}'_\mu(\neg \exists \vec{y}_\mu B_\mu(\vec{y}_\mu, \vec{z}_\mu) \vee B_\mu(\vec{y}'_\mu, \vec{z}_\mu)\}, \Gamma_2 \longrightarrow \Delta_2, Q^f\vec{x}C(\vec{x}) \end{array}$$

q.e.d.

In the same way, we can show the following theorem.

THEOREM 2. *If every quantifier in* $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ *is homogeneous and* $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ *is valid and does not contain* $=$, *and if* Γ_1, Δ_1 *and* Γ_2, Δ_2 *have at least one predicate constant in common, then there exists a formula C such that both the sequent*

$$C, \Gamma_1 \longrightarrow \Delta_1 \quad and \quad \Gamma_2 \longrightarrow \Delta_2, C$$

are valid and every free variable or predicate constant in C is contained in both Γ_1, Δ_1 *and* Γ_2, Δ_2.

Remark. 1. In theorems 1 and 2, we may add the condition that the hetrogeneous quantifier in C is only one in the front of C.

Remark. 2. As for Malitz's example in **§ 2,** we can construct an isomorphism between $\overset{1}{<}$ and $\overset{2}{<}$ by the following formula

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots (\bigwedge_i x_i \overset{1}{<} a \longrightarrow \bigwedge_i y_i \overset{2}{<} b \wedge \bigwedge_{i,j} (x_i \overset{1}{<} x_j \longleftrightarrow y_i \overset{2}{<} y_j) \wedge (x_i \overset{1}{=} x_j \longleftrightarrow y_i \overset{2}{=} y_j))$$

$$\wedge \forall y_1 \exists x_1 \forall y_2 \exists x_2 \cdots (\bigwedge_i y_i \overset{2}{<} b \longrightarrow \bigwedge_i x_i \overset{1}{<} a \wedge \bigwedge_{ij} (x_i \overset{1}{<} x_j \longleftrightarrow y_i \overset{2}{<} y_j) \wedge (x_i \overset{1}{=} x_j \longleftrightarrow y_i \overset{2}{=} y_j)).$$

The order type a of in $(\overset{1}{=}, \overset{1}{<})$ is denoted by $|a|_1$ and the order type of b in $(\overset{2}{=}, \overset{2}{<})$ is denoted by $|b|_2$. Then $|a|_1 \leq |b|_2$ is equivalent to

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots (\bigwedge_i x_i \overset{1}{<} a \longrightarrow \bigwedge_i y_i \overset{2}{<} b \wedge \bigwedge_{ij} (x_i \overset{1}{<} x_j \longleftrightarrow y_i \overset{2}{<} y_j) \wedge (x_i \overset{1}{=} x_j \longleftrightarrow y_i \overset{2}{=} y_j)).$$

This is easily shown by transfinite induction on $|a|_1$.

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